CHAPTER 2

LITERATURE SURVEY

2.1 INTRODUCTION

Topology is an important branch of mathematics. It has become a powerful topic in mathematical research. The study of topological spaces, their continuous mappings and general properties make up one branch of topology known as 'general topology'.

In this chapter, Section 2.2 presents some strong and weak forms of open sets and closed sets. Section 2.3 presents strong weaker forms of continuous functions. Section 2.4 and 2.5 explain the concepts of irresolute open maps and closed maps. Section 2.6 the notion of generalized homeomorphisms and Section 2.7 deals with minimal structure explained. Section 2.9 explains the fuzzy topological spaces. Section 2.10 deals with fuzzy topological space is explained. Section 2.11 explain the concept of fuzzy minimal space. Section 2.12 presents preliminary definitions and results on topological spaces. Section 2.12 outlines the contributions of the author.

2.2 STRONG AND WEAK FORMS OF OPEN SETS AND CLOSED SETS

(1987), Palaniappan and Rao (1993), Maki et al (1991) and Dontchev (1995) introduced respectively preopen sets, semi-open sets, g-open sets, \( \alpha \)-open sets, \( \beta \)-open sets, generalized semi-open sets, semi-generalized open sets, regular generalized open sets, generalized preopen sets and generalized semi-preopen sets which are some weak forms of open sets. Various types of sets have been studied in (Dontchev 1998, Dunham 1982, Maki 1996, Maki 1993, Maki 1994, Maki 1996). Ganster and Reily (1989) and Balachandran et al (1996) have introduced locally closed sets and generalized locally closed sets which are weaker than open and closed sets. Also in recent years, Ekici (2004, 2005, 2006, 2007, 2008, 2009) has studied some papers related with various classes of sets in topological spaces. Apart from the above said concepts the following important and relevant results are obtained.

1) Semi-open sets and semi-continuity in topological spaces:
   Levine (1963)
   The notion of semi generalized closed sets is introduced and the properties are explored.

2) Generalized closed sets in topology:
   Levine (1970)
   Author introduced a g-closed set and standard properties investigated in g-closed are normal space, complete uniform space, locally compact space Haustroff space. Also \( T_{1} \)-space implies \( T_{1/2} \)

3) A semi-open sets and semi-continuity in topological spaces:
   Noiri (1974)
   The notion of semi-open sets and semi-closed sets is introduced and interesting in topological properties are investigated.
4) A Note on semi continuity and pre-continuity:
The authors have introduced and proved some of properties of in preopen and preclosed sets.

5) Locally closed sets and LC-continuous functions
Ganster and Reilly (1989)
The authors introduced and investigated some properties of locally closed sets which are weaker than both open and closed sets.

6) Remarks on semi-generalized closed sets and generalized semi-closed sets:
Maki et al (1996)
The authors studied the product of generalized semi closed sets and proved a product theorem on GSO-compact spaces. A new class of topological spaces that is $T_{gs}$-spaces is introduced. The homeomorphic image of $T_{gs}$ space and relationships among $T_{1/2}$ spaces, semi-$T_{1/2}$ spaces and $T_{gs}$ spaces are investigated.

7) On generalizing semi open sets and preopen sets:
The authors generalized the notions of semi open sets and pre-open sets in a universal set and got a formula of the generalized closure of a subset. The new characterizations of semi-open sets, pre open sets and $\delta$ preopen sets and semi preopen sets are given.
2.3 STRONG AND WEAK FORMS OF CONTINUOUS MAPS

(1) A note on semi-continuous mappings
Noiri (1973)

The author gave three characterizations of semi continuous mappings and investigated some properties of such mappings. Also he investigated the properties of Hausdroff spaces and semi continuous mappings.

(2) Some properties of semi weakly continuous mappings:
Popa (1983)

The author introduced the concept of semi-weakly continuous mappings. Also he obtained the new characterizations of semi-weakly continuous mappings and investigated some properties of semi weakly continuous mappings.

(3) The pasting lemma for $\alpha$-continuous maps:
Maki and Noiri (1988)

The authors proved the pasting lemma for $\alpha$-continuous maps and introduced the notion of the $\alpha$-fundamental group.

(4) Locally closed sets and Lc-continuous functions:
Ganster and Reilly (1988)

The authors introduced and studied three different notions of generalized continuity, namely Lc-irresolutness, Lc-continuity and sub Lc-continuity. All three notions are obtained by using the concept of a locally closed set. Also they discussed some properties of these functions and showed that a functions between topological spaces is continuous if and only if it is sub Lc-continuous and newly continuous. Several examples are provided to illustrate the behaviors of these new classes of functions.
(5) On decomposition of continuity in topological spaces:

Tong (1989)

The author introduced the notion of $\beta$-set and $\beta$-continuity together with the notion of pre-continuity. He obtained another decomposition of continuity; a mapping $f: X \to Y$ is continuous if and only if it is pre-continuous and $\beta$-continuous.

6) Semi–Generalized continuous maps and semi $T_{1/2}$-spaces:

Sundaram et al (1990)

The authors introduced the concept of semi-generalized continuous maps, semi generalized irresolute maps and new semi closure operations. Further they studied the relations between semi-homeomorphisms and sg-irresolute maps and characterized semi–$T_{1/2}$ spaces by the new semi closure operations. Moreover they investigated the semi-homeomorphic image of semi $T_{1/2}$ spaces and obtained a product theorem for semi $T_{1/2}$ spaces.

2.4 IRRESOLUTE MAPS

Crossley and Hilderbrand (1972) introduced and investigated irresolute maps which are stronger than semi-continuous maps but are independent of continuous maps. Since then, several types of irresolute maps have been introduced by many authors (Balachandran 1996, Cammamaroto 1989, Dontchev 1996) Di-Maio and Noiri (1988), Cammaroto and Noiri (1989), Ganster and Reilly (1989), Maheswari and Thakur (1985), Dontchev et al (1996), Balachandran et al (1996), Devi et al (1998) and Arokiarani (1997) introduced and studied quasi-irresolute and strongly irresolute maps, weak irresolute maps and $\theta$-irresolute maps, LC-irresolute maps, $\alpha$-irresolute maps, $\delta g$-irresolute maps, gc-irresolute maps, $\alpha g$-irresolute maps and
goα-irresolute maps and gr-irresolute maps respectively. In this review a brief survey of some of the articles published on the generalizations of irresolute functions is given.

1) **On α-continuous mappings:**
Maheswari and Thakur (1985)
The authors introduced and investigated the properties of α-irresolute maps.

2) **On Strongly α-irresolute mappings:**
Giovanni Lo Faro (1987)
The author introduced the concept of strongly α-irresolute mappings and presented some properties of such mappings.

3) **Weak and strong forms of irresolute functions**
Di Maio and Noiri (1988)
The authors introduced and investigated weak and strong forms of irresolute functions, namely, quasi-irresolute functions and strongly irresolute functions. It will be shown that quasi-irresoluteness and semi continuity are independent of each other. Further more quasi irresolute and strongly irresolute functions turn out to be the natural tool for studying g-closed spaces and semi-compact spaces. Also they studied several characterizations of quasi –irresolute functions. They investigated the relationships among some weak forms of continuity, irresoluteness, quasi irresoluteness and strong irresoluteness.

4) **Almost irresolute functions:**
Cammamroto and Noiri (1989)
The authors introduced and studied the notion of almost irresolute functions in topological spaces.
(5) On $\delta$-generalized closed sets and $T_{3\frac{1}{2}}$ spaces
Dontchev and Ganster (1996)
The authors introduced and studied the properties of $\delta$-irresolute maps

(6) On regular generalized continuous maps in topological spaces:
Arockiarani and Balachandran (1997)
The authors introduced and investigated regular generalized continuous maps and irresolute functions in a topological space.

(7) On $\mu \alpha$-irresolute maps, Where $\mu \in \{\text{Strongly, Strongly semi-almost}\}$
The authors introduced new classes of functions called strongly generalized $\rho$-irresolute functions where $\rho \in \{\alpha, \alpha^*, \alpha^{**}\}$, strongly semi generalized $\alpha$-irresolute functions and almost generalized irresolute functions in topological spaces. Also they obtained some characterizations of these functions and several properties and investigated the relationships with other classes of functions in topological spaces.

2.5 OPEN MAPS CLOSED MAPS

semi-generalized closed maps and generalized semi- closed maps, $\alpha$-open maps, preopen maps and weak preopen maps, $\delta$-open maps and $\delta$-closed maps, pre-semi-preopen maps, $\alpha g$-closed maps, $\delta g$-closed maps and $rg$-closed maps and $(\gamma; \beta)$-closed maps respectively. Some studies on open maps and closed maps are given below.

(1) On some mappings in topological spaces:
   Biswas N. (1969)
   The author defined new class of sets and new class of functions with the following two objectives in mind:
   (1) To extend the class of semi open sets and set-continuous functions introduced by Levine
   (2) To see whether the new class of functions include the class of semi continuous functions in the usual sense.

(2) Note on some applications of semi-open sets:
   Phullendu Das (1971)
   The author introduced the second semi –open sets and studied some of their properties.

(3) Remarks on semi open mappings:
   Noiri (1973)
   The author studied some properties of semi-open mappings. Also he studied the semi-open sets in subspaces and semi opens sets in product spaces.

(4) A generaliazation of closed mappings:
   Noiri (1973)
   The author introduced a new class of mappings called semi – closed mappings which contains the class of closed mapping and to gave some characterizations of such mappings
Generalied closed maps:
Malghan (1982)

The generalized closed sets or g-closed sets defined by N. Levine are further investigated. The generalized closed maps are defined and their properties are investigated. Among theorems proved are the following:

1. Normality is preserved under g-closed continuous surjections
2. Regularity is preserved under g-closed continuous open surjections
3. For any g-closed set A of a topological spaces X, $\text{Ind} A \leq \text{Ind} X$.

Generalized $\alpha$-closed maps and $\alpha$-generalized closed maps:
Devi et al (1997)

The authors introduced and investigated the concept of generalized $\alpha$-closed maps, $\alpha$-generalized closed maps and $\alpha$-regular spaces as a generalizations of closed maps, generalized close maps and regular spaces respectively. By introducing the concept of pre-$\alpha$-closed maps, it is proved that normality and $\alpha$-regularity are preserved under the maps.

On $\theta$-generalized closed sets:
Dontchev and Maki (1999)

The authors studied the class of $\theta$-generalized closed sets which is properly placed between the class of generalized closed and $\theta$ closed sets. Furthermore, generalized $A$-sets are extended to $\theta$-generalized $A$-sets and Ro-$T_{1/2}$ and $T_1$-spaces were characterized. The relations with other notions directly or indirectly connected with generalized closed sets were investigated. The notion of TGo-connectedness is introduced.
2.6 GENERALIZED HOMEOMORPHISMS


(1) Somewhat continuous functions:
Gentry and Hoyee (1971)
The authors introduced and studied the properties of somewhat continuous functions and somewhat homeomorphisms.

(2) On semi-homeomorphisms:
Bigniew Piotrowski (1979)
The author introduced and proved the following theorems
1) Let spaces \( X \) and \( Y \) be a regular and \( Y \) be a dense in itself. Let a functions \( f: X \rightarrow Y \) be closed. Then \( f \) is a homeomorphsim if and only if \( f \) is a semi
homeomorphism in the sense of Crossley and Hildebrand

2) Let a topological space $Y$ be regular and dense in itself. If $f : X \rightarrow Y$ is a semi-homomorphism in the sense of Crossley and Hildebrand then $f$ is a semi-homomorphism in the sense of Biswas.

(3) A note on Homeomorphic image of a $T^*$-space:
Jan–hi Umehara and Maki (1989)

The authors proved the following theorem

1) The image of a $T^*$-space under a homeomorphism is a $T^*$-space.

2) If the product space of two compact topological spaces is a $T^*$-space, then each factor space is a $T^*$-space.

(4) Operations and topological spaces and associated topology:
Ogata (1991)

The author introduced and studied the properties of $(\alpha, \beta)$-homeomorphic, irresolute and continuous functions.

(5) On Generalized homomorphism in topological spaces:
Maki et al (1996)

The authors defined the notion of generalized homeomorphism and gc-homeomorphism which are generalizations of homeomorphisms and they investigated some properties of generalized homeomorphisms and gc-homeomorphism from the quotient space to other space.
Semi–generalized Homeomorphisms and generalized semi-homeomorphism in topological spaces:


The authors introduced two new class of mappings namely generalized semi homeomorphism and semi generalized homeomorphism and investigated the group structure of their subgroup. Moreover some properties of the mappings from the quotient space to other spaces are investigated.

2.7 MINIMAL STRUCTURE

Levine (1970) introduced the notion of generalized closed set in topological spaces and showed that compactness, locally compactness, countably compactness paracompactness and normality etc are g-closed hereditary. And he also introduced a separation axiom called $T_{1/2}$ between $T_1$ and $T_0$. Recently many modifications of g-closed sets were defined and investigated. Many authors introduced several low separation axioms. Maki (1996) introduced minimal structures and minimal spaces. Some results about minimal spaces can be found in Casazar (2002), Lugojan (1982) Maki (1996), Noiri (2003) and Popa (2000). Under the same setting m-continuous functions and weak continuous functions are also defined by Popa. The following are some of the references on minimal structure.

(1) On m-continuous functions:

Popa and Noiri (2000)

The authors introduced a new notion of M-continuous function as a function defined between sets satisfying some minimal conditions. They obtained some characterizations and some properties of such functions. Moreover they define m-compactness and m-connectness and investigated their properties.
(2) A unified theory of weak continuity for functions:
   Popa and Noiri (2002)
   The authors introduced a new notion of weakly M-continuous functions as a function from a set satisfying some minimal conditions. Also they obtained some characterizations and several properties of such functions. New form of weak M-continuity is also introduced.

(3) On $A_m$-sets and related spaces:
   Noiri (2003)
   The author introduced and studied $A_m$-sets and $(A,m)$ closed sets. Also he studied $(A,m)$ continuous functions.

(4) On Almost $m$-continuous multifunctions:
   Noiri and Popa (2005)
   The authors introduced and studied almost $m$-continuous multifunctions. They obtained several characterizations and properties concerning almost $m$-continuous multifunctions.

(5) A Unified theory of certain modifications of $g$-closed sets
   Fumie and Nobuyuki (2006)
   The author introduced and investigated the properties of $mg$-closed sets and locally $m$-closed sets and a new form of $mg$-closed sets.

(6) Minimal $\gamma$–closed sets:
   Nakaoka and Oda (2006)
   The authors introduced and studied the notion of $\gamma$-closed sets in minimal structure and studied some of their properties.
2.7 FUZZY TOPOLOGY SPACES


(1) Fuzzy topological product and quotient theorem:
   Wong (1974)
   The author introduced and studied the properties of product fuzzy topology and quotient fuzzy topology.

(2) On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity:
   Azad (1981)
   The author introduced and studied the properties of fuzzy semi-continuous, fuzzy almost continuous and fuzzy weakly continuous maps in fuzzy topological spaces.
(3) On Fuzzy topological spaces:
Malghan and Benchalli (1981)
The authors redefined the Hausdorff fuzzy topological spaces to weaker hypothesis. Regular fuzzy topological spaces are newly defined. These definitions help to characterize Hausdorff and regular topological spaces. Normal fuzzy topological spaces are further investigated. Fuzzy open and fuzzy closed maps are also studied. Also they studied countable compact and Lindelof property in fuzzy topological spaces.

(4) Fuzzy Closure spaces:
Mahshour and Ghanim (1985)
The authors explored fuzzy closure space. The notion of subspace sum and a product are extended in fuzzy closure space. The hereditary additivity and productivity behavior of compactness in fuzzy closure spaces were investigated and some weak forms of compactness and fuzzy continuous functions in fuzzy closure space were introduced.

(5) On Fuzzy strong semi continuity and fuzzy precontinuity:
Abdulla Bin Shahna (1991)
The author defined fuzzy $\alpha$-open ($\alpha$-closed) sets and fuzzy preopen (preclosed) sets. Also he introduced and made a preliminary study of fuzzy strongly semi continuous and fuzzy precontinuous mapping.

(6) Fuzzy preopen sets and fuzzy preseparation axioms:
Singal and Nitiprakash (1991)
The authors extended the notion of preopenness to fuzzy sets. Also they defined fuzzy preinteriors and preclosures and
investigated their properties. More attention is paid to the extension of separation notion to fuzzy topological spaces with the help of fuzzy preopen sets and obtained interesting characterization of fuzzy pre $T_0$, $T_1$ spaces. By means of counter examples they pointed out noncoincidence of different notions of preseparation.

(7) On some generalizations of fuzzy continuous functions:
Balasubramanian and Sundaram (1991)
The authors introduced the notion of generalized closed sets and generalized continuous maps in fuzzy topological spaces. Also they studied some basic properties of their fuzzy sets.

(8) Irresolute and almost open functions between fuzzy topological spaces:
Mukerjee and Sinha (1997)
The authors introduced and investigated irresolute and almost open sets in fuzzy topological spaces

(9) Degrees and fuzzy generalized closed sets:
Fukutake et al (2001)
The authors investigated fuzzy generalized closed sets, fuzzy generalized semi-closed sets and fuzzy semi generalized closed sets in Chang’s fuzzy topological spaces and characterized fuzzy $T_{1/2}$-spaces. Also the authors introduced and studied three kinds of fuzzy degrees of fuzzy sets. The fuzzy degree is used to characterize completely fuzzy generalized closed sets and fuzzy generalized semi-closed sets respectively.
(10) On generalizing fuzzy semi-open sets:

Harada et al (2005)

The authors introduced and studied the properties of generalized \((\varepsilon_X, \varepsilon^0_X)\)-semi open sets, fuzzy \((\varepsilon_X, \varepsilon^0_X)\)-semi closure operations and fuzzy \((\varepsilon_X, \varepsilon^0_X)\)-(\(F_Y, F^0_Y\))-irresolute functions and groups.

2.8 FUZZY MINIMAL SPACES

The concept of fuzzy sets was introduced by Zadeh (1965). Since then many authors have explosively developed the theory of fuzzy sets and their applications. Minimal structures and minimal space were introduced in Maki. Further results about minimal spaces can be found in Caszar (2002), Lugojan (1982), Maki (1996), Noiri (2003) and Popa (2000). Alimohammady (2006). Authors extended minimal structures to fuzzy minimal structures and established some results in this setting. They noted that fuzzy minimal structures may have very important applications in quantum particles physics, particularly in connection with string theory and \(\varepsilon^\infty\) theory El Naschie (1998, 2000). The detailed study on fuzzy minimal spaces is shown below.

1) Fuzzy minimal structure and fuzzy minimal vector spaces:

Alimohammady and Roohi (2006)

The authors introduced a fuzzy minimal space as a new generalization of the fuzzy topology. Moreover, they presented the continuity of fuzzy functions in this setting. Also they introduced and investigated the class of fuzzy minimal vector spaces as a generalization of the class of fuzzy topological vector spaces.
2) Fuzzy $U_m$–sets and fuzzy (u,m) continuous functions:
Alimohammady and Roohi (2006)
The authors introduced the notion of fuzzy $U_m$–sets and fuzzy (U,m)-open sets in fuzzy minimal spaces and derives some related basic results in this new setting are given. Further some types of fuzzy continuity for fuzzy functions are investigated.

3. Compactness in fuzzy minimal spaces:
Alimohammady and Roohi (2006)
The authors generalized the concept of fuzzy compactness. They introduced the concept of fuzzy (countably) compactness in fuzzy minimal spaces and some related basic results in these new setting. Further, some results on compactness for fuzzy topological spaces are obtained.

2.9 DEFINITIONS AND RESULTS ON TOPOLOGICAL SPACE

In this section we give the basic definition and results which are used for our study.

Definition 2.9.1: A subset $A$ of a space $(X,\tau)$ is called a preopen set (Mashour 1982) if $A \subseteq \text{int}(\text{cl}(A))$ and a preclosed set if $\text{cl}(\text{int}(A)) \subseteq A$.

Definition 2.9.2: A subset $A$ of a space $(X,\tau)$ is called a semi-open set (Levine 1963) if $A \subseteq \text{cl}(\text{int}(A))$ and a semi-closed set if $\text{int}(\text{cl}(A)) \subseteq A$.

Definition 2.9.3: A subset $A$ of a space $(X,\tau)$ is called a $\alpha$-open set (Njastad 1965) if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ and a $\alpha$-closed if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$. 
The pre-interior of a subset $A \subseteq X$ (briefly $\text{pint}(A)$) is the union of all preopen sets contained in $A$. The pre-closure (denoted by $\text{pcl}(A)$) is the intersection of all preclosed sets that contain $A$. Note that $\text{pcl}(A) = A \cup \text{cl}(\text{int}(A))$ and $\text{pint}(A)\subseteq A \cap \text{int}(\text{cl}(A))$ which is proved by Adrjevic (1986). The semi-closure and semi-interior of $A$ are analogously defined.

**Definition 2.9.4:** A subset $A$ of a space $(X, \tau)$ is called a generalized closed set (briefly g-closed) (Levine 1970) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

**Definition 2.9.5:** A subset $A$ of a space $(X, \tau)$ is called a semi-generalized closed set (briefly sg-closed) [Bhattacharyya(1987)] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open.

**Definition 2.9.6:** A subset $A$ of a space $(X, \tau)$ is called a generalized semi-closed set (briefly gs-closed) (Arya 1974) if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open,

**Definition 2.9.7:** A subset $A$ of a space $(X, \tau)$ is called a generalized pre-closed set (briefly gp-closed) (Maki 1996) if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

**Definition 2.9.8:** A subset $A$ of a space $(X, \tau)$ is called a generalized semi-preclosed set (briefly gsp-closed) (Dontchev 1995) if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

**Definition 2.9.9:** A space $(X, \tau)$ is called a submaximal space (Dontchev 1995) if every dense subset of $X$ is open in $X$. 


**Definition 2.9.10:** A space \((X, \tau)\) is called a door space (Dontchev 1995) if every subset of \(X\) is either open or closed in \(X\).

**Definition 2.9.11:** A space \((X, \tau)\) is called a \(T_{\frac{1}{2}}\) space (Dunham 1977) if every g-closed subset of \(X\) is closed in \(X\).

**Definition 2.9.12:** A space \((X, \tau)\) is called a semi-\(T_{\frac{1}{2}}\) space (Bhattacharyya 1987) if every sg-closed set of \(X\) is semi-closed in \(X\).

**Definition 2.9.13:** Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a map and \(\gamma\) and \(\beta\) be the operations defined on \(\tau\) and \(\sigma\) respectively. Then \(f\) is said to be semi-continuous (Levine 1963) if \(f^{-1}(V)\) is semi-closed in \(X\) for every closed set \(V\) of \(Y\).

**Definition 2.9.14:** Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a map and \(\gamma\) and \(\beta\) be the operations defined on \(\tau\) and \(\sigma\) respectively. Then \(f\) is said to be pre-continuous (Mashour(1982)) if \(f^{-1}(V)\) is pre-closed in \(X\) for every closed set \(V\) of \(Y\).

**Definition 2.9.15:** Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a map and \(\gamma\) and \(\beta\) be the operations defined on \(\tau\) and \(\sigma\) respectively. Then \(f\) is said to be g-continuous (Balachandran 1991) if \(f^{-1}(V)\) is g-closed in \(X\) for every closed set \(V\) of \(Y\).

**Definition 2.9.16:** Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a map and \(\gamma\) and \(\beta\) be the operations defined on \(\tau\) and \(\sigma\) respectively. Then \(f\) is said to be sg-continuous (Sundaram 1991) if \(f^{-1}(V)\) is sg-closed in \(X\) for every closed set \(V\) of \(Y\).

**Definition 2.9.17:** Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a map and \(\gamma\) and \(\beta\) be the operations defined on \(\tau\) and \(\sigma\) respectively. Then \(f\) is said to be \(\alpha\)-continuous (Mashour 1983) if \(f^{-1}(V)\) is \(\alpha\)-closed in \(X\) for every closed set \(V\) of \(Y\).
Definition 2.9.18: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a map and \( \gamma \) and \( \beta \) be the operations defined on \( \tau \) and \( \sigma \) respectively. Then \( f \) is said to be \((\gamma, \beta)\)-continuous (Ogata 1991) if for each point \( x \) of \( X \) and each open set \( V \) containing \( f(x) \) there exists an open set \( U \) such that \( x \in U \) and \( f(U) \subseteq V^\beta \).

Definition 2.9.19: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a map. \( f \) is said to be irresolute (Crossley 1972) if \( f^{-1}(V) \) is semi-open in \( X \) for every semi-open set \( V \) of \( Y \).

Definition 2.9.20: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a map. \( f \) is said to be pre-irresolute (Sen 1993) if \( f^{-1}(V) \) is preclosed in \( X \) for every preclosed set \( V \) of \( Y \).

Definition 2.9.21: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a map. \( f \) is said to be sg-irresolute (Sundaram 1991) if \( f^{-1}(V) \) is sg-closed in \( X \) for every sg-closed set \( V \) of \( Y \).

Definition 2.9.22: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a map. \( f \) is said to be gc-irresolute (Balachandran 1996) if \( f^{-1}(V) \) is g-closed in \( X \) for every g-closed set \( V \) of \( Y \).

Definition 2.9.23: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a map. \( f \) is said to be \( \alpha \)-irresolute (Maheswari 1985) if \( f^{-1}(V) \) is \( \alpha \)-closed in \( X \) for every \( \alpha \)-closed set \( V \) of \( Y \).

Definition 2.9.24: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a map. Then \( f \) is said to be semi-closed (Noiri 1973) if \( f(V) \) is semi-closed in \( Y \) for every closed set \( V \) of \( X \).

Definition 2.9.25: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a map. Then \( f \) is said to be preclosed (Sen 1993) if \( f(V) \) is preclosed in \( Y \) for every closed set \( V \) of \( X \).
**Definition 2.9.26**: Let \( f:(X,\tau) \rightarrow (Y,\sigma) \) be a map. Then \( f \) is said to be \( g \)-closed (Malghan 1982) if \( f(V) \) is \( g \)-closed in \( Y \) for every closed set \( V \) of \( X \).

**Definition 2.9.27**: Let \( f:(X,\tau) \rightarrow (Y,\sigma) \) be a map. Then \( f \) is said to be \( \alpha \)-open (Mashour 1983) if \( f(V) \) is \( \alpha \)-open in \( Y \) for every open set \( V \) of \( X \).

**Definition 2.9.28**: Let \( f:(X,\tau) \rightarrow (Y,\sigma) \) be a map. Then \( f \) is said to be \( (\gamma,\beta) \)-closed (Ogata(1991)) if for any \( \gamma \)-closed set \( A \) of \( X \), \( f(A) \) is a \( \beta \)-closed set of \( Y \).

**Definition 2.9.29** (Biswa 1969): A map \( f:(X,\tau) \rightarrow (Y,\sigma) \) is said to be a semi-homeomorphism (B) if \( f \) is continuous, semi-open, bijective.

**Definition 2.9.30** (Crossley 1972): A map \( f:(X,\tau) \rightarrow (Y,\sigma) \) is said to be a semi-homeomorphism (C.H) if \( f \) is irresolute, pre-semi-open and bijective.

**Definition 2.9.31** (Maki 1991): A map \( f:(X,\tau) \rightarrow (Y,\sigma) \) is said to be a generalized homeomorphism if \( f \) is \( g \)-continuous, \( g \)-open and bijective.

**Definition 2.9.32** (Maki 1991): A map \( f:(X,\tau) \rightarrow (Y,\sigma) \) is said to be a gc-homeomorphism if \( f \) is bijective, gc-irresolute and its inverse is also gc-irresolute.

**Definition 2.9.33** (Ogata 1991): A map \( f:(X,\tau) \rightarrow (Y,\sigma) \) is said to be a \( (\gamma,\beta) \)-homeomorphic, if \( f \) is bijective, \( (\gamma,\beta) \)-continuous and \( f^{-1} \) is \( (\beta,\gamma) \)-continuous.

**Fuzzy sets 2.9.34** Let \( X \) be a non-empty set. A fuzzy set \( \alpha \) in \( X \) is a function from \( X \) into the closed unit interval \([0, 1]\). We write \( \alpha \leq \beta \) if
\( \alpha(x) \leq \beta(x) \) for all \( x \in X \). By \( \alpha = \beta \), we mean \( \alpha \leq \beta \) and \( \beta \leq \alpha \). If \( \{ \alpha_i : i \in I \} \) is a collection of fuzzy sets in \( X \), then \( \cup \alpha_i \) and \( \cap \alpha_i \) are given by

\[
(\cup \alpha_i)(x) = \sup \{ \alpha_i(x) : i \in I \} \text{ for all } x \in X \text{ and }
\]
\[
(\cap \alpha_i)(x) = \inf \{ \alpha_i(x) : i \in I \} \text{ for all } x \in X.
\]

The complement \( \alpha^c \) of fuzzy set \( \alpha \) in \( X \) is given by \( \alpha^c(x) = 1 - \alpha(x) \) for all \( x \in X \). The fuzzy sets 0 and 1 are given by \( 0(x) = 0 \) and \( 1(x) = 1 \) for all \( x \in X \).

**Definition 2.9.35: (Chang 1968)** A fuzzy topology on a set \( X \) is a collection \( \tau \) of fuzzy sets such that

a) \( 0, 1 \in \tau \)

b) If \( \alpha, \beta \in \tau \) then \( \alpha \cap \beta \in \tau \)

c) If \( \alpha_i \in \tau \), then \( \cup \alpha_i \in \tau \)

Members of \( \tau \) are called fuzzy open sets and the pair \((X, \tau)\) is called a fuzzy topological space. Complements of fuzzy open sets are called fuzzy closed sets. The closure denoted by \( \text{cl}(\alpha) \) and interior denoted by \( \text{int}(\alpha) \) of a fuzzy set \( \alpha \) in a fuzzy topological space are given by \( \text{cl}(\alpha) = \cap \{ \beta : \beta \text{ is a fuzzy closed } \alpha \leq \beta \} \) and \( \text{int}(\alpha) = \cup \{ \beta : \beta \text{ is a fuzzy open set and } \beta \leq \alpha \} \).

We note the \( \text{cl}(\alpha) \) is the smallest fuzzy closed set containing \( \alpha \) and \( \text{int}(\alpha) \) is the largest fuzzy open set contained in \( \alpha \). Also, \( \text{cl}(\alpha \cup \beta) = \text{cl}(\alpha) \cup \text{cl}(\beta), \text{int}(1- \alpha) = 1 - \text{cl}(\alpha) \) and \( \text{cl}(1- \alpha) = \text{cl}(1 - \alpha) \). We denote a “fuzzy topological space by “fts”.
Definition 2.9.36 (Azad 1981) A fuzzy set $A$ in a fts $X$ is said to be

(a) fuzzy semi-open if and only if there exists a fuzzy open set $V$ in $X$ such that $V \subseteq A \subseteq \text{cl}(V),$

(b) fuzzy semi-closed if and only if there exists a fuzzy closed set $V$ in $X$ such that $\text{int}(V) \subseteq A \subseteq V.$

It is seen that a fuzzy set $A$ is fuzzy semi-open if and only if $A^c$ is fuzzy semi closed.

Definition 2.9.37 (Azad(1981)) A fuzzy set $\lambda$ of a fts $X$ is called

(i) a fuzzy regular open set of $X$ if $\text{int}\text{cl}(\lambda)) = \lambda.$

(ii) a fuzzy regular closed set of $X$ if $\text{cl}(\text{int}(\lambda)).$

Definition 2.9.38 (Chang 1968) Let $f$ be the function from a fts $Y$ and $\alpha$ be a fuzzy set in $T.$ Then the inverse image of $\alpha,$ written as $f^{-1}(\alpha),$ is a fuzzy set in $X$ whose membership function is given by $f^{-1}(\alpha)(x) = f(\alpha(x))$ for all $x$ in $X.$ Conversely let $\beta$ be a fuzzy set in $X.$ Then the image of $\beta$ written as $f(\beta),$ is a fuzzy set in $Y$ whose membership function is given by

$$F(\beta) = \left\{ \begin{array}{ll}
\sup \{ \beta(z) : z \in f^{-1}(y) \text{ if } f^{-1}(y) \neq \emptyset \\
0 \text{ otherwise,}
\end{array} \right. $$

where $f^{-1}(y) = \{ x \in X : f(x) = y \}.$

Definition 2.9.39 (Bin Shahana 1991) Let $X$ be a fts. Then a fuzzy set $\lambda$ in $X$ is called a fuzzy $\alpha$-open set if $\lambda \subseteq \text{int}(\text{cl}(\text{int}(\lambda))).$

Definition 2.9.40 (Bin Shahana 1991) Let $X$ be a fts. Then a fuzzy set $\lambda$ in $X$ is called a fuzzy $\alpha$-closed set if $\text{cl}(\text{int}(\text{cl}(\lambda))) \subseteq \lambda.$
Definition 2.9.41. Let $X$ be a fts. Then a fuzzy set $\lambda$ in $X$ is called a fuzzy pre-closed set (Singal 1991) if $\text{cl}(\text{int}(\lambda)) \leq \lambda$.

Definition 2.9.42 (Balasubramanian 1997) Let $X$ be a fts. A fuzzy set $\lambda$ in $X$ is called generalized fuzzy closed (gfc) if and only if $\text{cl}(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and $\mu$ is fuzzy open. A fuzzy set $\lambda$ is called generalized fuzzy open (gfo) if and only if $1 - \lambda$ is gfc.

Definition 2.9.43 (Chang 1968) A mapping $f:X \rightarrow Y$ from a fts $X$ to a fts $Y$ is said to be a fuzzy continuous mapping if $f^{-1}(\lambda)$ is fuzzy open in $X$ for each fuzzy open set $\lambda$ in $Y$.

Definition 2.9.44 (Chang 1968) A mapping $f:X \rightarrow Y$ from a fts $X$ to a fts $Y$ is said to be

(i) a fuzzy open mapping if $f(\lambda)$ is fuzzy open in $Y$ for each fuzzy open set $\lambda$ in $X$ and

(ii) a fuzzy closed mapping if $f(\mu)$ is fuzzy closed set of $Y$, for each fuzzy closed set $\mu$ of $X$.

Definition 2.9.45 (Azad 1981) A mapping $f:X \rightarrow Y$ from a fts $X$ to a fts $Y$ is said to be

(a) fuzzy semi-open (semi-closed) if and only if $f(A)$ is fuzzy semi-open (semi-closed) in $Y$ for each fuzzy open (closed) set $A$ in $X$.

(b) fuzzy semi-continuous if and only if $f^{-1}(B)$ is fuzzy semi-open in $X$, for each fuzzy open set $B$ in $Y$. 
Definition 2.9.46 Azad 1981) A mapping \( f: X \rightarrow Y \) from a fts X to a fts Y is said to be a fuzzy almost continuous mapping if \( f^{-1}(\lambda) \) is a fuzzy semi-open set B in Y.

Definition 2.9.47 (Mukerjee 1989) A mapping \( f: X \rightarrow Y \) is said to be fuzzy irresolute if and only if \( f^{-1}(B) \) is fuzzy semi-open in X, for any fuzzy semi-open set B in Y.

Definition 2.9.48 (Azad 1981) A mapping \( f: X \rightarrow Y \) is said to be almost fuzzy open if and only if the image of every fuzzy regularly open set in X is fuzzy open in Y.

Definition 2.9.49 (Balasubramanian 1991) A mapping \( f: X \rightarrow Y \) from a fts X to a fts Y is said to be generalized fuzzy continuous (gf-continuous) if the inverse image of every fuzzy closed set in Y is gf-closed in X.

Definition 2.9.50 (Balasubramanian 1997) A mapping \( f: X \rightarrow Y \) from a fts X to a fts Y is said to be fuzzy gc-irresolute if the inverse image of every gf-closed set in Y is gf-closed in X.

Definition 2.9.51 (Balasubramanian 1997) A mapping \( f: X \rightarrow Y \) from a fts X to a fts Y is said to be strongly fuzzy continuous if \( f^{-1}(\lambda) \) is both fuzzy open and fuzzy closed for each fuzzy set \( \lambda \) in Y.

Definition 2.9.52 (Balasubramanian 1997) A mapping \( f: X \rightarrow Y \) from a fts Y is said to be perfectly fuzzy continuous if \( f^{-1}(\lambda) \) is both fuzzy open and fuzzy closed for each fuzzy open set \( \lambda \) in Y.

Definition 2.9.53 (Bin Shahana 1991) A mapping \( f: X \rightarrow Y \) from a fts X to a fts Y is said to be fuzzy strongly semi continuous if \( f^{-1}(\lambda) \) is fuzzy \( \alpha \)-open in X for each fuzzy open set \( \lambda \) in Y.
**Definition 2.9.54 (Bin Shahana 1991)** A mapping $f: X \rightarrow Y$ from a fts $X$ to a fts $Y$ is said to be fuzzy pre-continuous if $f^{-1}(\lambda)$ is fuzzy pre-open in $X$ for each fuzzy open set $\lambda$ in $Y$.

**Definition 2.9.55 (Ferraro 1987)** A bijection $f: X \rightarrow Y$ from a fts $X$ to a fts $Y$ is said to be fuzzy homeomorphism if and only if $f$ and $f^{-1}$ are fuzzy continuous.

**Definition 2.9.56 (Noiri 2000)** Let $X$ be a nonempty set and $P(X)$ the power set of $X$. A subfamily $m_x$ of $P(X)$ is called a *minimal structure (briefly m-structure)* on $X$ [28] if $\emptyset \in m_x$ and $X \in m_x$.

A set $X$ with an $m$-structure $m_x$ is called an $m$-space and is denoted by $(X, m_x)$. Each member of $m_x$ is said to be an $m_x$-open.

**Definition 2.9.57 (Maki 1996)**: Let $X$ be a nonempty set and $m_x$ an $m$-structure on $X$. For a subset $A$ of $X$, the $m_x$-closer of $A$ and the $m_x$-interior of $A$ are defined as follows:

1. $m_x$-$\text{Cl}(A) = \{ F: A \subseteq f, X - f \in m_x \}$,
2. $m_x$-$\text{Int}(A) = \cap \{ U: U \subseteq m_x \}$.

**Remark 2.9.58 (Noiri 2000)**: Let $(X, m_x)$ be a topological space and $A$ a subset of $X$. If $m_x = \tau$ (resp. $SO(X)$, $PO(X)$, $a(X)$, $SPO(X)$, $B(O(X))$), then we have

1. $m_x$-$\text{Cl}(A) = \text{Cl}(A)$ (resp. $s\text{CL}(A)$, $p\text{Cl}(A)$, $a\text{Cl}(A)$, $sp\text{Cl}(A)$, $b\text{Cl}(A)$),
2. $m_x$-$\text{Int}(A) = \text{Int}(A)$ (resp. $s\text{Int}(A)$, $p\text{Int}(A)$, $a\text{Int}(A)$, $sp\text{Int}(A)$, $b\text{Int}(A)$),
Lemma 2.9.59 (Maki 1996): Let \( X \) be a nonempty set and \( m_x \), a minimal structure on \( X \). For subsets \( A \) and \( B \) of \( X \), the following properties hold:

1. \( m_x\text{-Cl}(X-A)=X- m_x\text{-Int}(A) \) and \( m_x\text{-}(X-A)=X- m_x\text{-Cl}(A) \),

2. if \( (x-A) \subseteq m_x \), then \( m_x\text{-Cl}(A)=A \) and if \( A \subseteq m_x \), then \( m_x\text{-Int}(A)=A \),

3. \( m_x\text{-Cl}(\emptyset)=\emptyset, m_x\text{-Cl}(X)=X, m_x\text{-Int}(\emptyset)=\emptyset \) and \( m_x\text{-Int}(X)=X \),

4. if \( A \subseteq B \), then \( m_x\text{-Cl}(A) \subseteq m_x\text{-Cl}(B) \) and \( m_x\text{-Int}(A) \subseteq m_x\text{-Int}(B) \),

5. \( A \subseteq m_x\text{-Cl}(A) \) and \( m_x\text{-Int}(A) \subseteq A \),

6. \( m_x\text{-Cl}(m_x\text{-Cl}(A))= m_x\text{-Cl}(A) \) and \( m_x\text{-Int}(m_x\text{-Int}(A))= m_x\text{-Int}(A) \).

Definition 2.9.60 (Alimohammady 2006): We say a family \( \mu \) of fuzzy sets on \( X \) is a fuzzy minimal structure on \( X \) if \( \alpha I_X \in \mu \) for any \( \alpha \in I \). In this case we consider \( (X,\mu) \) as a fuzzy minimal space.

Definition 2.9.61 (Alimohammady 2006): We say \( A \in I^X \) is a fuzzy m-open set if \( A \in \mu \) and \( B \in I^X \) is a fuzzy m-closed set if \( B^c \in \mu \). We set \( m\text{-int}(A)=\bigvee \{ U: U \subseteq A, U \in \mu \} \), \( m\text{-cl}(A)=\{ F: A \subseteq F, F \in \mu \} \).

Definition 2.9.62 (Alimohammady 2006): Let \( (X,M) \) and \( (Y,N) \) be two fuzzy minimal spaces. We say that a fuzzy function \( f: (X,M) \to (Y,N) \) is fuzzy minimal continuous (Briefly fuzzy m-continuous) if \( f^1(B) \in \mu \), for any \( B \in N \).
2.10 CONTRIBUTIONS OF THE AUTHOR

In the light of the above work, the author has obtained some interesting generalizations on the following topics.

(1) Further study on weakly generalized closed map.

(2) On mg-continuous function, irresolute and homeomorphism in minimal structure.

(3) Weakly generalized closed set in minimal structure.

(4) Fuzzy minimal generalized closed sets.

(5) Fuzzy minimal weakly generalized continuous map, irresolute and homeomorphism.


The rest of the thesis is the detailed study of the above topics.