3 The Cashless Economy

3.1 Introduction

In this chapter we investigate the existence of a monetary equilibrium in a model which has no role for money as a medium of exchange. Rather, money enters the economy as a unit of account in terms of which prices are quoted and debts are contracted. Moreover, nominal debt is also used as a store of wealth across periods. We show that given bounds on agents' price expectations, there exists in this model an equilibrium with a strictly positive value of money.

Our model follows the temporary equilibrium framework of Grandmont (1977) and Grandmont and Younes (1972). However, while Grandmont considers models with a exogenously given money stock, we extend the analysis to a situation where there is no special monetary asset whose supply is given from outside but rather agent’s prior nominal commitments to other agents which determines the equilibrium price level.
3.2 The model

We consider a simple competitive pure-exchange economy. Time is discrete and there are two dates 0 and 1. There is no uncertainty. There is a single consumption good in the economy which cannot be stored across periods. There are $H$ households. Household $h$ has endowments $\omega^h_0 > 0$ and $\omega^h_1 > 0$ in the two periods respectively.

Spot markets in the consumption goods open on both days. Prices in these markets are quoted in terms of a unit of account which we refer to as 'money'. There are no future markets for commodities.

Households in the economy can make and receive commitments denoted in money terms. At the beginning of period 0, household $h$ has a commitment, inherited from the past, to pay $D^h$ units of money. $D^h$ can be negative, in which case that particular household is entitled to receive money in the beginning of period 0.

In period 0 households can contract to borrow an amount $b$ in money terms at the nominal interest rate $i$. Repayments of these loans are made in period 1.

The household’s problem in period 0 is to decide on an optimum consumption plan $(x_0, x_1)$ for the two periods. Each household’s preference ordering over consumption plans is given by a continuous, monotonic and strictly quasi-concave utility function $u^h(x_0, x_1)$. The budget constraint faced
by each household is:

\[ p x_0 \leq p \omega_0 + b - D \]
\[ p^e_1 x_1 + (1 + i) b \leq p^e_1 \omega_1 \]
\[ x_0 \geq 0, \quad x_1 \geq 0 \]

Where \( p \) is the price of the consumption good in the spot market on day 0 and \( p^e_1 \) is the price expected to prevail in the spot market on day 1.

Since our agent live only on the two date 0 and 1 they do not borrow in period 1. The model could be extended to infinitely lived agents who could then borrow in each period. In that case the budget constraint above would have to be augmented with a no-Ponzi-game condition to keep borrowing bounded.

We can eliminate \( b \) from the day 0 and day 1 constraints above to obtain the single constraint:

\[ px_0 + \frac{p^e_1}{1 + i} x_1 \leq p \omega_0 + \frac{p^e_1}{1 + i} \omega_1 - D \quad x_0 \geq 0, \quad x_1 \geq 0 \quad (3.1) \]

### 3.3 Expectations

The expectation \( p^e \) of the spot-price on day-1 will in general depend on all information available to agents on day-0. Also, different agents will, in general, form different expectations even on the basis of the same information. While there is a considerable literature which has studied the existence of
monetary equilibrium under the assumption of rational expectations or perfect foresight, we do not feel the use of these assumptions justified in the absence of an argument showing the convergence over time of learning behaviour to rational expectations. In our model, where agents form expectations only once, on date 0, it is more natural to assume the expectation-formation mechanism to be given exogenously.

We therefore assume that $p_i$ is some function $\xi(p)$ of current prices. While expectations may also depend on other information including the past history of prices, these factors are fixed in our model and therefore the dependence of expectations on them does not need to be explicitly modelled.

With these assumptions, the household’s budget constraint now becomes:

$$px_0 + \frac{\xi(p)}{1+i} x_1 \leq p\omega_0 + \frac{\xi(p)}{1+i} \omega_1 - D \quad x_0 \geq 0, \quad x_1 \geq 0$$

For convenience we define an auxiliary function $\eta(p)$ as:

**Definition 2.**

$$\eta(p) = \frac{\xi(p)}{1+i}$$

This allows us to simplify the budget constraint to:

$$px_0 + \eta(p)x_1 \leq p\omega_0 + \eta(p)\omega_1 - D \quad x_0 \geq 0, \quad x_1 \geq 0 \quad (3.2)$$
3.3.1 Bankruptcy

In (3.2), if the day-0 price is such that \( p\omega_0 + \eta(p)\omega_1 - D < 0 \) then there are no consumption plans satisfying the budget constraint. We refer to this outcome as 'bankruptcy'. In effect, prices are so low that the earnings of the agent from selling its endowments is insufficient even to meet its prior nominal obligations. While we have not explicitly modeled the origin of \( D \), one way to understand it would be to see it as \( (1 + i)b_{-1} \) where \( b_{-1} \) is the amount borrowed by the agent in the period before period 0. In our model, agents never willfully plan to default on their borrowings. Therefore bankruptcy occurs only when prices are lower than those expected by the agents when formulating their plans in the previous period. Thus the occurrence of bankruptcy is a direct result of our not assuming any form of perfect foresight.

We handle bankruptcy in our model with two assumptions. First, agents who go bankrupt are assumed to consume nothing. Formally, we model this by setting their money income to zero. Second, the payments received by individuals with a negative \( D^h \) remains unchanged even if some other agents go bankrupt. We may motivate this by assuming that there is a perfect debt guarantee and choosing to ignore in the current setup the informational problems associated with implementing such a guarantee.

With these assumptions about bankruptcy in hand, we define an agent's expected money income as:

**Definition 3.** The expected money income of agent \( h \) is given by the func-
Now, an household's budget constraint can be written as:

\[ px_0 + \eta(p)x_1 \leq M(p) \quad x_0 \geq 0, \quad x_1 \geq 0 \]  \hspace{1cm} (3.3)

### 3.4 Equilibrium

Let the household's demand for consumption goods on day 0 be given by the function \( d_1^h(p) \) (we show in this appendix that this function is well-defined). Then that household's excess demand for is given by \( \zeta^h(p) = d_1^h(p) - \omega_0^h \). The aggregate excess demand is defined by,

\[ \zeta(p) = \sum_h \zeta^h(p) \]

A price \( \hat{p} > 0 \) is a temporary equilibrium for period 0 if it is the case that,

\[ \zeta(\hat{p}) = 0 \]

We investigate below the sufficient conditions for the existence of such a temporary equilibrium.
3.4.1 Continuity

We begin by assuming that expected future prices depend continuously on current prices.

**Assumption 1.** $\eta(p)$ is a continuous function of $p$.

Under this assumption we can show that,

**Lemma 1.** For all $h \zeta^h(p)$ is a continuous function of $p$.

**Proof.** Deferred to section 3.6. \qed

Since the aggregate excess demand function is a sum of household excess demand functions, it follows that the aggregate excess demand too is continuous.

**Proposition 4.** $\zeta(p)$ is a continuous function of $p$.

3.4.2 Boundary conditions

Since $\zeta(p)$ is a continuous function of $p$, we can obtain an equilibrium using the intermediate-value theorem provided we can find one price where excess demand is negative and another price where it is positive.

Naively, we would expect excess demand to go up as prices go to zero and excess demand to go down as prices go to infinity. However, the assumptions that we have made so far are not sufficient to establish this behaviour.
As is usual, we can divide the effect of a price change of demand into an income and a substitution effect. However, the existence of price expectations and nominal commitments requires some adjustments to this decomposition.

First, in addition to the usual income effect, we have an additional effect arising from changes in the real value of money commitments. This is the real balance effect. A rise in current prices raises the real income of creditors and reduces the real income of debtors.

Relative prices are given by,

\[
\frac{p_e}{p} = \frac{\zeta(p)}{p}
\]

Hence the substitution effect of a price change depends on the responsiveness of price expectations to current prices. Taking logarithms in the above equation and differentiating, we have,

\[
\frac{\dot{p}_e}{p} - \frac{\dot{p}}{p} = \frac{1}{p} \left( \frac{p \zeta''(p)}{\zeta(p)} - 1 \right)
\]

Thus the direction of the substitution effect depends on the elasticity of expectations, viz.

\[
e = \frac{p \zeta''(p)}{\zeta(p)}
\]

When \(e = 1\) there is no substitution effect at all since expected prices move in the same proportion to current prices. When \(e > 1\), the substitution effect of a current price increase actually leads to an increase in current...
consumption since the price of future consumption rises by an even greater proportion. It is only when \( e < 1 \) that the substitution effect works in the usual direction.

The presence of elastic expectations can cause our model to not have a solution since the reverse substitution effect can result in demand not becoming positive when price goes to 0 or not becoming negative when price goes to infinity. Thus any sufficient condition for existence of equilibrium in our model will require restrictions on admissible expectations functions. The assumption we choose is that expectations are bounded both above and away from zero below. More specifically,

**Assumption 2.** For each household \( h \), there are numbers \( L^h > 0 \) and \( U^h \) such that

\[
L^h(1 + i) \leq \xi(p) \leq U^h(1 + i)
\]

for all \( p \), which implies that

\[
L^h \leq \eta(p) \leq U^h
\]

for all \( p \).

An expectations function which satisfies this assumption cannot be elastic for all prices. One economic interpretation of this assumption is that agents have a notion of 'normal' prices and do not expect prices to go beyond these 'normal' limits. An extreme case of an expectation function which satisfies this assumption is zero-elastic expectations, i.e. an expectation which does not change at all in response to current prices. It is this boundedness of expectations which essentially guarantees the existence of monetary equilibrium in our model.

While our assumption takes care of the substitution effect, the income effect for debtors can still cause problems. As prices fall the real income of
debtor fall, and in the limit they become bankrupt and cannot consume anything. Therefore an economy consisting only of debtors may once again fail to have a non-negative demand at any prices as it may happen that before prices can fall enough to make demand positive, all agents have become bankrupt. To rule out this pathological case, we assume that there is at least one creditor household.

**Assumption 3.** There is at least one household $h$ with $D^h < 0$

Taken together, the assumptions above are sufficient to establish the existence of a monetary equilibrium in our model.

**Theorem 1.** There exists a $\bar{p} > 0$ such that $\zeta \bar{p} = 0$

*Proof:* Deferred to section 3.6.

## 3.5 Conclusion

The price equilibrium established in this chapter has two interesting properties. One, we showed the existence of equilibrium for an exogenously given nominal interest rate. In general, the equilibrium prices and allocations will be different for different nominal interest rates. Thus, interest rates can be used as an instrument of redistributive policy in this economy. Secondly, the classical dichotomy between monetary and real variables does not exist in this model since multiplying prices and nominal commitments by some factor does not necessarily multiply price expectations by the same factor and thus leads to a change in budget and equilibrium sets.
3.6 Proofs

3.6.1 The budget correspondence

Definition 4. For agent \( h \), the budget correspondence is a set-valued function of \( p \), given, for \( p \geq 0 \) by:

\[
B^h(p) = \{ (x_0, x_1) \in \mathbb{R}^2_+ | px_0 + \eta(p)x_1 \leq M^h(p) \}
\]

We now establish some properties of this correspondence. In what follows, we omit the subscript \( h \) for the household when this is not likely to cause any confusion.

**Lemma 2.** \( B(p) \) is compact for all \( p > 0 \)

*Proof.* Since \( x_1 \geq 0 \),

\[
x_0 \leq M(p)/p
\]

Similarly, since \( x_0 \geq 0 \),

\[
x_1 \leq M(p)/\eta(p)
\]

Thus \( B(p) \) is bounded.

We now show that \( B(p) \) is also closed by showing that every convergent sequence in \( B(p) \) converges to a point of \( B(p) \). Let \((x_0^n, x_1^n)\) be a sequence in
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$B(p)$ converging to $(x_0, y_0)$. Since $(x^n_0, x^n_1) \in B(p)$,

$$px^n_0 + \eta(p)x^n_1 \leq M(p)$$

Taking limits we have,

$$px_0 + \eta(p)x_1 \leq M(p)$$

Hence $(x_0, y_0)$ also belongs to $B(p)$.

Being a closed and bounded subset of an Euclidean space, $B(p)$ is compact.

**Lemma 3.** If $p^n$ is a sequence of prices that converge to $p$ and $(x^n_0, x^n_1) \in B(p^n)$ is a sequence of consumption plans that converge to $(x_0, x_1)$ then $(x_0, x_1) \in B(p)$.

**Proof.** From the definition of $B(p^n)$ we have,

$$p^n x^n_0 + \eta(p^n)x^n_1 \leq M(p^n)$$

Taking limits and making use of the continuity of $M(\cdot)$ and $\eta(\cdot)$, we have,

$$px_0 + \eta(p)x_1 \leq M(p)$$

From which it follows that

$$(x_0, x_1) \in B(p)$$
Lemma 4. If $p^n$ is a sequence of prices that converge to $p$ and, and $(x_0, y_0) \in B(p)$ then there is a sequence $(x^n_0, y^n_0)$ such that $\lim(x^n_0, x^n_1) = (x_0, x_1)$ and there is a number $N$ such that for $n > N$, $(x^n_0, x^n_1) \in B(p_n)$.

Proof. We denote the expected cost of a particular consumption plan $(x, y)$ at current price $q > 0$ by the function:

$$f(q, x, y) = qx + \eta(q)y$$

The function $f(\cdot)$ has the following properties,

1. $f(q, \lambda x, \lambda y) = \lambda f(q, x, y)$.
2. $f(q, x, y)$ is continuous: since $\eta(\cdot)$ is.
3. $f(q, x, y) \geq 0$.
4. $f(q, x, y) = 0$ if and only if $x = y = 0$.

Using $f(\cdot)$ we can write the consumer's budget set as,

$$B(q) = \{(x, y) \in \mathbb{R}_+^2 | f(q, x, y) - M(q) \leq 0\}$$

Now we construct the sequence required by the lemma.

If $x_0 = y_0 = 0$ then we let $N = 1$ and $(x^n_0, y^n_0) = (0, 0)$ for all $n$. Since $(0, 0) \in B(p^n)$ for all $n$ and $\lim(0, 0) = (0, 0) = (x_0, y_0)$, so this sequence satisfies all the requirements.
Otherwise, let,

$$\lambda_n = \frac{f(p, x_0, y_0) - M(p) + M(p^n)}{f(p^n, x_0, y_0)}$$

(3.4)

and,

$$(x_0^n, x_1^n) = (\lambda_n x_0, \lambda_n y_0)$$

Since $M(\cdot)$ is continuous, $p^n \to p$ and $f(p, x, y) > 0$, we can find a $N$ such that for $n > N$,

$$M(p) - M(p^n) < f(p, x, y)$$

From (3.4) this implies that for $n > N$, $\lambda_n > 0$.

We claim that $N$ and $(x_0^n, x_1^n)$ satisfy the requirements of the lemma.

For $n > N$,

$$f(p^n, x_0^n, x_1^n) - M(p^n) = f(p^n, \lambda_n x_0, \lambda_n x_1) - M(p^n)$$

$$= \lambda_n f(p^n, x_0, x_1) - M(p^n)$$

$$= f(p, x_0, x_1) - M(p) + M(p_n) - M(p^n)$$

$$= f(p, x_0, x_1) - M(p)$$

$$\leq 0 \quad \text{since } (x_0, x_1) \in B(p)$$

Thus $(x_0^n, x_1^n) \in B(p^n)$ for $n > N$.

Also, $\lim (x_0^n, x_1^n) = (x_0, x_1)$ since $\lim \lambda_n = 1$ by the continuity of $f(\cdot)$ and $M(\cdot)$.

□
3.6.2 Individual optimisation

Definition 5. If the solution to the household's maximisation problem

$$\max_{(x_0, x_1) \in B^h(p)} u^h(x_0, x_1)$$

is $(x_0, x_1)$ then we define $d^h(p) = (x_0, x_1)$ and $d_0^h(p) = x_0$ and $d_1^h(p) = x_1$.

Since $B(p)$ is compact and $u_h(\cdot)$ is continuous, this maximisation problem has a solution. Since $u_h(\cdot, \cdot)$ is strictly quasi-concave, this solution is unique. Hence the functions $d^h(\cdot)$, $d_0^h(\cdot)$ and $d_1^h(\cdot)$ are well defined.

Lemma 5. If $p_n$ is a sequence such that $\lim p_n = p$ and the optimal consumption bundles $(x_0^n, x_1^n) = d(p_n)$ converge to $(x_0, x_1)$ then $d(p) = (x_0, x_1)$.

Proof. By lemma 3, $(x_0, x_1)$ belongs to $B(p)$.

Let $(z_0, z_1)$ be an arbitrary bundle in $B(p)$. By lemma 4, there exists $N$ and a sequence $(z_0^n, z_1^n)$ such that $(z_0^n, z_1^n) \in B(p_n)$ for $n > N$ and $\lim (z_0^n, z_1^n) = (z_0, z_1)$.

Since $(x_0^n, x_1^n) = d(p_n)$ is the optimal bundle in $B(p_n)$ and it is the case that $(z_0^n, z_1^n) \in B(p_n)$ for $n > N$, it follows that for $n > N$,

$$u(x_0^n, x_1^n) \geq u(z_0^n, z_1^n)$$

Taking limits on both sides and using the continuity of $u(\cdot)$, we have

$$u(x_0, x_1) \geq u(z_0, z_1)$$
Since \((z_0, z_1)\) was chosen to be an arbitrary point in \(B(p)\) is follows that 
\((x_0, x_1)\) is the optimal point in \(B(p)\), i.e. 
\[ d(p) = (x_0, x_1). \]

Lemma 6. The demand function \(d(p)\) is continuous.

Proof. Let \(p^n > 0\) be a sequence of prices converging to \(p\). We shall prove that \(\lim d(p)\) exists and is equal to \(d(p)\).

We proceed by contradiction. Suppose \(d(p^n)\) does not converge to \(d(p)\). Then, there must be a \(\epsilon > 0\), and a subsequence \(p^{n_k}\) of \(p^n\) such that,

\[ |d(p^{n_k}) - d(p)| > \epsilon \quad (3.5) \]

We choose an arbitrary \(0 < \delta < \min(p, \eta(p))\). Since \(\lim p_n = p\) and \(\eta(\cdot)\) and \(M(\cdot)\) are continuous, there must exist a \(N\) such that for \(n > N\),

\[ p_n \geq p - \delta \]
\[ \eta(p_n) \geq \eta(p) - \delta \]
\[ M(p_n) \leq M(p) + \delta \]

Since \(x_1^n \geq 0, x_0^n \leq (M(p) + \delta)/(p - \delta)\) for \(n > N\). Similarly, since \(x_0^n \geq 0, x_1^n \leq (M(p) + \delta)/(\eta(p) - \delta)\) for \(n > N\). This means that \(d(p^n)\) and, hence \(d(p^{n_k})\), must be bounded. Being a bounded subset of an Euclidean space it must have a convergent subsequence. Let us call this subsequence \(d(p^{n_i})\). Since, \(p^{n_i}\) is a subsequence of \(p^n\) and hence converges to \(p\), \(d(p^{n_i})\) satisfies the conditions of lemma 3 and hence must converge to \(d(p)\). However, as \(d(p^{n_i})\) is a subsequence of \(d(p^{n_k})\), this contradicts (3.5) \(\Box\)
Lemma 7. The excess demand function $d_0(p)$ is continuous.

Proof. Since $d_0(p)$ is just the first component of the continuous function $d(p)$, it must be continuous. The excess demand function is given by,

$$\zeta^h(p) = d^h_0(p) - \omega^h_0$$

and hence it must be continuous too. \hfill \Box

3.7 Aggregate demand

Lemma 8. As $p \to 0$, $\zeta(p) \to \infty$.

Proof. We know that there is at least one agent with $D^h < 0$. We begin by analysing the aggregate demand of such agents.

Suppose $p^n$ is a sequence with $\lim p^n = 0$. Since $0 < L \leq \eta(p^n) < U$ by assumption, $\eta(p^n)$ is a bounded sequence and must have a convergent subsequence. We assume that such a convergent subsequence has been chosen so that,

$$\lim \eta(p^n) = p_1 \quad \text{(say)}$$

where

$$0 < L \leq p_1 \leq U$$

Let $(x^*_0, x^*_1) = d(p^n_0)$ be the corresponding optimal bundles. We first show that the sequence $(x^*_0, x^*_1)$ must be unbounded.
If \((x_0^n, x_1^n) = d(p^n)\) is bounded then it must have a convergent subsequence \((x_0^{n_k}, x_1^{n_k})\). Let the limit of this convergent subsequence be \((x_0, x_1)\). Since \(D < 0\), this particular agent can never go bankrupt and therefore his budget constraint is always:

\[
p^{n_k}x_0 + \eta(p^{n_k})x_1 \leq p^{n_k}\omega_0 + \eta(p^{n_k})\omega_1 - D
\]

Taking limits we have,

\[
p_1x_1 \leq p_1\omega_1 - D \tag{3.6}
\]

Choose some \(\lambda \in (0, 1)\) and let

\[
x_0^\lambda = \lambda(x_0 + 1), \quad x_1^\lambda = \lambda x_1
\]

Note that since \(D < 0\), the right hand side of (3.6) is strictly positive. Therefore if (3.6) holds with equality then \(p_1x_1 > 0\) and hence

\[
p_1x_1^\lambda = \lambda p_1x_1 < p_1x_1 = p_1\omega_1 - D
\]

On the other hand if (3.6) holds with strict inequality then,

\[
p_1x_1^\lambda = \lambda p_1x_1 \leq p_1x_1 < p_1\omega_1 - D
\]

In either case,

\[
p_1x_1^\lambda < p_1\omega_1 - D
\]
or,

\[ p_1(x_1^4 - \omega_1) < -D \]

Since \( p^n \to 0 \) and \( \eta(p^n) \to p_1 \), we can use continuity to argue from the above that there is some \( K \) such that for \( k > K \),

\[ p^{n_k}(x_0^4 - \omega_0) + \eta(p^{n_k})(x_1^4 - \omega_1) < -D \]

Which means that \((x_0^4, x_1^4) \in B(p^{n_k})\). From the definition of \((x_0^{n_k}, x_1^{n_k}) = d(p^{n_k})\) it follows that,

\[ u(x_0^{n_k}, x_1^{n_k}) \geq u(x_0^4, x_1^4) \]

Taking limits and using the continuity of \( u(\cdot) \) we have,

\[ u(x_0, x_1) \geq u(x_0^4, x_1^4) \]

or

\[ u(x_0, x_1) \geq u(\lambda(x_0 + 1), \lambda x_1) \]

Since this is true for all \( \lambda \in (0,1) \), we substitute \( \lambda_i = (1 - 1/i) \) for \( i = 2, 3, \ldots \), in the above inequality and take limits to get,

\[ u(x_0, x_1) \geq u(x_0 + 1, x_1) \]

This contradicts our assumption that \( u(\cdot) \) is strictly monotonic. Hence it must be the case that \((x_0^n, x_1^n)\) is unbounded.
Now we show that it is in fact $x_0$ which must become unbounded. Recall the form of the budget constraint,

$$px_0 + \eta(p)x_1 \leq p\omega_0 + \eta(p)\omega_1 - D$$

Since $L \leq \eta(p) \leq U$, and $x_1 \geq 0$, $\omega_1 > 0$,

$$px_0 + Lx_1 \leq p\omega_0 + B\omega_1 - D$$

Or, as $x_0 \geq 0$,

$$x_1 \leq \frac{p\omega_0 + B\omega_1 - D}{L}$$

Hence $x_1$ cannot become unbounded as $p^n$ tends to 0 and it must be $x_0$ which becomes unbounded.

We have seen that the excess demand of agents with $D_h < 0$ goes to infinity as $p$ tends to 0. On the other hand the excess demand of all agents are bounded below since consumption cannot be non-negative. Hence the aggregate excess demand $\zeta(\cdot)$ must go to infinity. □

**Lemma 9.** As $p \to \infty$, $d_1(p) \to \infty$.

*Proof.* Suppose $p^n$ is a sequence with $\lim p^n = \infty$. Let $(x^n_0, x^n_1) = d(p^n)$ be the corresponding optimal bundles. We first show that the sequence $(x^n_0, x^n_1)$ must be unbounded.

If $(x_0^n, x_1^n) = d(p^n)$ is bounded then it must have a convergent subsequence $(x_0^{nk}, x_1^{nk})$. Let the limit of this convergent subsequence be $(x_0, x_1)$.
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For a sufficiently high price no agent goes bankrupt and therefore we can write the budget constraint as:

\[ p^n k x_0 + \eta(p^n k) x_1 \leq p^n k \omega_0 + \eta(p^n k) \omega_1 - D \]

or,

\[ x_0 + \frac{\eta(p^n k)}{p^n k} x_1 \leq \omega_1 + \frac{\eta(p^n k)}{p^n k} \omega_1 - \frac{D}{p^n k} \]

Taking limits and using the fact that \( \eta(p) < B \),

\[ x_0 \leq \omega_0 \quad (3.7) \]

Choose some \( \lambda \in (0, 1) \) and let

\[ x_0^\lambda = \lambda x_0, \quad x_1^\lambda = \lambda (x_1 + 1) \]

Note that since by assumption \( \omega_0 > 0 \), the right hand side of (3.7) is strictly positive. Therefore if (3.7) holds with equality then \( x_0 > 0 \) and hence

\[ x_0^\lambda = \lambda x_0 < x_0 = \omega_0 \]

On the other hand if (3.7) holds with strict inequality then,

\[ x_0^\lambda = \lambda x_0 \leq x_0 < \omega_0 \]
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In either case,

\[ x_0^\lambda < \omega_0 \]

or,

\[ (x_0^\lambda - \omega_0) < 0 \]

Since \( \eta(p^n_k) / p^n_k \to 0 \), we can use continuity to argue from the above that there is some \( K \) such that for \( k > K \),

\[ (x_0^\lambda - \omega_0) + \eta(p^n_k) / p^n_k (x_1^\lambda - \omega_1) < -\frac{D}{p^n_k} \]

Which means that \((x_0^\lambda, x_1^\lambda) \in B(p^n_k)\). From the definition of \((x_0^n_k, x_1^n_k) = d(p^n_k)\) it follows that,

\[ u(x_0^n_k, x_1^n_k) \geq u(x_0^\lambda, x_1^\lambda) \]

Taking limits and using the continuity of \( u(\cdot) \) we have,

\[ u(x_0, x_1) \geq u(x_0^\lambda, x_1^\lambda) \]

or

\[ u(x_0, x_1) \geq u(\lambda x_0, \lambda (x_1 + 1)) \]

Since this is true for all \( \lambda \in (0, 1) \), we substitute \( \lambda_i = (1 - 1/i) \) for \( i = 2, 3, \ldots \), in the above inequality and take limits to get,

\[ u(x_0, x_1) \geq u(x_0, x_1 + 1) \]
This contradicts our assumption that \( u(\cdot) \) is strictly monotonic. Hence it must be the case that \((x_0^n, x_1^n)\) is unbounded.

Now we show that it is in fact \( x_1 \) which must become unbounded.

Recall the form of the budget constraint,

\[
px_0 + \eta(p)x_1 \leq p\omega_0 + \eta(p)\omega_1 - D
\]

Since \( L \leq \eta(p) \leq U \), and \( x_1 \geq 0, \omega_1 > 0 \),

\[
px_0 + Lx_1 \leq p\omega_0 + B\omega_1 - D
\]

or, as \( x_1 \geq 0 \),

\[
x_0 \leq \omega_0 + \frac{B\omega_1 - D}{pL}
\]

Hence \( x_0 \) cannot become unbounded as \( p^n \to \infty \) and it must be \( x_1 \) which becomes unbounded.

Hence the result is proved. \( \square \)

**Lemma 10.** There is a \( p > 0 \) such that \( \zeta(p) < 0 \).

**Proof.** For sufficiently high prices no agent goes bankrupt and we can write the budget constraint as,

\[
px_0 + \eta(p)x_1 \leq p\omega_0 + \eta(p)\omega_1 - D
\]

or,

\[
p(x_0 - \omega_0) \leq \eta(p)(\omega_1 - x_1) - D
\]

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We know from lemma 9 that the right hand side becomes negative as \( p \to \infty \). Hence for a sufficiently large \( \hat{p} \) it must be the case that,

\[
x_0 < \omega_0
\]

Repeating this analysis for all the \( H \) households and taking \( p \) greater than the largest of \( \hat{p} \) for all households, the result is proved.

Proof of Theorem 1. Follows from using lemmas 8 and 9 to apply the intermediate value theorem to the continuous function \( \zeta(\cdot) \).