Chapter 7
FUZZY GOAL PROGRAMMING BASED ON PIECEWISE LINEAR MEMBERSHIP FUNCTIONS

7.1. Introduction

Initially introduced as an application of fuzzy set theory to goal programming by Narasimhan [77] and Hannan [45], fuzzy goal programming (FGP) gained popularity in the area of multiple criteria optimization problems under fuzzy environment. Tiwari et al. [94] investigated how pre-emptive (lexicographic) priority structure could be used in FGP problems. In Tiwari et al. [95] an additive model for FGP problems has been introduced. Flavell [39] investigated a new goal programming formulation. Lohgaonkar et al. [69] introduced an additive fuzzy multiple goal programming model for unbalanced multi-objective transportation problem. An interactive fuzzy goal programming for multi-objective transportation problems has been introduced by Wahed and Lee [1]. Moreover, we can mention the work by Romero [84].

In the chapter, we introduce a brief survey of FGP problems in Multi-objective linear programming (MOLP) problems so as to establish a relationship between different approaches under fuzzy environment.

The chapter is organised as follows: In section 7.2, we present the concepts fuzzy decision and fuzzy goal programming (FGP) problems. In section 7.3, we briefly review some concepts of classical goal programming (GP) problems and the piecewise linear functions due to Charnes and Cooper [20] and Can and Houk [18]. In section 7.4, we propose the solution method for solving goal programming problem under fuzzy environment. An example is also illustrated to show the efficiency of the method.
7.2. Prerequisite:

Fuzzy Decision of Bellman and Zadeh [4]

Let X be a given set of all possible solutions to a decision problem. A fuzzy goal \( \tilde{G} \) is a fuzzy set on X characterized by its membership function
\[
\mu_{\tilde{G}} : X \rightarrow [0,1] \tag{7.2.1}
\]
A fuzzy constraint \( \tilde{C} \) is a fuzzy set on X characterized by its membership function
\[
\mu_{\tilde{C}} : X \rightarrow [0,1] \tag{7.2.2}
\]
Let \( \mu_{\tilde{C}_i}(x), i = 1,2,\ldots, m \) where \( x \in X \) be membership functions of constraints, defining the decision space and \( \mu_{\tilde{C}_j}(x), j = 1,2,\ldots, n, x \in X \) the membership functions of objective (utility) functions or goals. A fuzzy decision is then defined by its membership function (Zimmermann, [110]):
\[
\mu_{\tilde{D}}(x) = \otimes_i \mu_{\tilde{C}_i}(x) \ast \otimes_j \mu_{\tilde{G}_j}(x), i = 1,2,\ldots, m, j = 1,2,\ldots, n \tag{7.2.3}
\]
where \( \ast \otimes_i \) and \( \otimes_j \) denote possibly, appropriate context-dependent “aggregators” (connectives).

Model of the fuzzy goal Programming

The idea of GP is to establish a goal level of achievement of each criterion. For conventional treatment of goal programming, the reader is referred to sources such as Charnes and Cooper [19], Lee [68], Ignizio [49]. In this section, we considered a more critical look at fuzzy goal programming originated from MOLP problems.

A classical structure of multi-objective linear programming model with k linear objective functions is as follows:

Minimize \( z(x) = (z_1(x), z_2(x),\ldots, z_k(x)) \) 
subject to \( Ax \leq b \)
\[ x \geq 0 \tag{7.2.4} \]
where \( z_i(x) = \sum_{j=1}^{m} c_{ij} x_j, c_i = (c_{i1}, c_{i2},\ldots, c_{im}), i = 1,2,\ldots, k; x = (x_1, x_2,\ldots, x_n)^T \),
\( b = (b_1, b_2,\ldots, b_m)^T \), \( A \) is an \( m \times n \) matrix, \( T \) denote the transpose of a vector.
Assume that the DM proposes fuzzy goals such as “the objective function $z_i(x)$ should be substantially less than or equal to some goal value $G_i$.” In addition to above assumption, if it is possible to consider the RHS values of $b$ as a fuzzy resource, then model (7.2.4) can be written as

Find $x$
subject to $z(x) \geq \tilde{G}$  
$Ax \leq \tilde{b}$
$x \geq 0$

where $\equiv$ and $\leq$ mean “essentially equal to” and “essentially less than equal to” respectively.

Let the membership function of fuzzy goal $\tilde{G}$, $\mu_{\tilde{G}}(z(x))$ be $\mu_{\tilde{G}}: X \to [0,1]$. By using the fuzzy decision (max-min) of Bellman and Zadeh [4] and by introducing the auxiliary variable $\alpha, \alpha \in [0,1]$, a FGP adopt the following formation:

Max $\alpha$
Subject to $\alpha \leq \mu_{\tilde{G}}(z(x)), i = 1,2,...,k$

$Ax \leq \tilde{b}$
$x \geq 0$

(7.2.5)

7.3. The concept of classical goal programming

The traditional form of a goal programming (GP) problem with one of each type of goal criterion is expressed as

goal $\{f_i(x) = z_i\}, z_i \in \{z_i \geq t_i, z_i \leq t_i, z_i = t_i, z_i \in [t_i', t_i'']\}, i \in \{1,2,...,k\}$

subject to $x \in F$
$x \geq 0$

(7.3.1)

where $t_i$ is target level for the $i^{th}$ goal, $x \in R^n$ is vector of decision variables, $F$ is feasible set of constraints, $t_i'$ and $t_i''$ are the lower and upper limits of $z_i$ respectively.
A key element of GP model (7.3.1) is the achievement function that represents a mathematical expression of unwanted deviation variables. The three existing forms of GP achievement functions are (i) weighted GP (WGP), also known as Archimedean GP, (ii) lexicographic GP (LGP), also known as non-Archimedean or pre-emptive GP, and (iii) MINMAX GP (MGP), also known as Fuzzy programming or Chebyshev. The analytical structure of a WGP model of model (7.3.1) is expressed in the following (Ignizio [49]):

Achievement function:

\[ \text{Min } \sum_{i=1}^{k} (\alpha_i n_i + \beta_i p_i) \]

Goals and constraints:

\[ f_i(x) + n_i - p_i = t_i, \quad i \in \{1, 2, \ldots, k\} \]
\[ x \in F, \quad x \geq 0, \quad p \geq 0, \quad n \geq 0 \quad (7.3.2) \]

where \( n_i, p_i \) are negative and positive deviations from the target value \( t_i \) of \( i^{th} \) goal, \( \alpha_i = \frac{w_i}{q_i} \) if \( n_i \) is unwanted, otherwise \( \alpha_i = 0 \), \( \beta_i = \frac{w_i}{q_i} \) if \( p_i \) is unwanted, otherwise \( \beta_i = 0 \), \( w_i \) and \( q_i \) are the weights reflecting preferential and normalising purposes attached to the achievement of the \( i^{th} \) goal.

The solution of WGP provides the maximum aggregated achievement between the different goals (see e.g. Dyer [29]). The analytical structure of a LGP is the following (Ignizio [49]).

Achievement function:

\[ \text{Lex min } \left[ a = \sum_{i \in h_r} (\alpha_i n_i + \beta_i p_i), \ldots, \sum_{i \in h_r} (\alpha_i n_i + \beta_i p_i), \ldots, \sum_{i \in h_r} (\alpha_i n_i + \beta_i p_i) \right] \]

Goals and constraints:

\[ f_i(x) + n_i - p_i = t_i, \quad i \in \{1, 2, \ldots, k\}, \quad i \in h_r, \quad r \in \{1, 2, \ldots, Q\} \]
\[ x \in F, \quad n \geq 0, \quad p \geq 0 \quad (7.3.3) \]

where \( h_r \) denotes the index set of goals placed in the \( r^{th} \) priority level.
It is noted that in Lexicographic achievement functions, there is no finite trade-offs among goals placed in different priority levels (see. e.g. Romero,[83]). The analytical structure of MGP model is expressed as (Flavell [39]):

Achievement function:

\[
\text{Min } D
\]

Goals and constraints:

\[
(\alpha x_i + \beta y_i) - D \leq 0 \\
(\alpha x_i + \beta y_i) - D \leq 0 \\
f_i(x) + n_i - p_i = t_i, \quad i \in \{1, 2, ..., k\} \\
x \in F, n \geq 0, p \geq 0
\]

where D is the maximum deviation from any single goal.

MGP is the optimization of a utility function where the maximum deviation is minimized. Thus, this type of achievement function provides a solution that gives the maximum importance to the goal most displaced with respect to its target. An extensive analysis of MGP can be found in Romero [83,84]. Following Romero [84], other newer versions of formulation with achievement functions that generalize the three classic forms previously introduced are obtained as shown below:

(i) Achievement function:

\[
\text{Lex min } a = [D_1, D_2, ..., D_Q]
\]

Goals and constraints:

\[
(\alpha x_i + \beta y_i) - D_r \leq 0, \quad i \in h_r, \quad r \in \{1, 2, ..., Q\} \\
f_i(x) + n_i - p_i = t_i, \quad i \in \{1, 2, ..., k\} \\
x \in F, n \geq 0, \quad p \geq 0
\]

(ii) Achievement function:

\[
\text{Min } (1 - \lambda)D + \lambda(\alpha x_i + \beta y_i)
\]

Goals and constraints:

\[
(\alpha x_i + \beta y_i) - D \leq 0 \\
f_i(x) + n_i - p_i = t_i, \quad i \in \{1, 2, ..., k\} \\
x \in F, n \geq 0, \quad p \geq 0, \quad \lambda \in [0, 1]
\]

(iii) Achievement function:
Lex min \( a = [(1 - \lambda_i)D_i + \lambda_i \sum (\alpha_i n_i + \beta_i p_i),..., 
(1 - \lambda_q)D_q + \lambda_q \sum (\alpha_q n_q + \beta_q p_q),..., 
(1 - \lambda_r)D_r + \lambda_r \sum (\alpha_r n_r + \beta_r p_r)] \)

Goals and constraints:
\[
(\alpha_i n_i + \beta_i p_i) - D_i \leq 0 \\
f_i(x) + n_i - p_i = t_i, \ i \in \{1,2,...,k\} \tag{7.3.7} \\
x \in F, n \geq 0, p \geq 0, \lambda_r \in [0,1]
\]

In all the achievement functions presented above two basic assumptions underlie: (1) the decision maker (DM) determines a precise target for each attribute and (2) the deviations regarding the target are penalized with a constant weight independently of which is the distance from the target. With these assumptions we have a piecewise linear penalty function. Charnes \textit{et al.} [20] introduced a piecewise linear penalty function for GP in the following way (Fig 7.3.1).

\[
f(z) = \sum_{j=1}^{N} \alpha_j |z-t_j| + \beta z + \gamma \tag{7.3.8}
\]

where \( \alpha_j = \frac{k_{j+1} - k_j}{2}, \beta = \frac{k_{N+1} + k_1}{2}, \gamma = \frac{a_{N+1} + a_1}{2}, j = 1,2,...,N, \ k_j \) is the slope, \( a_j = f(z) \)- intercept, which is a constant, of the line segment initiated at \( t_{j-1} \) and terminated at \( t_j \).

\[
\text{Slope } k_i
\]

\[
\text{Fig 7.3.1}
\]
Assuming (7.3.8), Charnes and Cooper [20] showed how a GP with piecewise linear penalty functions can be formulated as a LP problem. Can and Houk [18] introduced a U-shaped piecewise linear penalty function, like those with five sides (Fig 7.3.2)

This penalty function can be represented in the following way:

\[
f(z) = |r_1| n_1 + |r_2| n_2 + |k_1| p_1 + |k_2| p_2
\]  

(7.3.9)

where:

\[
\begin{align*}
    z + n_1 + n_2 + n_3 - p_1 - p_2 &= t_4 \\
    n_1 &\leq t_2 - t_1, n_2 \leq t_3 - t_2, n_3 \leq t_4 - t_3 \\
    p_1 &\leq t_5 - t_4, p_2 \leq t_6 - t_5 \\
    n, p &\geq 0
\end{align*}
\]

Obviously, the above expression can be generalised in a straightforward manner for penalty functions with \( n \) sides.

7.4. Linear GP problem under fuzzy environment

In this section, we propose a method for solving GP problem under fuzzy environment. Assume that we have a Linear GP problem as defined in problem (7.3.1).

Assume that the DM proposes fuzzy goals such as “the objective function \( z_i(x) \) should be substantially greater than or equal to some value \( t_i \)”, \( (i=1,2,...,k) \). Then the corresponding fuzzy membership function:
\( \mu_i(z_i(x)), \mu_i : R \rightarrow [0,1] \) \tag{7.4.1}

provides the degree in which the \( i \)-th fuzzy goal is attained. By using the concept of fuzzy decision “max-min” of Bellman and Zadeh [4] and by introducing the auxiliary variable \( \lambda \), a fuzzy goal programming problem adopt the following formulation Zimmermann [110]

\[
\begin{align*}
\text{Max } & \lambda \\
\text{subject to } & \lambda \leq \mu_i(z_i(x)), \ i = 1,2,\ldots,k \\
& x \in F, x \geq 0, \lambda \in [0,1] \\
\end{align*}
\tag{7.4.2}
\]

Let us suppose that the membership function, \( \mu_i(z_i(x)) \), \( i = 1,2,\ldots,k \) is piecewise linear shaped increasing function, like on Fig. 7.4.1

Then, the membership function is constructed as follows:

\[
\mu_i(z_i(x)) = \sum_{j=0}^{N_i} \alpha_{ij} |z_i(x) - t_{ij}| + \mu_i(t_{ij}), \ j = 0,1,\ldots,N_i \tag{7.4.3}
\]

where \( \alpha_{ij} = (\mu_i(t_{ij+1}) - \mu_i(t_{ij}))/ (t_{ij+1} - t_{ij}) \) is the slope of the line segment originated from \( t_{ij} \) and terminated at \( t_{ij+1} \) and \( \mu_i(t_{ij}) \) is the membership value at \( z_i = t_{ij} \). To formulate model (7.4.2) as a linear programming problem, following the concepts and procedures as in classical GP, we introduce the nonnegative deviational variables \( n_{ij}, p_{ij} \).
\( z_i(x) + n_j - p_j = t_j, \ j = 0, 1, ..., N_j \quad (7.4.4) \)

\[ i = 1, 2, ..., k \]

Thus, each piecewise linear membership function \( \mu_i(z_i(x)) \) can be expressed as

\( \mu_i(z_i(x)) = \sum_{j=0}^{N_i} \alpha_j(n_j + p_j) + \mu_i(t_j) \quad (7.4.5) \)

Then problem (7.4.2) can be converted into the following problem:

Max \( \lambda \)

subject to

\[ \lambda \leq \sum_{j=0}^{N_i} \alpha_j(n_j + p_j) + \mu_i(t_j) \]

\[ z_i(x) + n_j - p_j = t_j \]

\[ x \in F, x \geq 0, \lambda \in [0, 1] \]

\[ n_j \geq 0, p_j \geq 0, \ i = 1, 2, ..., k, \ j = 0, 1, 2, ..., N_j \quad (7.4.6) \]

The solution of problem (7.4.6) will give the solution of the GP problem, (7.3.1) under fuzzy environment.

**7.5. Example:**

Consider the GP problem

\begin{align*}
\text{goal}\{x_1 = z_1\}, z_1 & \geq 4 \\
\text{goal}\{x_2 = z_2\}, z_2 & \geq 2 \\
\text{goal}\{x_3 = z_3\}, z_3 & \geq 3 \\
\text{s.t.} \quad 2x_1 + 3x_2 + 2x_3 & \leq 4 \\
& \quad x_1 + 2x_2 \leq 5 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}

Without lost of generality we define the membership values of the piecewise membership function of each goal as follows:

\( \mu_1(z_1) : 0 \quad 0.5 \quad 0.8 \quad 1 \)

\[ z_1 : 1 \quad 2 \quad 3 \quad 4 \]

\( \mu_2(z_2) : 0 \quad 0.5 \quad 0.8 \quad 1 \)

\[ z_2 : 0.5 \quad 1 \quad 1.5 \quad 2 \]
Then, we calculate $\alpha_y$ as follows:

\[
\begin{align*}
\alpha_{10} &= 0.5, \alpha_{11} = 0.3, \alpha_{12} = 0.2, \\
\alpha_{20} &= 1, \alpha_{21} = 0.6, \alpha_{22} = 0.4, \\
\alpha_{30} &= 1, \alpha_{31} = 0.6, \alpha_{32} = 0.2
\end{align*}
\]

We obtain

\[
\begin{align*}
\mu_1(z_1) &= z_1 + 0.8 \\
\mu_2(z_2) &= 2z_2 - 0.4 \\
\mu_3(z_3) &= 1.8z_3 - 1
\end{align*}
\]

The equivalent LP problem is obtained as

Max $\lambda$

Subject to

\[
\begin{align*}
\lambda &\leq (n_{11} + p_{11}) + 0.8 \\
\lambda &\leq 2(n_{21} + p_{21}) - 0.4 \\
\lambda &\leq 1.8(n_{31} + p_{31}) - 1 \\
x_1 + n_{11} - p_{11} &= 4 \\
x_2 + n_{21} - p_{21} &= 2 \\
x_3 + n_{31} - p_{31} &= 3 \\
2x_1 + 3x_2 + 2x_3 &\leq 4 \\
x_1 + 2x_2 &\leq 5 \\
x_1, x_2, x_3, n_{11}, n_{21}, n_{31}, p_{11}, p_{21}, p_{31} &\geq 0
\end{align*}
\]

$\lambda \in [0,1]$

Using TORA software, after 13 iterations, the solution is obtained as $x_1 = 0, x_2 = 0.07, x_3 = 1.87, \lambda = 1$.

### 7.6. Conclusion

The proposed method will provide a new strategy for solving FGP problems with piece-wise linear membership functions.