Chapter 3

A METHOD BASED ON THE FUZZIFICATION FOR SOLVING FUZZY NUMBER RELATIONSHIP LINEAR PROGRAMMING PROBLEMS

3.1. Introduction

The concept of fuzzy linear programming models was introduced as a special kind of fuzzy decision model: space, the goal and constraints were represented by fuzzy sets. Two types of fuzzy decision models: Bellman-Zadeh’s concepts of a symmetrical decision model Bellman and Zadeh [4] or specific models on the basis of a non-symmetrical basic model of a fuzzy decision Orlovsky [82] were considered in modelling fuzzy linear programming problems. The first attempt to fuzzify a linear program was due to Zimmermann [107]. Buckley [15,16], has considered about the modelling of fuzzy linear programming problems. Lai and Hwang [65], Shaocheng [87], and among others etc, have considered the situation where all parameters involved were in fuzzy sets. They introduced an auxiliary model which was solved by multi-objective linear programming methods. Yao and Wu [51] utilised ranking fuzzy numbers based on decomposition principle for comparison of fuzzy numbers and obtained on optimal solution. Nagoor Gani et al. [42] used the ranking method of triangular fuzzy numbers of LR- type based on signed distance, for comparing the fuzzy numbers and obtained the optimal solution through a fuzzy simplex technique.

In this chapter, we have considered a kind of fuzzy linear programming problem where the cost vector, constraint matrix and requirement vector are fuzzy sets. We have introduced the defuzzification method: “the centre of area”, to defuzzify the fuzzy cost vector, fuzzy constrained matrix and the fuzzy requirement vector of the right side, so that the fuzzy linear programming problem
reduced to a classical linear programming problem. The optimal solution can be obtained by using simplex method.

This chapter is organised as follows: in the section 3.2, we have given an overview of the definitions of triangular fuzzy number of LR-type, and some related results of fuzzy arithmetic, defuzzification methods, in section 3.3, we have discussed the mathematical models of fuzzy linear programming problems and their solution methods. We have proposed a solution technique based on defuzzification method to solve a FLPP. Using the proposed technique we solved a numerical example in section 3.4. Finally, we have given the conclusion.

3.2. LR-fuzzy numbers, Ordering of fuzzy numbers and Centroid defuzzification method

In the following, we briefly review some basic definitions of fuzzy sets theory from (Klir and Yuan [63], Ross [85], Zimmermann [110]).

Definition 3.2.1. For a given triplet of real numbers \((m,p,q)\) such that \(p > 0, q > 0\), the LR-fuzzy number \(\tilde{M}\) of the universal set \(E\) (of real numbers) associated with \((m,p,q)\) is defined by its membership function \(\mu_{\tilde{M}} : E \rightarrow [0,1]\) such that

\[
\mu_{\tilde{M}}(x) = \begin{cases} 
L\left(\frac{x-m}{p}\right), & \text{for } x \leq m \\
R\left(\frac{x-m}{q}\right), & \text{for } x \geq m 
\end{cases}
\]

where \(m\) is called the core value of \(\tilde{M}\), it is denoted as \(\text{cor}(\tilde{M})\), \(p\) and \(q\) are called the left and right spreads respectively. Symbolically \(\tilde{M}\) is denoted by \(\tilde{M} = (m, p, q)_{LR}\).

The functions \(L\) (and \(R\)), are defined by \(L\) (and \(R\)): \(E^+ \rightarrow [0,1]\) and are decreasing, are called left (and right) shape functions Buckley [15] if \(L(0)=1, L(x)<1\) for all \(x > 0; L(x) > 0\) for all \(x < 1; L(1)=0\). For \(L(z)\) (and \(R(z)\)), different functions can be chosen Zimmermann [110], for instance \(L(x) = \max(0,1-x^k)\) with \(k > 0\), \(L(x) = e^{-x}\) or \(L(x) = e^{-x^2}\).
Definition 3.2.2. An LR-fuzzy number, $\tilde{M}=(m,p,q)$ is called a triangular fuzzy number of LR-type if reference functions $L(x)$ (and $R(x)$) are linear functions and the membership function is defined as:

$$
\mu_M(x) = \begin{cases} 
L(x) = \frac{x-(m-p)}{p}, & \text{for } m-p \leq x \leq m \\
1, & \text{for } x = m \\
R(x) = \frac{m+q-x}{q}, & \text{for } m \leq x \leq m+q \\
0, & \text{otherwise.}
\end{cases}
$$

It is denoted by $\tilde{M}=(m,p,q)_{LR}$ or shortly by $(m,p,q)$. It is also noted that when $p=0, q=0$ then $\tilde{M}$ becomes a crisp number $m$, i.e., $\tilde{M}=(m,0,0)_{LR}=m$. When $p=0$ and $q=0$ then the reference functions $L$ (and $R$) are respectively 0.

Let $F_s$ be a family of triangular fuzzy numbers of LR-type on $E$. Then following Zimmermann [107], we define the following definitions and results.

Definition 3.2.3. For each $\alpha \in [0,1]$, the $\alpha$-level fuzzy interval of $\tilde{M} \in F_s$, is defined as $M(\alpha) = \{ x | \mu_M(x) \geq \alpha \} = [M(\alpha),\tilde{M}(\alpha)]$, which is unique and $M(\alpha) = m - p + p\alpha$ and $\tilde{M}(\alpha) = m + q - q\alpha$ are respectively left-point, right point of fuzzy interval $\tilde{M}(\alpha)$: $\tilde{M} = \bigcup_{\alpha \in [0,1]} \alpha \tilde{M}(\alpha)$. The integrations of $M(\alpha)$ and $\tilde{M}(\alpha)$ on the common range $[0, 1]$ exist.

Definition 3.2.4. The signed distance of $\tilde{M}=(m,p,q)_{LR}$ from $O=(0,0,0)_{LR}$ is defined as:

$$
d(\tilde{M},O) = \frac{1}{4} \int_0^1 [M(\alpha)+\tilde{M}(\alpha)]d\alpha = 1/4(4m+q-p).
$$

Definition 3.2.5. Suppose $\tilde{M}, \tilde{N} \in F_s$ with $\tilde{M} = (m,p,q)_{LR}$, $\tilde{N} = (n,p',q')_{LR}$. Then we define

(i) $\tilde{M} \oplus \tilde{N}=(m,p,q)_{LR} \oplus (n,p',q')_{LR} = (m+n,p+p',q+q')_{LR}$;

(ii) $-(m,p,q)_{LR} = (-m,p,q)_{LR}$;
(iii) \((m,p,q)_{LR} \otimes (n,p',q')_{LR} = (m-n, p+p', q+q')_{LR}\);

(iv) \((m,p,q)_{LR} \otimes (n,p',q')_{LR} \approx (mn, mp' + np, mq' + nq)_{LR}\) for \(M, N\) positive;

(v) \((m,p,q)_{LR} \otimes (n,p',q')_{LR} \approx (mn, np - mq', nq - mq')_{LR}\) for \(M\) positive, \(N\) negative; and

(vi) \((m,p,q)_{LR} \otimes (n,p',q')_{LR} \approx (mn, -np - mq', -mp - nq)_{LR}\), for \(M, N\) negative.

**Definition 3.2.6.** For \(\tilde{M}=(m,p,q)_{LR}\) and \(\tilde{N}=(n,p',q')_{LR}\) belong to \(F_{LR}\), the following ordering is defined:

\[
\tilde{M} < \tilde{N} \iff \begin{cases} 
\text{d}(\tilde{M},\tilde{O}) < \text{d}(\tilde{N},\tilde{O}) \\
\text{or} \\
\text{d}(\tilde{M},\tilde{O})=\text{d}(\tilde{N},\tilde{O}) \text{ and } m+q < n+q' \\
\text{or} \\
d(\tilde{M},\tilde{O})=d(\tilde{N},\tilde{O}) \\
\text{m}+p=n+q' \text{ and } m-p < n+q' 
\end{cases}
\]

If \(p=q\) and \(p'=q'\) then

\[
d(\tilde{M},\tilde{O}) < d(\tilde{N},\tilde{O}) \iff \begin{cases} 
m < n \\
\text{or} \\
m=n \text{ and } m+q < n+q' .
\end{cases}
\]

**Definition 3.2.7.** An L-R triangular fuzzy number \(\tilde{A}=(m,p,q)\) is said to be defuzzified with respect to centroid method (Lai and Hwang [64,65], Sugeno [90]) if \(\tilde{A}\) is expressed by the algebraic expression

\[
\int \mu_{\tilde{A}}(x)dx = m + \frac{1}{3}(q - p) = \bar{a} 
\]

(see Fig.3(a) and 3(b)) where \(\int\) denotes an algebraic expression. This method is shown in Fig.
3.3. Problem formulation

Fuzzy linear programming models shall be considered as a special kind of decision models under fuzzy environment: the decision space and the “goal” (objective function) are defined by fuzzy sets. These models shall be derived from many possibilities, depending on the assumptions or features of the real situations to be modelled. Bellman and Zadeh [4] proposed a fuzzy linear programming model by introducing the concepts of aspiration level of the objective function and small violations of constraints based on a standard classical linear programming problem:

Maximize $f(x) = C^T x$

Such that $Ax \leq b \quad (3.3.1)$

where $C, x \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$,

$C^T$ is the transpose of $C$ and $A = (a_{ij})_{m \times n}$

Case 1. Introducing subjective aspiration level, $z$, of the objective function, and fuzzifying the equations in appropriate linguistic interpretations, equation (3.3.1) is transformed into a fuzzified version:

Find $x$ such that

$Bx \leq d, x > 0 \quad (3.3.2)$
where $B = \begin{pmatrix} -C^T \\ A \end{pmatrix}$, $d = \begin{pmatrix} -z \\ b \end{pmatrix}$, $\leq$ denotes “essentially smaller than or equal to”.

We construct linearly increasing membership function $\mu_i(x), i = 1, 2, \ldots, m+1$ over the subjective “tolerance interval” $[d_i, d_i + p_i]$:

$$
\mu_i(x) = \begin{cases} 
1 & \text{for } B_i x \leq d_i \\
1 - \frac{B_i x - d_i}{p_i} & \text{for } d_i < B_i x \leq d_i + p_i, i = 1, 2, \ldots, m+1 \\
0 & \text{for } B_i x > d_i + p_i
\end{cases}
$$

(3.3.3)

where $p_i$ are subjectively chosen constants of admissible violations of the constraints. Then, the membership function of fuzzy decision Bellman and Zadeh [4] of model (3.3.2) is defined as

$$
\mu_{\beta}(x) = \min_i (\mu_i(x))
$$

(3.3.4)

In view of (3.3.3), introducing one new variable, $\lambda \in [0,1]$ the problem (3.3.1) considered under fuzzy environment, becomes the following classical optimization problem:

$$
\max \lambda \\
\text{Subject to } \lambda p_i + B_i x \leq d_i + p_i, \quad i = 1, 2, \ldots, m+1
$$

(3.3.5)

Now, if the optimal solution to (3.3.5) is the vector $(\lambda, x^*)$, then $x^*$ is the maximizing solution of (3.3.4) of model (3.3.2) assuming membership functions as specified in (3.3.3).

Case 2. In problem (3.3.1), if only the right-hand-side members $b_i$ are fuzzy numbers $\tilde{B}_i$ with membership function typically have the form
\[
\mu_{b_i}(x) = \begin{cases} 
1 & \text{when } x \leq b_i \\
\frac{b_i + p_i - x}{p_i} & \text{when } b_i < x < b_i + p_i \\
0 & \text{when } b_i + p_i \leq x
\end{cases}
\]

then we have the fuzzy L.P problem:

\[
\max \sum_{j=1}^{n} c_j x_j \\
\text{subject to } \sum_{j=1}^{n} A_{ij} x_j \leq \bar{B}_i, \ i=1,2,...,m \\
x_j \geq 0
\]

For each vector \(x=(x_1,x_2,...,x_n)\), we first calculate the degree \(\mu_{b_i}(x), \ i=1,2,...,m\)

where \(x\) satisfies the \(i^{th}\) constraint by the formula

\[
\mu_{b_i}(x) = \mu_{b_i}(\sum_{j=1}^{n} a_{ij} x_j), \ i=1,2,...m, \ j=1,2,...,n
\]

Now, the intersection \(\bigcap \mu_{b_i}(x)\) of these degrees, is a fuzzy feasible set.

Next, we determine the fuzzy set of optimal values. This is done by calculating the lower and upper bounds of the optimal values first. Let \(z_l\) and \(z_u\) be the lower and upper bounds of the optimal values. Then \(z_l\) is obtained by solving the linear programming problem:

\[
\max z = C^T x \\
\text{subject to } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i=1,2,...,m \\
x_j \geq 0, j=1,2,...,n
\]

The upper bound of the optimal values, \(z_u\), is obtained by a similar linear programming problem:
\[
\max z = C^T x \\
\text{subject to } \sum_{j=1}^{n} a_{ij}x_j \leq b_i + p_i, i = 1, 2, ..., m \\
x_j \geq 0, \quad j = 1, 2, ..., n
\] (3.3.8)

Then, the fuzzy set of optimal values, $\tilde{G}$, is defined by the membership function

\[
\mu_{\tilde{G}}(x) = \begin{cases} 
1 & \text{when } z_u \leq C^T x \\
\frac{C^T x - z_l}{z_u - z_l} & \text{when } z_l \leq C^T x \leq z_u \\
0 & \text{when } C^T x \leq z_l.
\end{cases}
\]

Now, the problem (3.3.6) becomes the following classical linear programming problem:

\[
\max \lambda \\
\text{subject to } \lambda(z_u - z_l) - C^T x \leq -z_l \\
\lambda p_i + \sum_{j=1}^{n} a_{ij}x_j \leq b_i + p_i, \quad i = 1, 2, ..., m \\
\lambda, x_j \geq 0, \quad j = 1, 2, ..., n
\] (3.3.9)

The above problem, (3.3.9) is actually a problem of obtaining $x \in \mathbb{R}^n$ such that $[(\cap \mu_{\tilde{D}}) \cap \mu_{G}](x)$ reaches the maximum value. The method employed here is known as a symmetrical method Bellman and Zadeh [4].

In the following we discuss the general problem of fuzzy linear programming defined by (3.3.6) and proposed a method of solving the problem. In this case, we assume that all constants are triangular fuzzy numbers. Any triangular fuzzy number $\tilde{T}$ can be represented by three real numbers, $m, p, q$ whose meanings are defined in Definition (3.2.1). Using this representation, we write $\tilde{T} = (m, p, q)$. Thus, problem (3.3.6) can be restated as
\[
\max \sum_{j=1}^{n} (c_{1j}, c_{2j}, c_{3j}) x_j \\
\text{subject to } \sum_{j=1}^{n} (m_{ij}, p_{ij}, q_{ij}) x_j \leq (t_i, u_i, v_i) \quad i = 1, 2, \ldots, m \quad (3.3.10)
\]

\[
x_j \geq 0 \quad j = 1, 2, \ldots, n
\]

where \( \tilde{c}_j = (c_{1j}, c_{2j}, c_{3j}) \), \( \tilde{a}_y = (m_{ij}, p_{ij}, q_{ij}) \) and \( \tilde{B}_i = (t_i, u_i, v_i) \) are triangular fuzzy numbers of LR type. Summation and multiplication are operations on fuzzy numbers, and partial order \( \leq \) is defined by \( \tilde{A} \leq \tilde{B} \iff \text{Max} (\tilde{A}, \tilde{B}) \leq \tilde{B} \) Klir and Yuan [63]. Assuming \( \tilde{c}_j \) in (3.3.10) as constant, \( c_j \), the problem can be rewritten as

\[
\max \sum_{j=1}^{n} c_j x_j \\
\text{Subject to } \sum_{j=1}^{n} m_{ij} x_j \leq t_i \\
\sum_{j=1}^{n} (m_{ij} - p_{ij}) x_j \leq t_i - u_i \\
\sum_{j=1}^{n} (m_{ij} + q_{ij}) x_j \leq t_i + v_i \quad i = 1, 2, \ldots, m \\
x_j \geq 0 \quad j = 1, 2, \ldots, n
\]

(3.3.11)

In the proposed method, each triangular number in (3.3.10) is transformed into single real value by employing the defuzzification centroid method (see Definition 3.2.6). Assuming \( \tilde{c}_j = \text{centroid} (\tilde{c}_j) \), \( \tilde{a}_y = \text{centroid} (\tilde{a}_y) \) and \( \tilde{B}_i = \text{centroid} (\tilde{B}_i) \), problem (3.3.10) can be rewritten as

\[
\max \sum_{j=1}^{n} \tilde{c}_j x_j \\
\text{subject to } \sum_{j=1}^{n} \tilde{a}_y x_j \leq \tilde{B}_i \quad i = 1, 2, \ldots, m \\
x_j \geq 0 \quad j = 1, 2, \ldots, n
\]

(3.3.12)
The solution of (3.3.12) which satisfies the constraints will give the optimal solution of the problem (3.3.10).

3.4. Numerical Example

In this section we present a simple illustrative example which reflects some characteristic features of the real application of optimal profit of production schedule of a manufacturing company. The problem is stated as follows:

A manufacturing company closed down the production of a certain unprofitable product line. This necessitated the rational utilization of released excess production capacity. Management has been contemplating to devote this excess capacity to the manufacture of one or more of three products A, B, and C. The details about available excess machine products are assessed as follows:

I) Machine type Available excess machine time
   Milling machine Around 250 hrs/week
   Lathe Around 150 hrs/week
   Grinder Around 50 hrs/week.

II) Machine type Machine hour requirements per unit of product.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milling</td>
<td>About 8</td>
<td>About 2</td>
<td>About 3</td>
</tr>
<tr>
<td>Lathe</td>
<td>About 4</td>
<td>Closed to 3</td>
<td>______</td>
</tr>
<tr>
<td>Grinder</td>
<td>Nearly 2</td>
<td>______</td>
<td>Nearly 1</td>
</tr>
</tbody>
</table>

It is estimated the profit would be Rs 20, Rs 16 and Rs 8 respectively for products A, B and C. The company desires a most profitable production schedule.

The problem is formulated as follows:

Maximize $\tilde{c}_1x_1 + \tilde{c}_2x_2 + \tilde{c}_3x_3$
Subject to \[
\sum_{j=1}^{3} \tilde{a}_{ij} x_j \leq \tilde{b}_i \quad i = 1, 2, 3
\]
\[
x_1, x_2, x_3 \geq 0
\]

where
\[
\tilde{c}_1 = (20, 0, 0), \quad \tilde{c}_2 = (16, 0, 0), \quad \tilde{c}_3 = (18, 0, 0),
\]
\[
\tilde{a}_{i1} = (8, 1.5, 2.5), \quad \tilde{a}_{i2} = (2, 1.5, 2.5), \quad \tilde{a}_{i3} = (3, 1.5, 2.5),
\]
\[
\tilde{a}_{21} = (4, 1.5, 2.5), \quad \tilde{a}_{22} = (3, 1, 2), \quad \tilde{a}_{23} = (0, 0, 0),
\]
\[
\tilde{a}_{31} = (2, 1, 1.5), \quad \tilde{a}_{32} = (0, 0, 0), \quad \tilde{a}_{33} = (1, 5, 1.5),
\]
\[
\tilde{b}_1 = (250, 5, 5), \quad \tilde{b}_2 = (150, 5, 5), \quad \tilde{b}_3 = (50, 3, 3).
\]

In the setting of problem (3.3.11), the above is formulated as follows:

Maximize \[
20x_1 + 16x_2 + 18x_3
\]
\[
8x_1 + 2x_2 + 3x_3 \leq 250
\]
\[
6.5x_1 + 0.5x_2 + 1.5x_3 \leq 245
\]
\[
10.5x_1 + 4.5x_2 + 5.5x_3 \leq 255
\]
\[
4x_1 + 3x_2 + 0.0x_3 \leq 150
\]
\[
2.5x_1 + 2x_2 + 0.0x_3 \leq 145
\]
\[
6.5x_1 + 5x_2 + 0.0x_3 \leq 155
\]
\[
2x_1 + 0.0x_2 + x_3 \leq 50
\]
\[
x_1 + 0.0x_2 + 0.5x_3 \leq 47
\]
\[
3.5x_1 + 0.0x_2 + 2.5x_3 \leq 53
\]
\[
x_1, x_2, x_3 \geq 0
\]

The optimal solution is obtained as: \(x_1 = 0, x_2 = 31, x_3 = 21\) and \(z = 874\).

According to the proposed model, the given problem can be rewritten as follows:

Maximize \[
20x_1 + 16x_2 + 18x_3
\]
In this case the optimal solution is obtained as \( x_1 = 0, \ x_2 = 45.05, \ x_3 = 37.59 \) and \( z = 1397.47 \)

### 3.5. Conclusion

We have provided a framework for the selection of optimal alternative of fuzzy linear programming models involving the state knowledge of the parameters obtained by non-deterministic triangular fuzzy numbers of LR- type. The triangular fuzzy numbers were defuzzified by employing centroid method and the fuzzy linear programming problem was transformed into a classical linear programming one. The proposed method provides a low cost effective solution procedure in comparison with the existing methods in the literature. Moreover, the procedure can be extended even when the fuzzy co-efficient parameters are replaced by fuzzy parameters obtained through fuzzy controllers for future research problems.