Chapter 4

An Analysis Of Ishkawa Iteration For Generalised Nonexpansive Mappings

4.1 Abstract

The object of this chapter is to study the behavior of Ishikawa iterative sequences of nonexpansive and generalized nonexpansive mapping respectively defined in Banach space under different conditions.

In Section-I of this chapter, main content is the result [Theorem 4.3.4] on strong convergence of Ishikawa iterative sequence of nonexpansive mappings in Banach space. It is proved when iteration has rank 3 with errors. Later, a corollary [4.3.5] is obtained from the said result for the sequence having rank 2 to include the result of Lie, Shenghong [10].

Next, weak convergence of the main result is derived with the help of Opial condition [Theorem 4.3.6]. It is underlined that weak convergence does not hold in the case of $L_p$ spaces if $1 < p \neq 2$ for the simple reason, such spaces does not satisfy Opial
condition. In order to overcome this situation, Banach space is replaced by reflexive Banach space with uniformly Gateaux differentiable norm wherein every subsets are nonempty, closed, convex, bounded and T- invariant having fixed point property [Theorem 4.3.7]. However, weakly but almost convergence could be established. Thus to attend weak convergence, a condition of weakly sequentially continuity is included in the hypothesis [Theorem 4.3.8]. It is not out of place to mention that Theorem 4.3.7 generalizes many known results including the results given by Jung, Cho and Lee [31]. Next, it is shown that Theorem 4.3.8 can be re-established even without the conditions of the subsets being nonempty, closed, convex, bounded and T- invariant having fixed point property. For this purpose, reflexive and strictly convex Banach space with uniformly Gateaux differentiable norm and a weakly sequentially continuous mapping is considered [Theorem 4.3.9]. Last but never the least, it is shown that if we want to drop the condition of strict convexity of reflexive Banach space from the Theorem 4.3.9, than, the condition of nonempty, closed, convex, bounded and T- invariant set equipped with a condition known as Normal structure has to be included [Theorem 4.3.10].

In Section-II of this chapter, the strong convergence theorem [Theorem 4.4.4] is proved for the iterative sequences with error terms. Under certain conditions, a convergence result given by Deng [40] is obtained as corollary from the main result.

Next, another convergence result [Theorem 4.4.5] in normed space is established by reducing the iterative condition given in the main result from six real sequences to two. It is shown that the said result generalizes an important lemma proved by Ishikawa [72].

Next is the weak convergence result [Theorem 4.4.6] for generalized nonexpansive
mappings defined in Banach space satisfying Opial condition for Ishikawa iterative sequences with error terms. It is shown that this result generalizes and improves an other convergence result given by Deng [40]. Finally, a weak convergence result is proved for generalized nonexpansive mappings in uniformly convex Banach space satisfying the Opial condition and the Iterative scheme is considered as in the main result of the chapter.

4.2 Introduction

In 1976, Ishikawa [71] designed an iteration process for the class of nonexpansive mapping. Incidentally in 1982 [18], the convergence of the Ishikawa iterative sequence was proved but in the metric space setting of nonexpansive mappings. In 1997, Ghosh and Debnath [51] discussed the convergence of the Ishikawa iteration of generalized nonexpansive mappings in a uniformly convex Banach space and subsequently in a strictly convex Banach space. Furthermore, a result of Wong (1976) [7] was considered and it was shown that the cases of Mann iterations (1953) [81] are also true for the cases of Ishikawa iterations.

In 1993, Tan and Xu [36] established the convergence using Ishikawa iterative sequence of nonexpansive mappings. Later, Lei and Shenghong [10] in 2000 added error terms with Ishikawa scheme and established its convergence for the class of nonexpansive mappings in uniformly convex Banach space. In a paper [40], Deng further generalized Ishikawa iteration process and proved its weak and strong convergence in Banach space. On the other hand, Tan and Xu [74] extended theorem 2 of reich [73] for Ishikawa iteration process in uniformly convex Banach space in [74], Zeng [38] gave refinement of iteration parameters appeared iteration process in Tan and Xu’s
results.

In [83], Xu introduced Ishikawa iteration process with errors. Departing from this line of investigation, Sahu [11] proposed a different structure of Ishikawa scheme using the concept of rank as below:

If $C$ is a convex of a real Banach space $E$ and $T$ is a mapping of $C$ into itself, the sequence $\{x_n\}$ in $C$ of rank 3 is defined by

$$x_0 \in C$$

$$x_{n+1} = (1 - a_n)x_n + a_nTy_n,$$

$$y_n = (1 - b_n)x_n + b_nTz_n,$$

$$z_n = (1 - a_n)x_n + a_nTx_n, \ n \geq 0,$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ satisfy $0 \leq a_n < 1$, $\sum_{n=0}^{\infty} a_n = 0$, $0 < b_n \leq 1$, $\lim_{n} b_n = 0$, $0 < c_n \leq 1 \forall n \geq 0$, $\sum_{n=0}^{\infty} \max\{a_n, 1 - a_n\}b_n < \infty$ and $\sum_{n=0}^{\infty} b_n c_n < \infty$.

In section-I of this chapter, we study the convergence of generalized Ishikawa Iteration sequence of non expansive having rank-3 by adding error term as below:

$$x_{n+1} = a_n x_n + b_n Ty_n + c_n u_n$$

(GI) $$y_n = a'_n x_n + b'_n z_n + c'_n v_n$$

$$z_n = a''_n x_n + b''_n T x_n + c''_n w_n, \ n \geq 0$$

for any given initial data $x_0 \in D$ where $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are arbitrary sequences in $D$ and $0 \leq a_n, b_n, c_n, a'_n, b'_n, c'_n, a''_n, b''_n, c''_n \leq 1$ satisfying certain restriction. It is first shown that the iterative sequence of our iteration process with errors is an approximate fixed point sequence (AFPS), i.e. the sequence $\{x_n\}$ in $D$ with $||x_n - Tx_n|| \to 0$ as $n \to \infty$ for nonexpansive mapping in general Banach
space. Then, we established the weak convergence of our iteration process with errors for nonexpansive mappings. Our results generalize and improve the results of [40, 53, 73, 38, 36, 5, 6].

In section-II, we study the convergence of the sequence of Ishikawa iteration process with errors for fixed points on generalized non expansive mapping in Banach spaces. Our results generalize and improve the results of Deng [40] and Tan and XU [74].

Let $D$ be a non empty subset of Banach space $X$ and $T : D \rightarrow D$ be a generalized nonexpansive mapping defined as follows:

$$||Tx - Ty|| \leq a||x - y|| + b(||x - Tx|| + ||y - Ty||) + c(||x - Ty|| + ||y - Tx||) \forall x, y \in D$$  \hspace{1cm} (4.2.1)

Where $a, b, c \geq 0$ with $a + 2b + 2c \leq 1$.

For $b = c = 0$ and $a = 1$, $T$ is called nonexpansive, i.e., $||Tx - Ty|| \leq ||x - y|| \forall x, y \in D$. In 1976 Ishikawa [71] provide the following theorem with out any assumption on convexity on domain of nonexpansive mappings in banach spaces.

**Theorem 4.2.1.** [71] Let $D$ be a nonempty subset of normed space $X$ and $T : D \rightarrow X$ be a nonexpansive mapping. Given a sequence $\{x_n\}$ in $D$ and a sequence $\{t_n\}$ in $[0,1]$ satisfying,

1. $0 \leq t_n \leq t < 1$ and $\sum_{n \geq 0} t_n = \infty$.
2. $x_{n+1} = (1 - t_n)x_n + t_nTx_n, n = 1, 2, 3...$

If $\{x_n\}$ is bounded, then $||x_n - Tx_n|| \rightarrow 0$ as $n \rightarrow \infty$

Deng [40] generalized Theorm 4.2.1 to Ishikawa iteration process and established weak and Strong convergence results for nonexpansive mappings in Banach spaces.
On the other hand, Tan and Xu [74] extended theorem 2 of Reich [73] for Ishikawa iteration process in uniformly convex Banach space in [74], Zeng [38] gave refinement of iteration parameters appeared iteration process in Tan and Xu's results.

In [83] Xu introduced Ishikawa iteration process with errors as follows. Let $D$ be a nonempty subset of Banach space $X$ and $T : D \to D$ be a non linear mapping. For any given $x_0 \in D$ then \{x_n\} is called Ishikawa iteration process with errors if it defined iteratively by

\[
\begin{align*}
  x_1 &\in D, \\
  x_{n+1} &= \alpha_n x_n + \beta_n Ty_n + \gamma_n u_n, \\
  y_n &= \alpha'_n x_n + \beta'_n Tx_n + \gamma'_n u_n, n \geq 0
\end{align*}
\] (4.2.2)

Where \{u_n\} and \{v_n\} are two bounded sequences in $D$ and \{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\alpha'_n\},\{\beta'_n\},\{\gamma'_n\} are six sequences in $[0,1]$ such that

\[
\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1, \forall n \geq 0
\] (4.2.3)

On Other hand Goebel, Kirk and Shimi [20] proved the following existence theorem.

\textbf{Theorem 4.2.2.} Let $X$ be a uniformly convex Banach space, $D$ is a nonempty bounded, closed and convex subset of $X$ and $T : D \to D$ a continuous mapping such that $||Tx - Ty|| \leq a||x - y|| + b(||x - Tx|| + ||y - Ty||) + c(||x - Ty|| + ||y - Tx||)$ for all $x, y \in D$, Where $a, b, c \geq 0$ and $a + 2b + 2c \leq 1$. Then $T$ has a fixed point in $D$. 
4.3 Section-I

4.3.1 Preliminaries

Recall that a Banach space $X$ is said to be smooth provided the limit

$$\lim_{t \to \infty} \frac{||x + ty|| - ||x||}{t}$$

exists for each $x$ and $y$ in $S = \{x \in X : ||x|| = 1||\}$. In this case, the norm of $X$ is said to be Gâteaux differentiable. It is said to be uniformly Gâteaux differentiable if for each $y \in S$, this limit is attained uniformly for $x \in S$. The norm is said to be Fréchet differentiable if for each $x \in S$, this limit is attained uniformly for $y \in S$. Finally, the norm is said to be uniformly Fréchet differentiable if the limit is attained uniformly for $(x, y) \in S \times S$. In this case $X$ is said to be uniformly smooth. Since the dual $X^*$ of $X$ is uniformly convex if and only if the norm of the $X$ is uniformly Fréchet differentiable, every Banach space with a uniformly convex dual is reflexive and has a uniformly Gâteaux differentiable norm, the reverse is false.

If $X$ is smooth, the duality mapping $J$ is said to be weakly sequentially continuous at 0 if $\{J(x_n)\}$ converges to 0 in the sense of the weak-star topology of $X^*$, as $\{x_n\}$ converges weakly to 0 in $X$.

we say that a Banach space $X$ satisfy the opial’s condition [85] if for each sequence $\{x_n\}$ in $X$ weakly convergent to a point $x$ and for all $y \neq x$.

$$\lim_{n} ||x_n - x|| < \lim_{n} ||x_n - y||.$$

The example of Banach space which satisfy the Opial’s condition are Hilbert spaces and all $L_p[0, 2\pi]$ with $1 < p \neq 2$ fails to satisfy Opial’s condition [85].
Let $D$ be a nonempty closed convex subset of a Banach space $X$. Then $I-T$ is demiclosed at zero if for any sequence $\{x_n\}$ in $D$ condition $x_n \to x$ weakly and $\lim_n ||x_n - Tx_n|| = 0$ implies $(I-T)x = 0$.

A Banach limit $\text{LIM}$ is a bounded linear functional on $l^\infty$ such that

$$\lim_n t_n \leq \text{LIM}_n t_n < \overline{\text{lim}}_n t_n$$

and

$$\text{LIM}_n t_n = \text{LIM}_{n+1} t_n$$

for all $\{t_n\}$ be a bounded sequence in $l^\infty$. Then we can define the real valued continuous convex function $f$ on a Banach space $X$ by

$$f(z) = \text{LIM}_n ||x_n - z||^2$$

for all $z \in X$, where $\{x_n\}$ is a bounded sequence in $X$.

To prove the main result of the paper, we also need the following Lemmas.

**Lemma 4.3.1.** [26] Let $X$ be a Banach space with uniformly Gâteaux differentiable norm and $u \in X$. Then

$$f(u) = \inf_z \in X f(z)$$

iff

$$\text{LIM}_n \langle z, J(x_n - u) \rangle = 0$$

for all $z \in X$, where $J : X \to X^*$ is the normalized duality mapping, and $\langle ., . \rangle$ denote the generalized duality pairing.

**Lemma 4.3.2.** [18] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of a normed space $X$. If there is a sequence $\{t_n\}$ in $[0,1]$ satisfying:
Lemma 4.3.3. [36] Let \( \{a_n\} \) and \( \{b_n\} \) be two nonnegative numbers such that \( a_{n+1} \leq a_n + b_n, n \geq 0 \). If \( \sum_{n=0}^{\infty} b_n \) converges, then \( \lim_n a_n \) exists.

4.3.2 Main Result

Now, we prove main result of section-I¹

Theorem 4.3.4. Let \( D \) be a nonempty subset of a Banach space \( X \) and \( T : D \to X \) a nonexpansive mapping. Given a sequence \( \{x_n\} \) in \( D \), three bounded sequences \( \{u_n\}, \{v_n\} \) and \( \{w_n\} \) in \( D \) and the real sequences \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}, \{a''_n\}, \{b''_n\} \) and \( \{c''_n\} \) of a real numbers in \([0,1]\) satisfying:

(i) \( a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1 \)

(ii) \( 0 < b'_n \leq 1, \forall n \geq 0, \lim_n b'_n = 0 \) and \( \sum_{n=0}^{\infty} b_n = \infty \),

(iii) \( \sum_{n=0}^{\infty} b_n b'_n < \infty, \sum_{n=0}^{\infty} b_n c'_n < \infty \) and \( \sum_{n=0}^{\infty} c_n < \infty \)

(iv) \( x_{n+1} = a_n x_n + b_n Ty_n + c_n u_n \)

\begin{align*}
y_n &= a'_n x_n + b'_n T z_n + c'_n v_n \\
z_n &= a''_n x_n + b''_n T x_n + c''_n w_n, \quad n \geq 0
\end{align*}

such that \( d = \max\{\|u_n - x_n\|, \|u_n - T x_n\|, \|x_n - v_n\|, \|x_n - w_n\|\} < \infty \), then

(a) \( \lim_n \|x_n - p\| \) exists if \( p \) is a fixed point of \( T \).

(b) If \( \{x_n\} \) is bounded, then \( \lim_n \|x_n - Tx_n\| = 0 \)

**Proof.** (a): Let \( p \) be a fixed point of \( T \), we have

\[ \|x_{n+1} - p\| < \|x_n - p\| \quad \forall n \geq 0 \]

It follows that \( \{\|x_n - p\|\} \) is non-increasing, and part (a) is proved.

(b): Since

\begin{align*}
\|x_n - z_n\| &= \|x_n - a''_n x_n - b''_n T x_n - c''_n w_n\| \\
&\leq b''_n \|x_n - T x_n\| + c''_n \|x_n - W_n\| \\
&\leq b''_n \|x_n - T x_n\| + c''_n d,
\end{align*}

(4.3.1)

\begin{align*}
\|x_n - y_n\| &\leq b'_n (\|x_n - T x_n\| + b'_n \|T x_n - T z_n\|) + c'_n d) \\
&\leq b'_n (\|x_n - T x_n\| + b'_n \|x_n - z_n\|) + c'_n d \\
&= b'_n (1 + b''_n) \|x_n - T x_n\| + d(b'_n c''_n + c'_n) from (4.3.1)
\end{align*}

(4.3.2)

\begin{align*}
\|x_n - x_{n+1}\| &= \|x_n - a_n x_n - b_n T y_n - c_n u_n\| \\
&\leq b_n \|x_n - T y_n\| + c_n d,
\end{align*}

(4.3.3)
and

\[ ||x_{n+1} - T x_{n+1}|| = ||a_n(x_n - T x_n) + b_n(T y_n - T x_n) + c_n(u_n - T x_n) + T x_n - T x_{n+1}|| \]
\[ \leq (1 - b_n)||x_n - T x_n|| + b_n||Ty_n - T x_n|| + c_n||u_n - T x_n|| + ||T x_n - T x_{n+1}|| \]
\[ \leq (1 - b_n)||x_n - T x_n|| + b_n||y_n - x_n|| + c_n d + ||x_n - x_{n+1}||. \]

From (4.3.3), we obtain,

\[ \leq (1 - b_n)||x_n - T x_n|| + b_n||y_n - x_n|| + c_n d \]
\[ \leq (1 - b_n)||x_n - T x_n|| + b_n||y_n - x_n|| + b_n(||x_n - T x_n|| + ||T x_n - Ty_n||) + 2c_n d \]
\[ \leq ||x_n - T x_n|| + 2b_n||y_n - x_n|| + 2c_n d. \]

From (4.3.2), we get

\[ ||x_{n+1} - T x_{n+1}|| \leq ||x_n - T x_n|| + 2b_n b'_n (1 + b''_n)||x_n - T x_n|| + 2b_n d(b'_n c''_n + c'_n) + 2c_n d \]
\[ \leq ||x_n - T x_n|| + 2d(3b_n b'_n + b_n c'_n + c_n). \]

Setting \( p_n = x_n - T x_n \) and \( K_n = d(3b_n b'_n + b_n c'_n + c_n) \),
we obtain, \( ||P_{n+1}|| \leq ||p_n|| + K_n \) \( \forall n \geq 0 \).

Since from (iii), we have \( \sum_{n=0}^{\infty} K_n \) is convergent, It follows from Lemma 4.3.3, we have \( \lim_n ||P_n|| \) exists and \( \lim_n ||P_n|| = p \).

Now set \( B_n = T y_n - T x_n + b^{-1}_n[T x_n - T x_{n+1} + c_n(u_n - T x_n)] \),
we have, $x_{n+1} - Tx_{n+1} = (1 - b_n)(x_n - Tx_n) + b_n B_n$.

\[
\|B_n\| \leq \|Ty_n - Tx_n\| + b_n^{-1}(\|Tx_n - Tx_{n+1}\| + c_n \|u_n - Tx_n\|)
\leq \|y_n - x_n\| + b_n^{-1}\|Tx_n - Tx_{n+1}\| + b_n^{-1}c_n u_n
\leq \|y_n - x_n\| + \|x_n - Tx_n\| + \|y_n - x_n\| + 2b_n^{-1}c_n d
\leq \|x_n - Tx_n\| + 2\|Ty_n - x_n\| + 2b_n^{-1}c_n d
\leq \{1 + 2b'_n(1 + b''_n)\}\|x_n - Tx_n\| + 2d(b'_n c''_n + c'_n + b_n^{-1}c_n).
\]

It follows that $\lim b'_n = 0$, that $\limsup \|B_n\| \leq P$. Moreover,

\[
\|\sum_{n=0}^{m} b_n B_n\| = \|\sum_{n=0}^{m} b_n(Ty_n - Tx_n) + c_n(u_n - Tx_n) + (Tx_n - Tx_{n+1})\|
\leq \sum_{n=0}^{m} b_n\|Ty_n - Tx_n\| + \sum_{n=0}^{m} c_n \|u_n - Tx_n\| + \sum_{n=0}^{m} \|Tx_n - Tx_{n+1}\|
\leq \sum_{n=0}^{m} b_n\|y_n - x_n\| + \sum_{n=0}^{m} c_n d + \|x_0 - x_{m+1}\|
\leq \|x_0 - x_{m+1}\| + \sum_{n=0}^{m} b_n b'_n (1 + b''_n) \|x_n - Tx_n\|
\quad + \sum_{n=0}^{m} b_n (b'_n c''_n + c'_n) + d\sum_{n=0}^{m} c_n
\leq \|x_0 - x_{m+1}\| + d\sum_{n=0}^{m} (3b_n b'_n + b_n c'_n + c_n)
\]
by(iii), we have $\sum_{n=0}^{m} (3b_n b'_n + b_n c'_n + c_n)$ convergent, it follows that $\{\|\sum_{n=0}^{m} B_n b_n\|\}$ is bounded.

Hence from lemma 4.3.2, we have $\lim_{n \to 0} \|x_n - Tx_n\| = 0$. Completing the proof.

If $b'_n = c'_n = 0 \quad \forall n \geq 0$, we obtain the following interesting result:
Corollary 4.3.5. Let $D$ be a nonempty subset of Banach space $E$ and $T : D \to E$ be a nonexpansive mapping. Given a sequence $\{x_n\}$ in $D$, two arbitrary sequences $\{u_n\}$ and $\{v_n\}$ in $D$ and six real sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$ and $\{c'_n\}$ satisfying:

(i) \[a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \quad 0 < b_n < 1 \quad \text{for} \quad n \geq 0 \quad \text{and} \quad \lim_{n \to \infty} b'_n = 0.\]

(ii) \[\sum_{n=0}^{\infty} a_nb_n < \infty, \quad \sum_{n=0}^{\infty} a_nb'_n < \infty, \quad \sum_{n=0}^{\infty} c_n < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} c'_n < \infty\]

If $\{x_n\}$ is bounded Ishikawa iteration sequence defined as follows:

\[
x_{n+1} = a_n x_n + b_n Ty_n + c_n u_n
\]

\[
y_n = a'_n x_n + b'_n Tx_n + c'_n v_n, \quad n \geq 0
\]

Then \( \lim_{n \to \infty} ||x_n - Tx_n|| = 0. \)

Remark 4.3.1. Corollary 4.3.5 is an improvement of the results of [10].

Theorem 4.3.6. Let $X$ be a Banach space satisfying Opial’s condition, $D$ a nonempty weakly compact subset of $X$ and $T : D \to X$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a sequence $\{x_n\}$ as in Theorem 4.3.4, then $\{x_n\}$ converges weakly to a fixed point of $T$.

Proof. First we show that $w_w(x_n) \subset F(T)$. Let $x_{n_k} \weakto x$. By theorem 4.3.4(b), we have $\lim_{n} ||x_n - Tx_n|| = 0$. Since $I-T$ is demiclosed at zero. Hence $x \in F(T)$. By Opial’s condition $\{x_n\}$ possesses only weak limit point, i.e., $\{x_n\}$ converges weakly to a fixed point of $T$. \qed

We remark that theorem 4.3.6 does not apply to any $L_p$ space if $1 < p \neq 2$, since none of these spaces satisfy Opial’s condition [85]. The following result can be applied to all uniformly convex Banach Spaces and hence all $L_p$ spaces.
Theorem 4.3.7. Let $X$ be a reflexive Banach space with uniformly Gâteaux differentiable norm and $T : X \to X$ a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that every nonempty closed convex bounded $T$-invariant subset of $X$ has fixed point property for $T$. Given a sequence $\{x_n\}$ as in theorem 4.3.4, then there exists a point $u \in F(T)$ such that $\{J(x_n - u)\}$ converges weakly to zero.

Proof. Let LIM be a Banach limit and define a real-valued function $f$ on $X$ by

$$f(z) = \text{LIM}_n ||x_n - z||^2$$

For each $z \in D$. Then $f$ is a continuous convex functional and $f(z) \to \infty$ as $||Z|| \to \infty$. Since $X$ is reflexive, $f$ attains its infimum over $X$. Let

$$M = \{u \in X : f(u) = \inf_{z \in X} f(z)\}.$$ 

Then $M$ is a nonempty closed convex bounded set. Also $M$ is invariant under $T$. In fact, by Theorem 4.3.4(b), we have $\lim_n ||x_n - Tx_n|| = 0$ and hence we have for each $y \in M$

$$f(Ty) = \text{LIM}_n ||Tx_n - Ty||^2 
\leq \text{LIM}_n ||x_n - y||^2 = f(y).$$

Therefore, by assumption, $T$ has a fixed point in $M$, denote such a point by $u$.

On the other hand, by Theorem 4.3.4(a), $\lim_n ||x_n - p||$ exists for all $p \in F(T)$, then $f(P)$ is independent of Banach limits. Thus, we may assume that $u$ minimizes $f$ for any Banach limit LIM. It follows Lemma 4.3.1, such that

$$\text{LIM}_n \langle z, J(x_n - u) \rangle = 0,$$

for all $z \in X$ and any LIM. Thus, $\{\langle z, J(x_n - u) \rangle = 0\}$ is almost convergent [22] to zero, i.e., $\{J(x_n - u)\}$ is weakly almost convergent to zero.  

\qed
Applying Theorem 4.3.4 and 4.3.7, we obtain the following weak convergence theorem:

**Theorem 4.3.8.** Let $X$ be a reflexive Banach space $X$ with uniformly Gâteaux differentiable norm and $T : X \to X$ a nonexpansive mapping with $F(T) \neq \emptyset$ and let $J^{-1} : X^* \to X$ be weakly sequentially continuous at zero. Suppose that every nonempty closed convex bounded $T$-invariant subset of $X$ has fixed point property for $T$. Given a sequence $\{x_n\}$ as in Theorem 4.3.4 such that $x_n - x_{n+1} \to 0$ as $n \to \infty$, then there exists a point $u \in F(T)$ such that $\{x_n\}$ converges weakly to $u$.

**Proof.** Since the norm of $X$ is uniformly Gâteaux differentiable, the duality mapping is uniformly continuous on bounded subset of $X$ from the strong topology of $X$ to weak* topology of $X$. Thus, since $x_n - x_{n+1} \to 0, \{J_n(x_n - u), J(x_{n+1} - u)\}$ converges weakly to zero. However, this is a Tauberian condition for almost convergence, so $\{J(x_n - u)\}$ converges weakly to zero. Since $J^{-1}$ is weakly sequentially continuous at zero, $\{x_n\}$ converges weakly to $u$. This completes the proof. \qed

We remark that Theorem 4.3.7 generalizes several recent results of this nature. Particularly, it extends Theorem 4.3.7 Jung, Cho and Lee [31].

Next we substitute the Fixed Point Property (FPP) assumption mentioned in Theorem 4.3.8, by assuming that space $X$ is reflexive and strictly convex.

**Theorem 4.3.9.** Let $X$ be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and $T : X \to X$ a nonexpansive mapping with $F(T) \neq \emptyset$ and let $J^{-1} : X^* \to X$ a weakly sequentially continuous at zero. Give a sequence $\{x_n\}$ as in Theorem 4.3.4 such that $x_n - x_{n+1} \to 0$ as $n \to \infty$, then there exists a point $u \to F(T)$ such that $\{x_n\}$ converges weakly to $u$. 
Proof. To be able to use the argument of Theorem 4.3.6, we just need to show that the set $M$ contains a fixed point $T$. To see this let, $w \in F(T)$ and define

$$M_0 = \{u \in D : ||u - w|| = d(w, M)\},$$

where $d(w, M) = \inf\{||x - w|| : x \in M\}$.

Then since $X$ is strictly convex, $M_0$ is a singleton. Let $M_0 = \{v\}$. Then $||Tv - w|| \leq ||v - w||$ and so $Tv = v$. \qed

On the other hand, it is easy to find examples of spaces which satisfy the FPP for nonexpansive self mappings, which are not strictly convex.

As a consequence of theorem 4.3.7, we may derive the following results.

**Theorem 4.3.10.** Let $X$ be reflexive Banach space with uniformly Gâteaux differentiable norm and $T : X \to X$ a nonexpansive mapping with $F(T) \neq \emptyset$. Given a sequence $\{x_n\}$ as in theorem 4.3.4, then there exists a nonempty closed convex bound $T$ invariant set $M$ defined by

$$M = \{u \in X : f(u) \inf_z \in xf(z)\}.$$ 

Suppose in addition that $x_n - x_{n+1} \to 0$ as $n \to \infty$, $J^{-1} : X^* \to X$ is weakly sequentially continuous at zero and $M$ has normal structure. Then $\{x_n\}$ converges weakly to a fixed point of $T$.

**Proof.** The existence of fixed point of $T$ in $M$ follows from Kirk [78].

**EXAMPLE 1.** For the parameters of our theorems, we can make the following choice:

$$a_n = 1 - \frac{1}{4n\sqrt{n+1}} - \frac{1}{n\sqrt{n+1}},$$
where
\[ b_n = \frac{1}{4n\sqrt{n+1}}, c_n = \frac{1}{n\sqrt{n+1}} \]
\[ a'_n = 1 - \frac{1}{n+3} - \frac{1}{(n+3)^2}, \text{ where } b'_n = \frac{1}{n+3}, c'_n = \frac{1}{(n+3)^2}, \]
\[ a''_n = 1 - \frac{1}{(n+1)} - \frac{1}{(n+1)}, \text{ where } b''_n = c''_n = \frac{1}{(n+1)}, n \geq 0 \]

then \( \lim n b'_n = 0 \) and \( \sum_{n=0}^{\infty} b_n c'_n < \infty \), these choices satisfy all the conditions of our theorems.

\[ \square \]

4.4 Section-II

4.4.1 Preliminaries

We give the following definitions and lemmas which we shall need in the sequel.

**Definition 4.4.1.** A Banach space \( X \) is called uniformly convex if for each \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \( x, y \in X \) with \( ||x|| \leq 1, ||y|| \leq 1 \) and \( ||x - y|| \geq \epsilon \) it follows that \( \frac{1}{2}||x + y|| \leq 1 - \delta(\epsilon) \)

**Definition 4.4.2.** Recall that a Banach space \( X \) satisfies the Opial's condition [85] if for each sequence \( \{x_n\} \) in \( X \) is weakly convergent to a Point \( x \) and for all \( y \neq x \)

\[ \limsup_n ||x_n - x|| < \limsup_n ||x_n - y|| \]

**Definition 4.4.3.** Let \( D \) be a nonempty subset of normed space \( X \). A mapping \( T : D \to X \) is called demiclosed with respect to \( y \in X \) if for each sequence \( \{x_n\} \) in \( D \) and each \( x \in X, x_n \to x \) and \( T x_n \to y \) imply that \( x \in D \) and \( T x = y \).
Lemma 4.4.1. Let \( \{a_n\}, \{b_n\} \) be sequences of nonnegative real numbers satisfying the inequality \( a_{n+1} \leq (1 + \delta_n)a_n + b_n \) for all \( n \geq 1 \). If \( \sum_{n=1}^{\infty} \delta_n < \infty \) and \( \sum_{n=1}^{\infty} b_n < \infty \) then \( \lim a_n \) exists.

Lemma 4.4.2. Suppose \( \{a_n\} \) and \( \{b_n\} \) are two sequences in a normed space \( E \) if there is a sequence \( \{t_n\} \) of real number satisfying.

(i) \( 0 \leq t_n < t < 1 \) and \( \sum_{n \geq 1} t_n = \infty \)

(ii) \( a_{n+1} = (1-t_n)a_n + t_nb_n \), \( \forall \ n \geq 1 \)

(iii) \( \lim_{n \to \infty} ||a_n|| = a \)

(iv) \( \lim_{n} \sup ||b_n|| \leq a \) and \( \{\sum_{i=1}^{n} t_ib_i\} \) is bounded, then \( a = 0 \)

Lemma 4.4.3. ([55]) Let \( X \) be a uniformly convex Banach space, \( D \) be a nonempty closed convex bounded subset of \( X \) and let \( T \) be a continuous generalized nonexpansive mapping. If \( \{u_i\} \) is weakly convergent sequence in \( D \) with weak limit \( u_0 \) and \( (I-T)u_i \) converges strongly to an element \( w \) in \( X \), then \( (I-T)U_0 = w \).

4.4.2 Main Results

Now, we prove main result of section-II

Theorem 4.4.4. Let \( D \) be nonempty subset of normed space \( X \) and \( T : D \to D \) be a generalized nonexpansive mapping defined as in theorem (4.2.1) Given a sequence \( \{x_n\} \) in \( D \), two bounded sequences \( \{u_n\} \) and \( \{v_n\} \) in \( D \) and six real sequences \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\} \) and \( \{\gamma'_n\} \) in \( [0,1] \), satisfying the following conditions:

(i) \( \alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1, \forall n \geq 0 \)

(ii) \(0 \leq \beta_n < \beta < 1\), for \(n \geq 0\), \(\sum_{n=0}^{\infty} \beta_n = \infty\) and \(\lim_n \beta'_n = 0\),

(iii) \(\sum_{n=0}^{\infty} \beta_n \beta_n' < \infty\), \(\sum_{n=0}^{\infty} \gamma < \infty\) and \(\sum_{n=0}^{\infty} \gamma' < \infty\),

and (iv) \(x_{n+1} = \alpha_n x_n + \beta_n Ty_n + \gamma_n u_n\)

\[ y_n = \alpha'_n x_n + \beta'_n T x_n + \gamma'_n v_n \quad \forall n \geq 0. \]

If \(\{x_n\}\) is bounded, then \(\lim_n ||x_n - Tx_n|| = 0\).

**Proof.** Note that,

\[
||x_{n+1} - Tx_{n+1}|| = ||\alpha_n x_n + \beta_n Ty_n + \gamma_n u_n - Tx_{n+1}||
\]

\[
= ||\alpha_n (x_n - Tx_n) + \beta_n (Ty_n - Tx_n) + \gamma_n (u_n - Tx_n) + (Tx_n - Tx_{n+1})||
\]

\[
||x_{n+1} - Tx_{n+1}|| = ||\alpha_n x_n + \beta_n Ty_n + \gamma_n u_n - Tx_{n+1}|| = ||\alpha_n (x_n - Tx_n) + \beta_nTy_n - Tx_n) + \gamma_n (u_n - Tx_n) + (Tx_n - Tx_{n+1})||
\]

\[
||x_{n+1} - Tx_{n+1}|| \leq (1 - \beta_n)||x_n - Tx_n|| + \beta_n||Ty_n - Tx_n||
\]

\[
+ \gamma_n||u_n - Tx_n|| + ||Tx_n - Tx_{n+1}||
\]

(4.4.1)

Since \(T\) is generalized nonexpansive mapping, then

\[
||Ty_n - Ty_m|| \leq \rho||x_n - y_n|| + b\{||x_n - Tx_n|| + ||y_m - Ty_m||\} + c\{||x_n - Ty_m||
\]

\[
+ \|y_n - Ty_m\| \leq (a + b + c)||x_n - y_n|| + c\{||x_n - Ty_m||
\]

\[
+ ||y_n - Ty_m|| \leq (2b + 2c)||x_n - Ty_m||
\]

(4.4.2)
from (4.4.1) and (4.4.2), we obtain
\[
\|x_{n+1} - Tx_{n+1}\| \leq (1 - \beta_n)\|x_n - Tx_n\| + \beta_n\{\|x_n - y_n\| + \frac{2b + 2c}{1 - b - c} \|x_n - Tx_n\| \}
\]
\[+ \gamma_n\|u_n - Tx_n\| + \|x_n - x_{n+1}\| + \frac{2b + 2c}{1 - b - c} \|x_n - Tx_n\| \]
\[\leq \{1 - \beta_n + (\beta_n + 1) \frac{2b + 2c}{1 - b - c}\} \|x_n - Tx_n\| + \beta_n\|x_n - y_n\|
\]
\[+ \gamma_n d + \|x_n - (\alpha_n x_n + \beta_n Ty_n + \gamma_n u_n)\| \]
\[\leq \{1 - \beta_n + (\beta_n + 1) \frac{2b + 2c}{1 - b - c}\} \|x_n - Tx_n\| + \beta_n\|x_n - y_n\|
\]
\[+ \gamma_n d + \beta_n\|x_n - Ty_n\| + \gamma_n\|x_n - u_n\|
\]
\[\leq \{1 - \beta_n + (\beta_n + 1) \frac{2b + 2c}{1 - b - c}\} \|x_n - Tx_n\| + \beta_n\|x_n - y_n\|
\]
\[+ \beta_n\|x_n - Tx_n\| + \beta_n\|Tx_n - Ty_n\| + 2\gamma_n d.
\]
\[\leq \{1 + (2\beta_n + 1) \frac{2b + 2c}{1 - b - c}\} \|x_n - Tx_n\| + 2\beta_n\|x_n - y_n\| + 2\gamma_n d
\]
\[\leq \{1 + (2\beta_n + 1) \frac{2b + 2c}{1 - b - c}\} \|x_n - Tx_n\| + 2\beta_n\beta'_n\|x_n - Tx_n\|
\]
\[+ 2\gamma_n d + 2\beta_n\gamma'_n d
\]
\[\leq \{1 + (2\beta_n + 1) \frac{2b + 2c}{1 - b - c}\} \|x_n - Tx_n\| + 2d(\beta_n\beta'_n + \gamma_n + \gamma'_n)
\]
Setting \(p_n = x_n - Tx_n\), \(k_n = d(\beta_n\beta'_n + \gamma_n + \gamma'_n)\)then,
\[
\|p_{n+1}\| \leq \{1 + (2\beta_n + 1) \frac{2b + 2c}{1 - b - c}\} \|p_n\| + k_n, \forall n \geq 0.
\]
Since from (iii), \(\sum_{n=0}^{\infty} k_n < \infty\). It follows from Lemma (4.4.1) that \(\lim_n \|p_n\|\) exits.

Suppose \(\lim_n \|p_n\| = p\).

Setting
\[
p_{n+1} = (1 - \beta_n)p_n + \beta_n(Ty_n - Tx_n) + \gamma_n(u_n - Tx_n) + (Tx_n - Tx_{n+1})
\]
\[
p_{n+1} = (1 - \beta_n)p_n + \beta_n B_n.
\]
\[ B_n = (T_{x_n} - T_{y_n}) + \beta_n^{-1}(T_{x_{n+1}} - T_{x_n}) + \beta_n^{-1}\gamma_n(u_n - T_x) \]  
(4.4.3)

\[ ||B_n|| \leq ||T_{x_n} - T_{y_n}|| + \beta_n^{-1}||T_{x_{n+1}} - T_{x_n}|| + \beta_n^{-1}\gamma_n||(u_n - T_x)|| \]

\[ \leq ||x_n - y_n|| + \frac{2b + 2c}{1 - b - c} ||x_n - T_{x_n}|| + \beta_n^{-1}||x_{n+1} - x_n|| \]

\[ + \frac{2b + 2c}{1 - b - c} ||x_n - T_{x_n}|| \}

\[ \leq \{1 + (1 + \beta_n^{-1}) \frac{2b + 2c}{1 - b - c} \} ||x_n - T_{x_n}|| + 2\beta_n^{-1}||x_n - y_n|| + 2\beta_n^{-1}\gamma_n \]

\[ \leq \{1 + 2(1 + \beta_n^{-1}) \frac{2b + 2c}{1 - b - c} \} ||x_n - T_{x_n}|| + 2d(\beta_n^{-1}\gamma_n + \gamma'_n). \]

Using (ii)and (iii) and taking lim superior both sides, we have

\[ \limsup_n ||B_n|| \leq p \]

Since from (4.4.3), we have

\[ ||\sum_{n=0}^{m} B_n\beta_n|| = ||\sum_{n=0}^{m} \{(T_{x_{n+1}} - T_{x_n}) + \gamma_n(u_n - T_{x_n}) + \beta_n(T_{x_n} - T_{y_n})\}|| \]

\[ \leq ||\sum_{n=0}^{m} \{(T_{x_{n+1}} - T_{x_n})|| + ||\sum_{n=0}^{m} \gamma_n(u_n - T_{x_n})|| + ||\sum_{n=0}^{m} \beta_n(T_{x_n} - T_{y_n})|| \]

\[ \leq ||x_{m+1} - x_0|| + \frac{2b + 2c}{1 - b - c} ||x_0 - T_{x_0}|| + d \sum_{n=0}^{m} \gamma_n + \sum_{n=0}^{m} \beta_n\beta'_n ||x_n - T_{x_n}|| \]

\[ + \sum_{n=0}^{m} \beta_n \frac{2b + 2c}{1 - b - c} ||x_n - T_{x_n}|| + d \sum_{n=0}^{m} (\beta_n\gamma'_n) \]
\[
\leq \|x_{m+1} - x_0\| + \frac{2b + 2c}{1 - b - c}\|p_0\| + d \sum_{n=0}^{m} (\beta_n \beta'_n + \gamma_n + \gamma'_n) \\
+ \sum_{n=0}^{m} \beta_n \frac{2b + 2c}{1 - b - c}\|p_n\|
\]

by (ii) and (iii) \(0 \leq \beta_n < 1\) and \(\sum_{n=0}^{m} (\beta_n \beta'_n + \gamma_n + \gamma'_n) < \infty\), it follows that \(\{\sum_{n=0}^{m} \beta_n B_n\}\) is bounded. Hence from Lemma 4.4.2

\[
\lim_n \|x_n - Tx_n\| = 0
\]

completing the proof. \(\square\)

**Remark 4.4.1.** If we put \(a = 1\) and \(b = c = 0\), then Theorem 2 due to Deng [40] becomes corollary of the above Theorem.

**Theorem 4.4.5.** Let \(D\) be a nonempty subset of normed space \(X\) and \(T: D \to D\) be a generalized non expansive mapping defined as Theorem 4.2.1. Given a sequence \(\{x_n\}\) in \(D\) and two real sequences \(\{\alpha_n\}\) and \(\{\beta_n\}\) satisfying:

(i) \(0 \leq \alpha_n \leq 1\) for \(n \geq 0\) and \(\sum_{n=0}^{\infty} \alpha_n = \infty\).

(ii) \(0 \leq \beta_n \leq 1\) for \(n \geq 0\) and \(\sum_{n=0}^{\infty} \alpha_n \beta_n\) is convergent.

(iii) \(x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n\)  
\(y_n = (1 - \beta_n)x_n + \beta_n Tx_n, n = 0, 1, 2...\)

If \(\{x_n\}\) is bounded, then \(\{x_n - Tx_n\}\) converges strongly to zero.

**Remark 4.4.2.** (i) Theorem 4.4.5 generalizes Lemma 2 of Ishikawa [71]

(ii) We observe that \(\sum_{n=0}^{\infty} \beta_n < \infty\) implies \(\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty\)

Theorem 4.4.5 improves the Theorem 1 of Deng [40], where it is assumed that \(\sum_{n=0}^{\infty} \beta_n\) be convergent.
**Theorem 4.4.6.** Let \( x \) be a Banach space satisfying Opial’s condition, \( D \) weakly compact subset of \( X \) and let \( T \) and \( \{x_n\} \) be as in Theorem 4.4.4 Then \( \{x_n\} \) converges weakly to a fixed point of \( T \).

**Proof.** First we show that \( w_w(x_n) \subset F(T) \). Let \( x_{n_k} \to x \) weakly. By Theorem 4.4.4 we have \( \lim_n ||x_n - Tx_n|| = 0 \) and since \( I-T \) is demiclosed at zero. Hence \( x \in F(T) \). By Opial’s condition \( \{x_n\} \) possesses only one weak limit point, i.e., \( \{x_n\} \) Converges weakly to a fixed point of \( T \).

**Remark 4.4.3.** Theorem 4.4.6 generalizes and improves the Theorem 2 of Deng [40] under remarks 2 (ii).

**Theorem 4.4.7.** Let \( D \) be a closed convex bounded subset of a uniformly convex Banach space \( X \) which satisfies Opial’s condition, \( T : D \to D \) be a generalized nonexpansive mapping with a fixed point such that \( F(T) \neq \phi \). Given a sequence \( \{x_n\} \) as in Theorem 4.4.4 then \( \{x_n\} \) converges weakly to fixed point of \( T \).

**Proof.** Let \( w_w(x_n) \subset F(T) \). Let \( x_{n_k} \to x \) weakly by Lemma 4.4.3 and Theorem 4.4.4 \( w_w(x_n) \) is contained in \( F(T) \). By Opial’s condition \( \{x_n\} \) Possesses only one weak limit point, i.e. \( \{x_n\} \) converges weakly to a fixed point of \( T \).

**Remark 4.4.4.** Theorem 4.4.4 generalizes the results of Tan and Xu [74] [Theorem 3.1].

**4.5 Conclusion**

We conclude by saying that Ishikawa Iterative Sequence with error term for nonexpansive and generalized nonexpansive mappings are converges strongly in domain of
Banach space. Similarly, Ishikawa iterative sequence of nonexpansive and generalized nonexpansive mapping satisfying Opial’s condition converges weekly.