8.1 INTRODUCTION


In this chapter we introduce the new class of wg-closed sets in digital line and digital planes. Further we study some of its basic properties.

8.2 wg-CLOSED SET IN DIGITAL LINE

In this section, we study some properties of wg-closed sets in digital line.

Theorem 8.2.1: A function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is wg-continuous if and only if

1. $f$ is Lip -1.
2. for all even $x$, $f(x) \sim x \Rightarrow f(x \pm 1) = f(x)$. 
**Proof:** These conditions are necessary. For the converse, let $A = \{y-1, y, y+1\}$ where $y$ is even be any sub-base element. We must show that $f^{-1}(A)$ is open. If $x \in f^{-1}(A)$ is odd, then $\{x\}$ is a neighborhood of $x$. If $x$ is even, then we have two cases. First, if $f(x)$ is odd, then condition (2) $f(x \pm 1) = f(x)$ so that $\{x-1, x, x+1\} \subseteq f^{-1}(A)$ is a neighborhood of $x$. Second, if $f(x)$ is even, then $f(x)$ is odd, and the Lip -1 condition implies $|f(x \pm 1) - y| \leq 1$ so that again $\{x-1, x, x+1\} \subseteq f^{-1}(A)$ is a neighborhood of $x$. Thus $f$ is wg-continuous.

**Proposition 8.2.2:** A wg-continuous function $f : Z^n \to Z$ is Lip -1 with respect to the $l'$ metric.

**Proof:** We use induction over the dimension. Assume that the statement holds in $Z^{n-1}$. Let $f : Z^n \to Z$ be wg-continuous, $x' \in Z^{n-1}$, $x_n \in Z$ and $x = (x', x_n) \in Z^n$, Assume that $f(x) = 0$. We consider the cases $x_n$ being odd and $x_n$ even. If $x_n$ is odd, then $f(x+(0, 0, ..., 1)) = 0$. and by induction hypothesis $f(x+(1, 1, 1, ..., 1)) \leq 1$. On the other hand, it is always true, by induction hypothesis that $f(x+(1, 1, 1, ..., 1, 0)) \leq 1$. If $x_n$ is even, $f(x+(1, 1, 1, ..., 0)) = 1$. Then also $f(x+(1, 1, 1, ..., 0)) = 1$. This shows that $f$ can increase at most 1 if we take a step in every co-ordinate direction and by a trivial modification of the argument, also if we step only in some directions. By a similar argument, we can get a lower bound and hence $f$ is Lip -1.

**Theorem 8.2.3:** $f : Z^n \to Z$ be wg-continuous if and only if $f$ is separately wg- continuous.

**Proof:** The only if part is a general topological property. For the other direction, it suffices to verify that the inverse image of a sub-basis element, $A = \{y-1, y, y+1\}$ where $y$ is even, is open. Suppose that $x \in f^{-1}(A)$. We show that $N(x) \subseteq f^{-1}(A)$. It is easy to see that
\[ N(X) = \begin{cases} 
\mathbb{Z} \in \mathbb{Z}^n |x_i - z_i| \leq 1 & \text{if } x_i \text{ is even} \\
z_i = x_i & \text{if } x_i \text{ is odd.}
\end{cases} \]

Let \( Z \in N(x) \) and \( I = \{i_0, i_1, \ldots, i_k\} \) be the indices for which \(|x_i - z_i| = 1\).

Let \( x_0, x_1, \ldots, x_k \) be the sequence of points in \( \mathbb{Z}^n \) such that \( x_0 = x, x_k = z \) and \( x^{j+1} = x^j + (0, 0, \ldots, \pm 1, 0, 0, \ldots) \) for \( j = 0, 0, \ldots, k-1 \) so that \( x^{j+1} \) is one step closer to \( z \) than \( x^j \) in the \( i_j \) co-ordinate direction. Now, if \( f(x) \) is odd, then by separate continuity and above theorem, it follows that \( f(x^{j+1}) = f(x^j) \). In particular, \( f(z) = f(x) \) and hence \( z \in f^{-1}(A) \). If \( f(x) \) is even, it may happen that \( f(x^{j+1}) = f(x^j) + 1 \) for some index \( j \). But then \( f(x^{j+1}) \) is odd, and must be constant on the remaining elements of the sequence. Therefore \( f(z) \in A \), and so \( z \in f^{-1}(A) \).

**Definition 8.2.4:** Let \( A \subseteq \mathbb{Z} \). A gap of \( A \) is an ordered pair of integers \((p, q) \in \mathbb{Z} \times \mathbb{Z} \) such that \( q \geq p + 2 \) and \([p, q] \cap A = \{p, q\}\).

**Proposition 8.2.5:** Let \( A \subseteq \mathbb{Z} \) and \( f : A \rightarrow \mathbb{Z} \) be wg-continuous. Then \( f \) has a wg-continuous extension if and only if for every gap \((p, q)\) of \( A \), one of the following conditions holds:

(i) \(|f(q) - f(p)| < q - p|.

(ii) \(|f(q) - f(p)| = q - p| \text{ and } p \sim f(p)|.

**Proof:** There are two possibilities for a point \( x \) not in \( A \). Either \( x \) is in the gap of \( A \) or it is not. In the latter case, one of \( x > a \) or \( x < a \) holds for every \( a \in A \). Let \((p, q)\) be any gap. We try to extend \( f \) to a function \( g \) that is defined also on the gap. It is clear that the function can jump at most one step at the time. If \( p \sim f(p) \), then it must remain constant in the first step \( g(p + 1) = f(p) \), so \( p \sim f(p) \) is clearly necessary when \(|f(q) - f(p)| = q - p| \). It is also sufficient since the condition implies \( q \sim f(q) \). If \(|f(q) - f(p)| \leq q - p| \). It does
not matter whether \( p \sim f(p) \) the function can always be extended. If 
\[ |f(q) - f(p)| < q - p - 1 \]
then let \( p_2 = q - 1 - |f(q) - f(p)| \) and define \( g(i) = f(p) \) for \( i = p + 1, p + 2, \ldots, p + 2 \). Thus we consider the pair \((p_2, q)\) where 
\[ |g(q) - g(p_2)| = q - p_2 - 1. \]
If \( g(p_2) \sim p_2 \), then define \( g(p_2 + 1) = g(p_2) \) so that 
\( (p_2 + 1) \sim g(p_2 + 1) \) and we are in the situation described in condition (ii).

Similarly for the case \( f(q) \sim q \). Finally, if \( |f(q) - f(p)| > q - p \), the function is not globally Lip -1 and thus, cannot be extended. If there is a largest element \( a \) in \( A \), then \( f \) can always be extended for all \( x > a \) by \( g(x) = f(a) \) and similarly if there is a smallest element in \( A \), since every possibility for on \( x \in A \) is now covered.

**Definition 8.2.6:** Let \( A \subseteq \mathbb{Z}^n \) and \( f: A \rightarrow Z \) be wg-continuous. Let \( x \) and \( y \) be two distinct points in \( A \). If one of the following conditions is fulfilled for some \( i = 1, 2, \ldots, n \) 
1. \( |f(x) - f(y)| < |x_i - y_i| \)
2. \( f(x) = f(y) = |x_i - y_i| \)
and \( x_i \sim f(x_i) \), then we say that the function is strongly Lip -1 with respect to (the points) \( x \) and \( y \). If the function is strongly Lip -1 with respect to every pair of distinct points in \( A \), then we simply say that \( f \) is strongly Lip -1.

**Theorem 8.2.7:** If \( f: \mathbb{Z}^n \rightarrow Z \) is wg-continuous, then \( f \) is strongly Lip -1.

**Proof:** Suppose that \( f \) is not strongly Lip -1. Then there are distinct points \( x \) and \( y \) in \( \mathbb{Z}^n \) such that \( f \) is not strongly Lip -1 with respect to \( x \) and \( y \). Define \( d \) by 
\[ d = |f(x) - f(y)| \]
since \( x \neq y \), it is clear that \( d > 0 \). Let \( J \) be an enumeration of the (finite) set of indices for which \( |x_i - y_i| = d \) but \( x_i \sim f(x) \).

Let \( k = |J| \). Define \( x^0 = x \) and for each \( i_j \in J, j = 1, 2, \ldots, k \), let \( x^j \in \mathbb{Z}^n \) be a point one step closer to \( y \) in the \( i_j \) the co-ordinate direction 
\[ x^{j+1} = x^j + (0, 0, \ldots, 0, \pm 1, 0, 0\ldots) \]
where the co-ordinate with \( \pm 1 \) is determined by \( i_j \) and the sign by the direction toward \( y \). If \( J \) happens to be empty, only \( x^0 \) is defined of course. Now we note that, for all \( j = 1, 2, \ldots, k \),
\( f(x^{i+1}) = f(x^i) \). This is because \( f(x^i) \approx x^i_{ij} \) by construction and since \( f \) is necessary by separately continuous. Thus \( f(x^i) = f(x) \). Also, for all \( i = 1, 2...n \) it is true that \( |x^i_k - y^i| < \delta = 0 \) that \( |f(x^i) - f(y)| \). This contradict the fact that \( f \) must be Lip-1 for \( l' \) the metric (Theorem 7.2.2). Therefore \( f \) is strongly Lip-1.

**Lemma 8.2.8:** Let \( x \) and \( y \) in \( Z^n \) be two distinct points and \( f: \{x, y\} \rightarrow Z \) be a function that is strongly Lip-1. Then it is possible, for any point \( p \in \mathbb{Z}^n \) to extend the function to \( F: \{x, y, p\} \rightarrow Z \) so that \( F \) is still strongly Lip-1.

**Proof** Let \( i \) be the index of a co-ordinate for which one of the conditions in the definition of strongly Lip-1 functions is fulfilled. Then there is a wg-continuous function \( g: Z \rightarrow Z \) such that \( g(x_i) = f(x) \) and \( g(y_i) = f(y) \) by theorem 8.2.9. Define \( h: \mathbb{Z}^n \rightarrow Z \) by \( h(z) = g(z_i) \), obviously \( h \) satisfies the strongly Lip-1 condition in the \( i \)th co-ordinate direction for any pair of points and therefore \( h \) is strongly Lip-1. By construction, \( h(x) = g(x_i) = f(x) \) and similarly \( h(y) = f(y) \). The restriction of \( h \) to \( \{x, y, p\} \) is the desired function.

**Theorem 8.2.9:** Suppose \( A \subset \mathbb{Z}^n \) and \( f: A \rightarrow Z \) is strongly Lip-1. Then \( f \) can be extended to all of \( \mathbb{Z}^n \) so that the extended function is still strongly Lip-1.

**Proof:** If \( A \) is the empty set or \( A \) is all of \( \mathbb{Z}^n \), then the Lemma 7.2.8 is trivially true, so we need not consider these cases further. First we show that for any point where \( f \) is not defined we can define it so that the new function is still strongly Lip-1.

To this end, let \( p \) be any point in \( \mathbb{Z}^n \) not in \( A \). For every \( x \in A \), it is possible to extend \( f \) to \( f_x \) defined on \( A \cup \{p\} \) so that the new function is
strongly Lip-1 with respect to \(x\) and \(p\), for example, by letting \(f^x (p) = f(x)\). It is also clear that there is a minimal (say \(a^x\)) and a maximal (say \(b^x\)) value that \(f^x (p)\) can attain if it is still to be strongly Lip-1 with respect to \(x\). It is obvious that \(f^x (p)\) may also attain every value in between \(a^x\) and \(b^x\). Thus the set of possible values is in fact an interval \([a^x, b^x]\) \(\cap Z\). Now define \(R = \cap \{x \in A | [a^x, b^x]\} \cap Z\).

If \(R=\emptyset\), then there is an \(x\) and a \(y\) such that \(b^x < a^y\). This means that it is impossible to extend \(f\) at \(p\) so that it is strongly Lip-1 with respect to both \(x\) and \(y\). But this cannot happen according to Theorem 7.2.10. Therefore \(R\) cannot be empty. Define \(\check{f}(p)\) to be say the smallest integer in \(R\) and \(f(x) = f(x)\) if \(x \in A\). Then \(f : A \cup p \rightarrow Z\) is still strongly Lip-1.

**Theorem 8.2.10:** Suppose \(A \subset Z^n\) and that \(f : A \rightarrow Z\) is strongly Lip-1. Then \(f\) is wg-continuous.

**Proof:** Since we can always extend \(f\) to all of \(Z^n\) by Theorem 7.2.11 and the restriction of a wg-continuous function is wg-continuous, it is sufficient to consider the case \(A = Z^n\). But it is clear, from the definition of strongly Lip-1 function and in view of Theorem 8.2.1, that such a function is separately wg-continuous, thus by Theorem 8.2.5 \(f\) is wg-continuous.

**Theorem 8.2.11:** Let \(A \subset Z^n\) and \(f : A \rightarrow Z\) be any function. Then \(f\) can be extended to a wg-continuous function on all of \(Z^n\) if and only if \(f\) is strongly Lip-1.

**Proof:** That it is necessary that the function is strongly Lip-1 follows from Theorem 8.2.6. For the converse first use the Theorem 8.2.8 to find a strongly Lip-1 extension to all of \(Z^n\) and then Theorem 8.2.10 to conclude that this extension is in fact wg-continuous.
8.3  \textbf{wg-CLOSED SET IN THE DIGITAL PLANE}

In this section we introduce and study some properties in \(wg\)-closed set in digital plane.

We begin recalling the definition and properties of the digital line. The Khalimsky line or so called the digital line is the set of the integers \(Z\) equipped with the topology \(k\) having \(\{\{2n - 1, 2n, 2n + 1\} \mid n \in Z\}\) as a sub base. It is denoted by \((Z, k)\). Then, a set \(U\) is open in \((Z, k)\) if and only if whenever \(x \in U\) and \(x\) is an even integer, then \(x - 1, x + 1 \in U\). Clearly, a singleton \(\{2s+1\}\) is open and a subset \(\{2m - 1, 2m, 2m + 1\}\) is the smallest open set containing \(\{2m\}\) where \(s\) and \(m\) are any integers. Thus, the digital line \((Z, k)\) is a typical example of the \(T_{\frac{1}{2}}\) space which is not \(T_1\). A space \((X, \tau)\) is \(T_{\frac{1}{2}}\) if and only if every singleton \(\{x\}\) is open or closed where \(x \in X\). Let \((Z^2, k^2)\) be the topological product of two digital lines \((Z, k)\) where \(Z^2 = Z \times Z\) and \(k^2 = k \times k\). This space \((Z^2, k^2)\) is called the digital plane. This space \((Z^2, k^2)\) is a mathematical model of the computer screen. By the product topology in \((Z^2, k^2)\) every singleton \(\{(2n, 2m)\}\) is closed and every singleton \(\{(2s + 1, 2k + 1)\}\) is open where \(n, m, s, k \in Z\). It is noted that singletons \(\{(2n, 2m + 1)\}, \{(2s + 1, 2k)\}\) are neither open nor closed where \(n, m, s, k \in Z\) (they are called mixed points). For a pure point \(x = (2n + 1, 2m + 1),\) \(n, m \in Z\) there exists the smallest open set \(U(x) = \{(2n + 1) \times \{2m+1\}\}\) containing \(x\). For a pure point \(x = (2n, 2m)\) \(n, m \in Z\) there exists the smallest open set \(U(x) = \{2n - 1, 2n, 2n + 1\} \times \{2m-1, 2m, 2m+1\}\) containing \(x\). And, for a mixed point \(x = (2n, 2m + 1)\) (respectively \(x = (2n + 1, 2m)\)) \(n, m, \in Z\). There exist the smallest open set \(U(x) = \{2n - 1, 2n, 2n + 1\} \times \{2m+1\}\) (respectively \(U(x) = \{2n + 1\} \times \{2m - 1, 2m, 2m + 1\}\)) containing \(x\).
Theorem 8.3.1: In digital plane \((Z^2, k^2)\) every preopen set is semiopen.

Proof: Let \(A\) be a pre-open set in \((Z^2, k^2)\). Let \(x\) be any point such that \(x \in \text{cl}(\text{int}(A))\). Then, there exists, the smallest open set \(U(x)\) containing \(x\) such that \(U(x) \cap \text{int}(A) = \emptyset\). We claim that \(x \notin A\). We have the following three cases.

Case I: \(x\) is a pure point such that \(x = (2^n + 1, 2^m + 1)\) for some integers \(n\) and \(m\). Then \(U(x) = \{x\}\) and \(x \notin \text{int}(A)\) hold. Suppose that \(x \in A\). Since \(\{x\}\) is an open set containing \(x\), \(x \in \text{int}(A)\). This is a contradiction.

Case II: \(x\) is a pure point such that \(x = (2^n, 2^m)\) for some integers \(n\) and \(m\). Then, the smallest open set \(U(x) = U^1(2n) \times U^1(2m)\), where \(U^1(2n) = \{2n - 1, 2n, 2n + 1\}\) and \(U^1(2m) = \{2m - 1, 2m, 2m + 1\}\) and \(U(x) \cap \text{int}(A) = \emptyset\) holds. Suppose that \(x \in A\). Then \(x \in \text{int}(\text{cl}(A))\) because \(A\) is pre-open in \((Z^2, k^2)\). It follows from minimality of \(U(x)\) that \(U(x) \subseteq \text{cl}(A)\) and so \(\{y_1, y_2, y_3, y_4\} \subseteq \text{cl}(A)\), where

\[
y_1 = (2n + 1, 2m + 1), \quad y_2 = (2n + 1, 2m - 1), \quad y_3 = (2n - 1, 2m - 1), \quad y_4 = (2n - 1, 2m + 1).
\]

Since \(\{y_i\} (i = 1, 2, 3, 4)\) are open singletons, we have \(y_i \in A\) and so \(\{y_i\} = \text{int}(\{y_i\}) \subseteq \text{int}(A)\) for \(i = 1, 2, 3, 4\). Therefore, \(\emptyset \neq \{y_1, y_2, y_3, y_4\} \cap \text{int}(A) \subseteq U(x) \cap \text{int}(A)\) and hence \(U(x) \cap \text{int}(A) \neq \emptyset\). This is a contradiction.

Case III: \(x\) is a mixed point: Let \(x = (2n, 2m + 1)\) for some integers \(n\) and \(m\). Then the smallest open set \(U(x) = U^1(2n) \times \{2m + 1\}\) where \(U^1(2n) = \{2n - 1, 2n, 2n + 1\}\). Suppose that \(x \in A\). Then \(x \in \text{int}(\text{cl}(A))\).

It follows from the minimality of \(U(x)\) that \(U(x) \subseteq \text{cl}(A)\) and so \((2n - 1, 2m + 1), (2n + 1, 2m + 1) \subseteq \text{cl}(A)\). These pure points are open in \((Z^2, k^2)\) say \(y_i (i = 1, 2)\). Then, we have \(y_i \in A\) and so \(\{y_i\} = \text{int}(y_i) \subseteq \text{int}(A)\) for \(i = 1, 2\). Therefore \(\emptyset \neq \{y_1, y_2\} \cap \text{int}(A) \subseteq U(x) \cap \text{int}(A)\) and hence
The proof for a mixed point \( x = (2n + 1, 2m) \) is similar to the above. By Cases 1, 2 and 3 it is proved that \( x \not\in A \) if \( x \not\in cl(int(A)) \). Therefore we have \( A \subseteq cl(int(A)) \) and so \( A \) is semiopen.

**Remark 8.3.2:** The converse of the above theorem is not true in general.

**Example 8.3.3:** In \((\mathbb{Z}^2, k^2)\) a subset \( A = \mathbb{Z}^2/\{(0, 0), (-1, 1), (-1, 0), \ (1, -1), (-1, -1)\} \) is semi open ; \( A \) is not pre-open. We note that the following set \( B \) is not open but it is pre-open: \( B = \{(0, 0), (1, 1), (-1, 1), (1, -1), (-1, -1)\} \).