Chapter 5

Existence Of Fixed Points For A Class Of ANT Mappings

5.1 Abstract

The object of this chapter is to study the existence of fixed points of the new class mappings of k- ANT in a Banach space with uniformly normal structure and the asymptotic behavior of the iterates \( \{T^n x\} \) for a fixed point of mapping T of ANT in a Banach space satisfying uniform Opial condition. Our results improve and generalize upon results of \([25, 24, 64, 52, 75, 29, 34, 82, 56]\).

5.2 Introduction

Let D be a nonempty subset of a Banach space X and let \( T : D \to D \) a mapping. Then T is said to be asymptotically nonexpansive \([25]\) if there exists a sequence \( \{k_n\} \subset [1, \infty) \) of positive numbers with \( \lim_{n \to \infty} k_n = 1 \) such that

\[
\|T^n x - T^n y\| \leq k_n \|x - y\|
\]
for all \( x, y \in D \) and \( n \geq 1 \). \( T \) is said to be \textit{mapping of asymptotically nonexpansive type (ANT)} if for each \( x \in D \), there holds the inequality:

\[
\limsup_{n \to \infty} \left\{ \sup \left( ||T^n x - T^n y|| - ||x - y|| \right) : y \in D \right\} \leq 0.
\]

In 1974, the class of mapping of ANT was introduced by Kirk [80] as a generalization of the class of asymptotically nonexpansive mappings and he proved that \( T \) has fixed point if \( D \) is a closed convex bounded subset of a Banach space with its characteristic of convexity less than one. In [24], Xu extended the Kirk’s results to nearly uniformly convex (NUC) Banach space.

On the other hand, the asymptotic behavior of the iterates \( \{T^n x\} \) for an asymptotically nonexpansive mapping has been studied by Xu [24], Bose[64], Hirano [52], Lim and Xu [75], Gornicki[29], Tan and Xu [34] and others in uniformly convex Banach spaces.

We now introduce the notion of \textit{mappings of k-asymptotically nonexpansive type (k-ANT)}.

A mapping \( T : D \to D \) is said to be mapping of k-ANT if for each \( x \in D \), there exists a sequence \( \{k_n\} \) of nonnegative real numbers with \( \lim_{n \to \infty} k_n = k > 0 \) such that

\[
\limsup_{n \to \infty} \left\{ \sup \left( ||T^n x - T^n y|| - ||x - y|| \right) : y \in D \right\} \leq \limsup_{n \to \infty} k_n ||x - T^n x||.
\]

The mapping of k-ANT is called uniformly k-ANT if \( k_n = k \ \forall n \geq 1 \). The class of mappings of uniformly 0-ANT is the class of mappings of ANT. It is important to note that the class of mappings of uniformly k-ANT is essentially wider than the
class of mappings of ANT. Without loss of generality, we shall assume that \( k \in (0,1) \).

**Remark 5.2.1.** If \( D \) is bounded and \( T : D \to D \) a mapping satisfying the following:

\[
||T^{n} x - T^{n} y|| \leq L_n ||x - y|| + k_n ||x - T^n x||
\]

for all \( x, y \in D \) and \( n \geq 1 \), where \( L_n \) and \( k_n \) are nonnegative real number such that \( \lim_{n \to \infty} L_n = 1 \) and \( \lim_{n \to \infty} k_n \) exists, then \( T \) is mapping of k-ANT.

**Proof.** Since \( D \) is bounded, then for each \( x \in D \),

\[
[||T^{n} x - T^{n} y|| - ||x - y||] \leq ||L_n - 1|| ||x - y|| + k_n ||x - T^n x||
\]

and so

\[
\sup_{y \in D} [||T^{n} x - T^{n} y|| - ||x - y||] \leq ||L_n - 1|| \sup_{x,y \in D} ||x - y|| + k_n ||x - T^n x||.
\]

Therefore,

\[
\limsup_{n \to \infty} \sup_{y \in D} [||T^{n} x - T^{n} y|| - ||x - y||] \leq \limsup_{n \to \infty} k_n ||x - T^n x||,
\]

i.e., \( T \) is mappings of k-ANT.

\[\square\]

### 5.3 Section-I

**5.3.1 Existence Result**

In this section\(^1\), we prove the existence of fixed points of mappings of k-ANT in a Banach spaces with uniformly normal structure. To do so, we first recall some

definitions:

The uniformity $\tilde{N}(X)$ of normal structure of a Banach space $X$ is defined (cf. [82]) by

$$\tilde{N}(X) = \sup \{ R_c(C) : C \text{ is a convex bounded subset if } X \text{ with } \text{diam } C > 0 \},$$

where $R_c(C) = \inf_{x \in C} (\sup_{y \in C} ||x - y||)$ is Chebyshev radius of $C$ relative to itself and $C = \sup_{x,y \in C} ||x - y||$ is the diameter of $C$.

We say that $X$ has uniformly normal structure if $\tilde{N}(X) < 1$. It is well known that a space with uniformly normal structure is reflexive and that all uniformly convex and uniformly smooth Banach spaces have uniformly normal structure (cf. [1]). Recently, Pichugov [59] calculated that

$$\tilde{N}(L^p) = \max \{ 1/2^p, 1/2^{(p-1)/p} \}, 1 < p < \infty.$$

Recall that a subset $D$ of a Banach space $X$ is said to have the property (P):

(P) $x \in D$ implies $\omega_{w}(x) \subset D$, where $\omega_{w}(x)$ is the weak $\omega$-limit set of $T$ at $x$, i.e.,

$$\{ y \in X : y = \text{weak } \lim_{j \to \infty} T^{n_j} x \text{ for some } n_j \to \infty \}$$

Following lemma is needed to prove our results of this section.

**Lemma 5.3.1.** Let $D$ be a nonempty closed convex subset of a Banach space $X$ and $T : D \to D$ an asymptotically regular mapping such that for some $m \geq 1$, $T^m$ is continuous. If there is an increasing sequence $\{n_i\}$ of natural numbers such that $\lim_{i \to \infty} ||z - T^{n_i}x|| = 0$ for some $x \in D$ and $z \in D$, then $z = Tz$.

**Theorem 5.3.2.** Let $X$ be a Banach space with uniformly normal structure, $D$ a nonempty subset of $X$ and $T : D \to D$ an asymptotically regular mapping of $k$-ANT such that $\liminf_{n \to \infty} k_n = k < \frac{1 - \tilde{N}(X)}{1 + \tilde{N}(X)}$ and for some $N \geq 1$, $T^N$ is continuous.
Suppose that there is a closed convex bounded subset $C$ of $D$ with property $(P)$ and for each $x \in C, \{T^n x\}$ is bounded. Then $T$ has a fixed point in $C$.

Proof. Let $x_0$ be an arbitrary element of $C$. Since the sequence $\{T^n x_0\}$ is bounded, then by definition of $\tilde{N}(X)$, we have a $z_n \in \bar{co}\{T^e x_0 : e \geq n\}$ such that

$$\limsup_{e \to \infty} ||z_n - T^e x_0|| \leq \tilde{N}(X) \limsup_{n \to \infty} (sup\{||T^i x_0 - T^j x_o|| : i, j \geq n\}) (5.3.1)$$

By reflexivity of $X$, we have a sequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\omega-\lim_{k \to \infty} z_{n_k} = x_1$, it follows from (5.3.1) and $\omega$-1.s.c of functional $\limsup_{j \to \infty} ||x - T^j x_0||$ that

$$\limsup_{n \to \infty} ||x_1 - T^n x_0|| \leq \tilde{N}(X) \limsup_{n \to \infty} (sup\{||T^i x_0 - T^j x_o|| : i, j \geq n\}).$$

It can be easily seen that $x_1 \in \bigcap_{n \geq 1} \bar{co}\{T^j x_0 : j \geq n\}$ and

$$||x - x_1|| \leq \limsup_{n \to \infty} ||x - T^n x_0|| \text{ for all } x \in X.$$

Using property $(P)$, the fact $\bigcap_{n \geq 1} \bar{co}\{T^j x_0 : j \geq n\} = \bar{co}\omega_0(X)$ and the Separation Theorem [59] , we have $x_1 \in C$. So, we can inductively define a sequence $\{x_m\}$ in the following manner : for all integers $m \geq 1$ and $x \in X$

$$\limsup_{n \to \infty} ||x_m - T^n x_{m-1}|| \leq \tilde{N}(X) \limsup_{n \to \infty} (sup\{||T^i x_{m-1} - T^j x_{m-1}|| : i, j \geq n\}) (5.3.2)$$

$$||x - x_m|| \leq \limsup_{n \to \infty} ||x - T^n x_{m-1}||. \quad (5.3.3)$$

Now consider a sequence $\{n_i\}$ of natural numbers such that

$$k = \liminf_{n \to \infty} k_n = \lim_{i \to \infty} k_{n_i},$$

For each $m \geq 1$, we set,

$$D_m = \limsup_{i \to \infty} ||x_m - T^{n_i} x_m||,$$
\( r_m = \limsup_{i \to \infty} ||x_{m+1} - T^m x_m||. \)

Since T is a mapping of k-ANT, then, we have,

\[
\sup_{n, n_j \geq n} ||T^n x_m - T^{n_j} x_m|| \leq \sup_{n, n_j \geq n} \left\{ \left[ ||T^n x_m - T^n (T^{n_j-n} x_m)|| - ||x_m - T^{n_j-n} x_m|| \right] + \left[ ||x_m - T^{n_j-n} X_m|| \right] \right\}
\]

\[
\leq \sup_{n, n_j \geq n} \left\{ \left[ \sup_{y \in D} \left[ ||T^n x_m - T^n y|| - ||x_m - y||: y \in D \right] \right] + \left[ ||x_m - T^{n_j-n} X_m|| \right] \right\}
\]

\[
\leq \sup_{n \geq n_j} \left\{ \sup_{n_j \geq n} \left[ ||T^n x_m - T^{n_j} y|| - ||x_m - y||: y \in D \right] \right\} + \sup_{n, n_j \geq n} \left( ||X_m - T^{n_j} X_m|| + ||T^{n_j-n} X_m - T^{n_j} X_m|| \right),
\]

it follows from (5.3.2) and asymptotic regularity of T that

\[
r_m \leq (1 + k) \tilde{N}(X) D_m.
\]

Moreover, from (5.3.3) for \( i > 1 \), we have

\[
||x_{m+1} - T^{m} x_{m+1}|| \leq \limsup_{j \to \infty} ||T^{n_j} x_m - T^{n_i} x_{m+1}||
\]

\[
\leq \limsup_{j \to \infty} \{ \left[ ||T^{n_i} (T^{n_j-n} x_m) - T^{n_i} x_{m+1}|| - ||T^{n_j-n} x_m - x_{m+1}|| \right] + \left[ ||T^{n_j-n} x_m - x_{m+1}|| \right] \}
\]

\[
\leq \sup_{y \in D} ||T^{n_i} y - T^{n_i} x_{m+1}|| - ||y - x_{m+1}||
\]

\[
+ \limsup_{j \to \infty} ||T^{n_j-n} x_m - x_{m+1}||
\]

Taking limit superior as \( i \to \infty \) on each side and using asymptotic regularity, we get

\[
D_{m+1} \leq k \cdot D_m + r_m,
\]

Which implies

\[
D_{m+1} \leq (1 - k)^{-1} r_m.
\]
Thus, we conclude that
\[ D_{m+1} \leq \frac{1 + k}{1 - k} \bar{N}(X) D_m \leq A D_m \forall m = 0, 1, 2, \]
where \( A = \frac{1 + k}{1 - k} \bar{N}(X) < 1 \). Since
\[ ||x_{m+1} - x_m|| \leq r_m + D_m \leq 2D_m \leq \ldots \leq 2A^m D_{0-0} \]
as \( m \to \infty \), it follows that \( \{x_m\} \) is a cauchy sequence and it converges to some point \( z \) in \( C \). Then, we have that
\[ ||z - T^m z|| \leq ||z - z_m|| + ||z_m - T^m z_m|| + ||T^m z_m - T^m z|| \]
\[ \leq 2||z - z_m|| + D_m + \sup \{||T^m y - T^m z|| : y \in D\} \]
Taking the limit superior as \( i \to \infty \) on each side, we obtain
\[ \limsup_{i \to \infty} ||z - T^i z|| \leq 2||z - z_m|| + D_m + k \limsup_{i \to \infty} ||z - T^i z|| \]
and so,
\[ (1 - k) \limsup_{i \to \infty} ||z - T^i z|| \leq 2||z - z_m|| + D_m \to 0 \text{ as } m \to 0 \]
Since \( T \) is an asymptotically regular such that for some \( N \geq 1, T^N \) is continuous mapping, it follows from lemma 5.3.1 that \( z = Tz \).

Now, we are able to remove the asymptotically regularity assumption on \( T \) by strengthening the assumption on sequence \( \{k_n\} \).

**Theorem 5.3.3.** Let \( X \) be a Banach space with uniformly normal structure, \( D \) a nonempty subset of \( X \) and \( T : D \to D \) a mapping of \( k \)-ANT such that \( \liminf_{n \to \infty} k_n = k < \frac{1 - \bar{N}(X)}{1 + \bar{N}(X)} \) and for some \( N \geq 1, T^N \) is continuous. Suppose that there is a closed convex bounded subset \( C \) of \( D \) with property \((P)\) and for each \( x \in C, \{T^n x\} \) is bounded. Then \( T \) has a fixed point in \( C \).
Proof. As Theorem 5.3.1 there is a sequence \( \{x_m\} \) in \( C \) defined by (5.3.2) and (5.3.3). For each \( m \geq 1 \), we set

\[
D_m = \limsup_{i \to \infty} ||x_m - T^i x_m||,
\]

\[
r_m = \limsup_{i \to \infty} ||x_{m+1} - T^i x_m||.
\]

Since \( T \) is a mapping of k-ANT, then, we have

\[
\sup_{i,j \geq n} ||T^i x_m - T^j x_m|| \leq \sup_{i,j \geq n} \left\{ ||T^i x_m - T^i(T^{j-i} x_m)|| + ||x_m - T^{j-i} x_m|| \right\}
\]

\[
\leq \sup_{i \geq n} \sup_{y \in D} ||T^i x_m - T^i y|| - ||x_m - y|| + \sup_{i,j \geq n} ||x_m - T^{j-i} x_m||,
\]

it follows from (5.3.2) that

\[
r_m \leq (1 + k) \tilde{N}(X) D_m
\]

Moreover, from (5.3.3) for \( i > 1 \), we have

\[
||x_{m+1} - T^i x_{m+1}|| \leq \limsup_{j \to \infty} ||T^j x_m - T^i x_{m+1}||
\]

\[
\leq \limsup_{j \to \infty} \left\{ ||T^j(T^{j-i} x_m) - T^i x_{m+1}|| + ||T^{j-i} x_m - x_{m+1}|| \right\}
\]

\[
\leq \sup_{j \to \infty} \left\{ ||T^j y - T^i x_{m+1}|| - ||y - x_{m+1}|| : y \in D \right\}
\]

Taking limit superior as \( i \to \infty \) on each side, we get

\[
D_{m+1} \leq \frac{r_m}{1 - k}.
\]
Thus we conclude that

\[ D_{m+1} \leq \frac{1 - \tilde{N}(X)}{1 + \tilde{N}(X)} \tilde{N}(X) D_m \leq A.D_m \quad \forall m = 0, 1, 2, \ldots, \]

where \( A = \frac{1 - \tilde{N}(X)}{1 + \tilde{N}(X)} \tilde{N}(X) < 1 \). Since

\[ ||x_{m+1} - x_m|| \leq r_m + D_m \leq 2D_m \leq \ldots \leq 2A^m D_0 \to 0 \]
as \( m \to \infty \), it follows that \( \{x_m\} \) is a Cauchy sequence and it converges to some point \( z \) in \( C \). Then, we have that

\[ ||z - T^i z|| \leq ||z - z_m|| + ||z_m - T^i z_m|| + ||T^i z_m - T^i z|| \]

\[ \leq 2||z - z_m|| + D_m + \sup \{ ||T^i y - T^i z|| - ||y - z|| : y \in D \}. \]

Taking the limit superior as \( i \to \infty \) on each side, we get,

\[ \limsup_{i \to \infty} ||z - T^i z|| \leq 2||z - z_m|| + D_m + k \limsup_{i \to \infty} ||z - T^i z|| \]

and so,

\[ (1 - k) \limsup_{i \to \infty} ||z - T^i z|| \leq 2||z - z_m|| + D_m \to 0 \quad \text{as} \quad m \to \infty. \]

Hence \( T^i z \to z \) as \( i \to \infty \), i.e., \( T^{N_i} z \to z \) as \( i \to \infty \). By continuity of \( T^N \), we have \( T^N z = z \). Thus \( T z = T(T^{N_i} z) = T^{N_i+1} z \to z \) and \( T z = z \).

\[ \square \]

**Corollary 5.3.4.** Let \( X \) be a Banach space with uniformly normal structure, \( D \) a nonempty subset of \( X \) and \( T : D \to D \) a mapping of ANT such that for some \( N \geq 1, T^N \) is continuous. Suppose that there is a closed convex bounded subset \( C \) of \( D \) with property \((P)\) and for each \( x \in C \), \( \{T^n x\} \) is bounded. Then \( T \) has a fixed point in \( C \).
Remark 5.3.1. In [8], Casini and Maluta remarked that the condition \( \tilde{N}(X) < 1 \) is weaker than \( \epsilon_0(X) < 1 \). So, **Corollary 5.3.4** improves the result of Kirk [80] and also partially improves the result of Xu [24].

**Corollary 5.3.5.** Let \( D \) and \( X \) be as in **Corollary 5.3.4** and let \( T : D \rightarrow D \) be an asymptotically nonexpansive mapping. Suppose that there is a closed convex bounded subset \( C \) of \( X \) with property \((P)\) and for each \( x \in C, \{T^nx\} \) is bounded. Then \( T \) has a fixed point in \( C \).

Remark 5.3.2. No closedness, convexity and boundedness assumptions are made on \( D \). So, **Corollary 5.3.4** improves the results of Goebel and Kirk [25], Gornicki [29] and Lim and Xu [75].

For James spaces \( X_M = (1^2, |.|_M), \text{where} |.|_M = \max\{|.|_2, M||.|_\infty\}, (M \geq 1) \), we have

\[
\tilde{N}(X_M) = \frac{M}{\sqrt{2}} \text{ for } 1 \leq M \leq \sqrt{2} \quad [67]
\]

Using the above fact, we get the following:

**Corollary 5.3.6.** Let \( D \) be a nonempty subset of a James space \( X_M, 1 \leq M < \sqrt{2} \) and

\( T : D \rightarrow D \) an asymptotically regular mapping of \( k\text{-ANT} \) such that with \( \liminf_{n \to \infty} k_n = k < \frac{\sqrt{M}}{\sqrt{2} + M} \) and for some \( N \geq 1, T^n \) is continuous. Suppose there is a closed convex bounded subset \( C \) of \( D \) with property \((P)\) and for each \( x \in C, \{T^nx\} \) is bounded. Then \( T \) has a fixed point in \( C \).
5.4 Section-II

5.4.1 Asymptotic Behavior

In this section, we investigate the asymptotic behavior of the iterates for a mapping of ANT in Banach space satisfying the uniform opial’s condition.

A Banach space $X$ is said to satisfy the uniform Opial’s condition [67] if for each $c > 0$, there exists an $r > 0$ such that

$$1 + r \leq \liminf_{n \to \infty} ||x + x_n||,$$

for each $x \in X$ with $||x|| \geq c$ and each sequence $\{x_n\}$ in $X$ such that $w - \lim_{n \to \infty} x_n = 0$ and $\liminf_{n \to \infty} ||x_n|| \geq 1$. We now define Opial’s modulus of $X$, denoted by $r_x$, as follows

$$r_x(c) = \inf \{\liminf_{n \to \infty} ||x + x_n|| - 1\},$$

where $c \geq o$ and the infimum is taken over all $x \in X$ with $||x|| \geq c||$ and sequences $\{x_n\}$ in $X$ such that $w - \lim x_n = o$ and $\lim ||x_n|| \geq 1$. It is easy to see that the function $r_x$ is nondecreasing and that $X$ satisfies the uniform Opial condition if and only if $r_x(c) > 0$ for all $c > 0$.

**Theorem 5.4.1.** Let $X$ be a Banach space satisfying the uniform Opial condition, $D$ a nonempty weakly compact convex subset of $X$, and $T : D \to D$ a continuous mapping of ANT. Then given an $x \in D$, $\{T_n x\}$ converges weakly to a fixed point of $T$ if and only if $T$ is weakly asymptotically regular at $x$, i.e., $w - \lim_{n \to \infty} (T^n x - T^{n+1} x) = 0$. 
Proof. First, we observe that for any $p \in F(T)$, the $\limsup_{n \to \infty} ||T^n x - p||$ exists. In fact for all integers $m \geq 1$,

$$\limsup_{n \to \infty} ||T^n x - p|| = \limsup_{n \to \infty} ||T^{n+m} x - p||$$

$$\leq \limsup_{n \to \infty} \{ ||T^n (T^m x) - T^m p|| - ||T^n x - p|| + ||T^m x - p|| \}$$

$$\leq \limsup_{n \to \infty} \{ \sup_{y \in D} \{ ||T^n y - T^m p|| - ||y - p|| : y \in D \} + ||T^m x - p|| \}$$

$$\leq ||T^m x - p||,$$

which implies that

$$\limsup_{n \to \infty} ||T^n x - p|| \leq \liminf_{m \to \infty} ||T^m x - p||,$$

it means that $\lim_{n \to \infty} ||T^n x - p||$ exists.

From the definition of Opial’s modulus $r_x$ of $X$, we have,

$$r_x(c) = \inf \{ \limsup_{n \to \infty} ||x_n + x|| - 1 : ||x|| \geq c, x_n \to 0 \text{ weakly}, \limsup_{n \to \infty} ||x_n|| \geq 1 \}$$

Suppose that $\{T^n x\}$ converges weakly (to fixed point of $T$) then we can show that $T$ is weakly asymptotically regular at $x$. Conversely, assume that $T$ is weakly asymptotically regular, i.e., $(T^n x - T^{n+1} x) \to 0$ weakly. We shall show that $w_w(x) \subset F(T)$. Let $y \in w_w(x)$, then by weak asymptotic regularity of $T$, we have for all integers $n' \geq 0$,

$$T^{n_i + n'} x \to y \text{ weakly as } i \to \infty.$$

Let $r_{n'} = \limsup_{i \to \infty} ||T^{n_i + n'} x - y||$, then for all integers $n', n'' \geq 0$: by Opial’s
condition of $x$, we have,

$$r_{n' + j} = \limsup_{i \to \infty} ||T_{n' + n''} x - y||$$

$$\leq \limsup_{i \to \infty} ||T_{n' + n''} x - T_{n''} y||$$

$$= \limsup_{i \to \infty} \{ ||T_{n' + n''} (T_{n'} x) - T_{n''} y|| - ||T_{n' + n''} x - y|| $$

$$+ ||T_{n' + n''} x - y|| \},$$

$$\leq \sup \{ ||T_{n''} u - T_{n''} y|| - ||u - y|| : u \in D \} + r_{n'},$$

it follows that the $\lim_{n' \to \infty} r_{n'} = r$ exists and $r_{n'} \geq r$ for all $n' \geq 0$. If $r = 0$, then

$\{T_{n'} x\}$ converges strongly to $y$ and hence $y = Ty$. Suppose now that $r > 0$. Let

$u_{n'} = \frac{(T_{n'} + n'' x - u)}{r_{n'}}$. Then for each fixed $n' \geq 0$, $u_{n'} \to 0$ weakly as $i \to \infty$ and

$$\limsup_{i \to \infty} ||u_{n'}|| = 1.$$ Hence by (5.3.1), we have

$$\limsup_{i \to \infty} ||u_{n'} + z|| \geq 1 + r_x(c) \text{ for all } z \in x, \text{ with } ||z|| \geq c.$$

Taking $z = (y - T_{n''} y)/r_{2n'}$, we obtain

$$r_{2n'} (1 + r_x(||y - T_{n''} y||/r_{2n'})) \leq \limsup_{i \to \infty} ||T_{n' + 2n''} x - T_{n''} y|| \leq r_{n'},$$

Since $r_x$ is nondecreasing and continuous [56] [proposition 2.1], taking the limit as $n' \to \infty$, we obtain

$$r (1 + r_x(\limsup_{n' \to \infty} ||y - T_{n''} y||/r)) \geq r,$$

it follows that $\limsup_{n' \to \infty} ||y - T_{n''} y|| = 0$ and hence by continuity of $T, y = Ty$.

Therefore, we have $w_w(x) \subset F(T)$. Since $\lim_{n' \to \infty} ||T_{n'} x - p||$ exists for any $p \in F(T)$, then by Opial’s condition $w_w(x)$ is singleton.

**Remark 5.4.1.** Theorem 5.4.1 extends the results of Xu [24], Bose [64], Hirano [52], Lim and Xu [75], Gornicki [29], Tan and Xu [34], and Lin, Tan and Xu [56] to the
larger class of mappings of ANT and from uniformly convex Banach space to the more general Banach space.

Remark 5.4.2. It is no clear whether the uniform Opial condition in Theorem 5.4.1 can be weakened to the local uniform Opial condition [56].

As an immediate consequence of Theorem 3.5 of [56] and Theorem 5.3.1, we have the following corollary.

Corollary 5.4.2. Let $X$ be a $k$-UR Banach space for some integer $k \geq 1$ satisfying Opial’s condition, $D$ a nonempty weakly compact convex subset of $X$, and $T : D \rightarrow D$ a continuous mapping of ANT. Then given an $x \in D$, $\{T^n x\}$ converges weakly to a fixed point of $T$ if and only if $T$ is weakly asymptotically regular at $x$.

5.5 Conclusion

We conclude by saying that ANT- mapping in a Banach space with uniformly normal structure ensures the existance of a fixed point and such mapping in Banach space when satisfies the uniform Opial condition converges weakly when it is weakly asymptotically regular.