CHAPTER 3
SEMI-OPEN, PRE-OPEN, α-OPEN, β-OPEN MAPPINGS IN TOPOLOGICAL ORDERED SPACES.

Introduction: For semi-open sets we define the following.

\[ A^{iso} = \bigcup \{ G / G \text{ is an increasing semi-open subset of } X \text{ contained in } A \} , \]

\[ A^{dso} = \bigcup \{ G / G \text{ is a decreasing semi-open subset of } X \text{ contained in } A \} \text{ and} \]

\[ A^{bso} = \bigcup \{ G / G \text{ is a balanced semi-open subset of } X \text{ contained in } A \} . \]

Clearly \( A^{iso} \) (resp. \( A^{dso}, A^{bso} \)) is the largest increasing (resp. decreasing, balanced) semi-open set contained in \( A \).
3.1 I-SEMI-OPEN, D-SEMI-OPEN, B-SEMI-OPEN MAPS.

We introduce the following.

**DEFINITION 3.1.01.** A function $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ is called an I–semi-open [5] (resp.a D-semi-open, a B-semi-open) map if $f(G) \in \text{ISO}(X^*)$ (resp. $f(G) \in \text{DSO}(X^*)$, $f(G) \in \text{BSO}(X^*)$) whenever $G$ is an open subset of $(X, \tau)$

It is evident that every $x$-semi-open map is a semi-open map for $x=I$, $D$, $B$ and every B-semi-open is both I–semi-open and D-semi-open. The following example shows that a semi-open map need not be a $x$-semi-open map for $X=I$, $D$, $B$.

**EXAMPLE 3.1.1.** Let $X=\{a, b, c\}$, $\tau=\{\emptyset, X, \{a\} , \{b\}, \{a, b\}\}$ and $\leq=\{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. Let $f$ be the identity map from $(X, \tau, \leq)$ onto itself. Since $f$ is the identity map from $(X, \tau, \leq)$ on to itself, every open set in $(X, \tau)$ is mapped onto an open set and hence it is a semi-open set in $(X, \tau)$ (since every
open set is a semi-open set). Therefore \( f \) is a semi-open map.

\( \{b\} \) is an open set in \((X, \tau)\). \( f(\{b\}) = \{b\}, i(\{b\}) = \{b, c\} \neq \{b\} \). Therefore \( f(\{b\}) \notin ISO(X) \). Hence \( f \) is not an I-semi-open map. \( d(\{b\}) = \{a, b\} \neq \{b\} \).

=> \( f(\{b\}) \notin DSO(X) \). Therefore \( f \) is not a D-semi-open map and hence \( f \) is not a B-semi-open map.

Thus a semi-open map need not be a \( x \)-semi-open map for \( x = I, D, B \).

Following example shows that D-semi-open map need not be a B-semi-open map.

**EXAMPLE 3.1.02** Let \( X = \{a, b, c\} = X^* \), \( \tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} = \tau^* \), \( \leq = \{(a, a), (b, b), (c, c), (a, c)\} \) and \( \leq^* = \{(a, a), (b, b), (c, c), (a, c), (b, c)\} \). Let \( f \) be the identity map from \((X, \tau, \leq)\) onto \((X^*, \tau^*, \leq^*)\). \( \phi \) is the open set in \((X, \tau)\), \( f(\phi) = \phi \) is a semi-open set in \((X^*, \tau^*)\), \( d(\phi) = \phi \). \( X \) is the open set in \((X, \tau)\), \( f(X) = X^* \) is a semi-open set in \((X^*, \tau^*)\), \( d(X^*) = X^* \). \( \{a\} \) is an open set in \((X, \tau)\), \( f(\{a\}) = \{a\} \) is a semi-open set in \((X^*, \tau^*)\), \( d(\{a\}) = \{a\} \). \( \{b\} \) is an open set in \((X, \tau)\),
f({b}) = {b} is a semi-open set in $(X^*, \tau^*)$, d({b}) = {b}. 
G = {a, b} is an open set in $(X, \tau)$, f({a, b}) = {a, b} is a semi-open set in $(X^*, \tau^*)$, d({a, b}) = {a, b}. Therefore $f(G) \in \text{DSO}(X^*)$, for every open set $G$ in $(X, \tau)$. 
Therefore $f$ is a D-semi-open map. 

G = {a} is an open set in $(X, \tau)$, f({a}) = {a}, i({a}) = {a, c} ≠ {a}. => f({a}) $\notin \text{ISO}(X^*)$. Therefore $f$ is not an I-semi-open map and hence $f$ is not a B-semi-open map.

Thus a D-semi-open map need not be a B-semi-open map.

**EXAMPLE 3.1.03.** Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{a, c\}\} = \tau^*$, $\leq = \{(a, a), (b, b), (c, c), (c, a), (b, c), (b, a)\} = \leq^*$. Define $f:(X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ be the identity map. $\phi$ is the open set in $(X, \tau)$, $f(\phi) = \phi$ is a semi-open set in $(X^*, \tau^*)$, i(\phi) = \phi. X is the open set in $(X, \tau)$, $f(X) = X^*$ is a semi-open set in $(X^*, \tau^*)$, i(X*) = X*. \{a\} is an open set in $(X, \tau)$, $f(\{a\}) = \{a\}$ is a semi-open set in $(X^*, \tau^*)$, i(\{a\}) = \{a\}. \{a, c\} is an open set in $(X, \tau)$, $f(\{a, c\}) = \{a, c\}$ is a semi-open set in $(X^*, \tau^*)$, i(\{a, c\}) = \{a, c\}. Therefore
f(G) \in \text{ISO}(X^*), \text{ for every open set } G \text{ in } (X, \tau) \text{ and consequently } f \text{ is an I-semi-open map.}

\{a\} \text{ is an open set in } (X, \tau), \ f(\{a\}) = \{a\}, \ d(\{a\}) = \{a, b, c\} \neq \{a\}. \ \{a\} \text{ is not a decreasing set in } (X^*, \tau^*, \leq^*). \Rightarrow f(\{a\}) \notin \text{DSO}(X^*). \text{ Therefore } f \text{ is not a D-semi-open map and hence } f \text{ is not a B-semi-open map.}

Thus an I-semi-open map need not be a B-semi-open map.

3.01. The above observations are given in the following diagram.

For a function f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)
LEMMA 3.1.01. Let $A$ be any subset of a topological ordered space $(X, \tau, \leq)$. Then

1. $C(dscl(A)) = [C(A)]^{iso}$.  
2. $C(iscl(A)) = [C(A)]^{iso}$.  
3. $C(bscl(A)) = [C(A)]^{bso}$. 

Proof. 1) \( C(dscl(A)) = C\{ \cap F / F \text{ is decreasing semi-closed subset of } X \text{ containing } A \} \)

\[ = \cup \{ C(F) / F \text{ is decreasing semi-closed subset of } X \text{ containing } A \} \]

\[ = \cup \{ G / G \text{ is an increasing semi-open subset of } X \text{ contained in } C(A) \} \]

\[ = C(A)^{iso}. \]

Proofs of (2) and (3) are parallel to that of (1).

Following theorem characterizes I-semi-open functions.

THEOREM 3.1.01. For any function $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$, the following statements are equivalent.

1) $f$ is an I–semi-open map.

2) $f(A^\circ) \subseteq f(A)^{iso}$ for any $A \subseteq X$. 

3). \([ f^{-1}(B)]^\circ = f^{-1}(B^{iso})\) for any \(B \subseteq X^*\).

**Proof.** (1) \(\Rightarrow\) (3) Since \((f^{-1}(B))^0\) is open in \(X\) and \(f\) is I-semi-open, \(f((f^{-1}(B))^0) \in \text{ISO}(X^*)\) Clearly \(f[(f^{-1}(B))^0] \subseteq f[f^{-1}(B)] \subseteq B\). Then \(f[(f^{-1}(B))^0] \subseteq B^{iso}\), since \(B^{iso}\) is the largest increasing semi-open set contained in \(B\) Therefore \((f^{-1}(B))^0 \subseteq f^{-1}(B^{iso})\).

(3) \(\Rightarrow\) (2) Replacing \(B\) by \(f(A)\) in (3), we have \([ f^{-1}(f(A))]^0 \subseteq f^{-1}(f(A)^{iso})\).

Since \(A^\circ \subseteq [ f^{-1}(f(A))]^0\) we have \(A^\circ \subseteq f^{-1}(f(A)^{iso}) \Rightarrow f(A^\circ) \subseteq f(f^{-1}[f(A)^{iso}]) \subseteq [f(A)]^{iso}\).

Hence \(f(A^\circ) \subseteq [f(A)]^{iso}\).

(2) \(\Rightarrow\) (1) Let \(G\) be any open set in \(X\). Then \(f(G) = f(G^\circ) \subseteq [f(G)]^{iso}\). But \([f(G)]^{iso} \subseteq f(G)\). Therefore \(f\) is an I-semi-open map.

In the light of above theorem the following characterizations of D-semi-open, B-semi-open maps are obtained trivially.
THEOREM 3.1.02. For any function \( f: (X, \tau, \leq) \to (X^*, \tau^*, \leq^*) \), the following statements are equivalent.

1. \( f \) is D–semi-open map.
2. \( f(A^\circ) \subseteq [f(A)]^{dso} \) for any \( A \subseteq X \).
3. \( [f^{-1}(B)]^\circ \subseteq f^{-1}(B^{dso}) \) for any \( B \subseteq X^* \).

THEOREM 3.1.03. For any function \( f: (X, \tau, \leq) \to (X^*, \tau^*, \leq^*) \) the following statements are equivalent.

1. \( f \) is B – semi-open map.
2. \( f(A^\circ) \subseteq [f(A)]^{bso} \) for any \( A \subseteq X \).
3. \( [f^{-1}(B)]^\circ \subseteq f^{-1}(B^{bso}) \) for any \( B \subseteq X^* \).

THEOREM 3.1.04. Let \( f : (X, \tau, \leq_1) \to (Y, \sigma, \leq_2) \) and \( g : (Y, \sigma, \leq_2) \to (Z, \eta, \leq_3) \) be any two mappings, then \( g \circ f : (X, \tau, \leq_1) \to (Z, \eta, \leq_3) \) is \( x \)-semi-open if \( f \) is open and \( g \) is \( x \)-semi-open for \( x = I, D, B \).

**Proof.** Let \( G \) be an open set in \((X, \tau)\). Since \( f \) is open map, \( f(G) \) is an open set in \((Y, \sigma)\). Since \( g \) is \( x \)-semi open map, \( g(f(G)) \) is \( x \)-semi-open set in \((Z, \eta)\), for \( x = I, D, B \). \( \Rightarrow \) \( g \circ f (G) \) is \( x \)-semi open set in \((Z, \eta)\) for
x-I,D,B. Therefore gof is x-semi open map for x=I,D,B.

3.2 I-PRE-OPEN, D-PRE-OPEN AND B-PRE-OPEN MAPS.

Introduction: We define the following for pre-open sets.

\[ A_{ipo} = \bigcup \{ G / G \text{ is an increasing pre-open subset of } X \text{ contained in } A \}, \]

\[ A_{dpo} = \bigcup \{ G / G \text{ is a decreasing pre-open subset of } X \text{ contained in } A \} \]

\[ A_{bpo} = \bigcup \{ G / G \text{ is a balanced pre-open subset of } X \text{ contained in } A \}. \]

Clearly \( A_{ipo} \) (resp. \( A_{dpo}, A_{bpo} \)) is the largest increasing (resp. decreasing, balanced) pre-open set contained in A.

We introduce the following

**DEFINITION 3.2.01.** A function \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) is called an I-pre-open map [4] (resp. D-pre-open, B-pre-open map if \( f(G) \in IPO(X^*) \) (resp. \( f(G) \)


\[ \in DPO(X^*), f(G) \in BPO(X^*) \text{ whenever } G \text{ is an open subset of } (X, \tau, \leq). \]

It is evident that every \( x \)-pre-open map is an \( x \)-pre-open map for \( x = I, D, B \) and that every \( B \)-pre-open map is both \( I \)-pre-open and \( D \)-pre-open.

Following example shows that a \( \text{pre-open map} \) need not be \( x \)-pre-open for \( x = I, D, B \). It needs reference from example 1.1.01.

**EXAMPLE 3.2.01.** Let \( X = \{a, b, c\} = X^*, \tau = \{\emptyset, X, \{a\}, \{b\}\}, \{a, b\} = \tau^* \) and \( \leq = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\} \) clearly \((X, \tau, \leq)\) is a topological ordered space. Let \( f \) be the identity map from \((X, \tau, \leq)\) onto itself. Since \( f \) is the identity map, every open set in \((X, \tau)\) is mapped onto an open set and hence it is a pre-open set (Since every open set is a pre-open set). Therefore \( f \) is a pre-open map.

\{b\} is an open set in \((X, \tau)\), \( f(\{b\}) = \{b\}, i(\{b\}) = \{b\} \). \( \Rightarrow f(\{b\}) \notin IPO(X) \).

Therefore \( f \) is not an \( I \)-pre-open map, \( \{b\} \) is an open set in \((X, \tau)\), \( f(\{b\}) = \{b\}, d(\{b\}) = \{a, b\} \neq \{b\} \). \( \Rightarrow f(\{b\}) \)

\[\notin \text{DPO}(X).\] Therefore \(f\) is not a D-pre-open map and consequently \(f\) is not a B-pre-open map.

Thus a pre-open map need not be a \(x\)-pre-open map for \(x=I, D, B\).

Following example shows that a D-pre-open map need not be a B-pre-open map. It needs reference from example 1.1.01.

**EXAMPLE 3.2.02.** Let \(X=\{a, b, c\} = X^*, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} = \tau^*, \leq =\{(a, a), (b, b), (c, c), (a, c)\} \) and \(\leq^* =\{(a, a), (b, b), (c, c), (a, c), (b, c)\}\). Let \(f\) be the identity map from \((X, \tau, \leq)\) onto \((X^*, \tau^*, \leq^*)\). \(\emptyset\) is the open set in \((X, \tau)\), \(f(\emptyset) = \emptyset\) is a pre-open set in \((X^*, \tau^*)\) \(d(\emptyset) = \emptyset\). \(X\) is the open set in \((X, \tau)\), \(f(X) = X^*\) is a pre-open set in \((X^*, \tau^*)\), \(d(X^*) = X^*\). \(\{a\}\) is an open set in \((X, \tau)\), \(f(\{a\}) = \{a\}\) is a pre-open set in \((X^*, \tau^*)\), \(d(\{a\}) = \{a\}\). \(\{b\}\) is an open set in \((X, \tau)\), \(f(\{b\}) = \{b\}\) be an a pre-open set in \((X^*, \tau^*)\), \(d(\{b\}) = \{b\}\). \(\{a, b\}\) is an open set in \((X, \tau)\), \(f(\{a, b\}) = \{a, b\}\) is a pre-open set in \((X^*, \tau^*)\), \(d(\{a, b\}) = \{a, b\}\). \(\Rightarrow f(G) \in \text{DPO}(X^*)\),
whenever \( G \) is an open set in \((X, \tau)\). Therefore \( f \) is a D-pre-open map.

\{a\} is an open set in \((X, \tau)\), \( f(\{a\}) = \{a\}, i(\{a\}) = \{a, c\} \neq \{a\} \Rightarrow f(\{a\}) \notin IPO(X^*). \) Therefore \( f \) is not an I-pre-open map and hence \( f \) is not a B-pre-open map.

Thus a D-pre-open map need not be a B-pre-open map.

Following example shows that I-pre-open map need not be a B-pre-open map.

It needs reference from example 1.1.04.

**EXAMPLE 3.2.03.** Let \( X = \{a, b, c\} = X^* \), \( \tau = \{\phi, X, \{a\}, \{a, c\}\} = \tau^* \), \( \leq = \{(a, a), (b, b), (c, c), (c, a), (b, c), (b, a)\} = \leq^* \). Let \( f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) be the identity map. \( \phi \) is the open set in \((X, \tau)\), \( f(\phi) = \phi \) is a pre-open set in \((X^*, \tau^*)\), \( i(\phi) = \phi \). \( X \) is the open set in \((X, \tau)\), \( f(X) = X^* \) is a pre-open set in \((X^*, \tau^*)\), \( i(X^*) = X^* \).

Let \{a\} be an open set in \((X, \tau)\), \( f(\{a\}) = \{a\} \) is a pre-open set in \((X^*, \tau^*)\), \( i(\{a\}) = \{a\} \).

\{a, c\} is an pre-open set in \((X, \tau)\), \( f(\{a, c\}) = \{a, c\} \) is a pre-open set in \((X^*, \tau^*)\), \( i(\{a, c\}) = \{a, c\} \). \Rightarrow
f(G) ∈ IPO(X*), whenever G is an open set in (X, τ). Therefore f is an I-pre-open map.

{a} is an open set in (X, τ), f({a}) = {a}, d({a}) = {a, b, c} ≠ {a}. f([{a}]) ∉ DPO(X*). Therefore f is not a D-pre-open map and hence f is not a B-pre-open map.

Thus an I-pre-open map need not be a B-pre-open map.

3.2.01 The above observations are given in the following diagram.

For a function f:(X, τ, ≤) → (X*, τ*, ≤*),

**LEMMA 3.2.01.** Let A be any subset of a topological ordered space (X, τ, ≤). Then

1) C(dpcl(A)) = (C(A))_{ipo}.
2) C(ipcl(A)) = (C(A))_{dpo}.
3) \( C(\text{bpcl}(A)) = (C(A))^{\text{ip}}. \)

**Proof.**

\[
C(\text{dpcl}(A)) = C\{ \cap \mathcal{F}/\mathcal{F} \text{ is a decreasing pre-closed subset of } X \text{ containing } A \}
= \bigcup \{C(F)/\mathcal{F} \text{ is a decreasing pre-closed subset of } X \text{ containing } A \}
= \bigcup \{G/G \text{ is an increasing pre-open subset of } X \text{ contained in } C(A) \}
= (C(A))^{\text{ipo}}.
\]

Proofs of (2) and (3) are parallel to that of (1).

The following theorem characterizes I-pre-open functions.

**THEOREM 3.2.01.** For any function \( f:(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \), the following statements are equivalent.

1) \( f \) is an I-pre-open map.

2) \( f(A^0) \subseteq [f(A)]^{\text{ipo}} \) for any \( A \subseteq X \).

3) \( [f^{-1}(B)]^0 = f^{-1}(B^{\text{ipo}}) \) for any \( B \subseteq X^* \).

**Proof.** \((1) \Rightarrow (3)\) Since \( [f^{-1}(B)]^0 \) is open in \( X \) and \( f \) is an I-pre-open, \( f((f^{-1}(B))^0) \subseteq f(f^{-1}(B)) \subseteq B \) and
f([f^{-1}(B)]^0) \text{ is I-pre-open in } X^*. \text{Then } f((f^{-1}(B))^0) \subseteq B_{ipo} \text{ since } B_{ipo} \text{ is the largest increasing pre-open set contained in } B. \text{Therefore } [f^{-1}(B)]^0 \subseteq f^{-1}(B_{ipo})

(3) => (2) \quad \text{Replacing } B \text{ by } f(A) \text{ in (3), we have } [f^{-1}(f(A))]^0 \subseteq f^{-1}([f(A)]_{ipo}). \text{Since } A^0 \subseteq [f^{-1}(f(A))]^0, \text{we have } A^0 \subseteq f^{-1}([f(A)]_{ipo}). f(A^0) \subseteq f(f^{-1}([f(A)]_{ipo})) \subseteq [f(A)]_{ipo}. \text{Hence } f(A^0) \subseteq [f(A)]_{ipo}.

(2) => (1) \quad \text{Let } G \text{ be any open set in } X. \text{Then } f(G) = f(G^0) \subseteq [f(G)]_{ipo} \subseteq f(G). \text{Therefore } f(G) \text{ is an increasing pre-open set in } X^* \quad f \text{ is an I-Pre-open map.}

We can obtain the following two theorems regarding D-pre-open map and B-pre-open maps trivially.

**THEOREM 3.2.02.** For any function \( f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \), the following statements are equivalent.

1) \( f \) is D-pre-open map.

2) \( f(A^0) \subseteq [f(A)]^{dpo} \) for any \( A \subseteq X \).
3) \([ f^{-1}(B)]^0 \subseteq f^{-1}(B^{bpo})\) for any \(B \subseteq X^*\).

**THEOREM 3.2.03.** For any function \(f:(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)\), the following statements are equivalent.

1) \(f\) is \(B\)-pre-open map.

2) \([f(A^0)] \subseteq [f(A)]^{bpo}\) for any \(A \subseteq X\).

3) \([ f^{-1}(B)]^0 \subseteq f^{-1}(B^{bpo})\) for any \(B \subseteq X^*\).

**THEOREM 3.2.04.** Let \(f: (X, \tau, \leq_1) \rightarrow (Y, \sigma, \leq_2)\) and \(g: (Y, \sigma, \leq_2) \rightarrow (Z, \eta, \leq_3)\) be any two mappings. Then \(gof: (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)\) is \(x\)-pre-open if \(f\) is open and \(g\) is \(x\)-pre-open for \(x = I, D, B\).

**Proof.** Let \(F\) be any open set in \((X, \tau)\). Since \(f\) is open map, \(f(F)\) is open in \((Y, \sigma)\). Given \(g\) is \(x\)-pre-open map, \(g(f(F))\) is \(x\)-pre-open set in \((Z, \eta, \leq_3)\) for \(x = I, D, B\). Hence \(gof\) is \(x\)-pre-open map for \(x = I, D, B\).

### 3.3  I-\(\alpha\)-OPEN, D-\(\alpha\)-OPEN AND B-\(\alpha\)-OPEN MAPS.

**Introduction:** We introduce the following for \(\alpha\)-open sets.
\( A^{i\alpha o} = \bigcup \{G/G \text{ is an increasing } \alpha\text{-open subset of } X \text{ contained in } A\} \),
\( A^{d\alpha o} = \bigcup \{G/G \text{ is a decreasing } \alpha\text{-open subset of } X \text{ contained in } A\} \) and
\( A^{b\alpha o} = \bigcup \{G/G \text{ is a balanced } \alpha\text{-open subset of } X \text{ contained in } A\} \).

Clearly \( A^{i\alpha o}(A^{d\alpha o}, A^{b\alpha o}) \) is the largest increasing (decreasing, balanced) \( \alpha\)-open set contained in \( A \).

We introduce the following.

**DEFINITION 3.3.01.** A function \( f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*) \) is called an I-\( \alpha\)-open map [6] (resp. D-\( \alpha\)-open, B-\( \alpha\)-open map if \( f(G) \in \text{I}^{\alpha O}(X^*) \) (resp. \( f(G) \in \text{D}^{\alpha O}(X^*) \), \( f(G) \in \text{B}^{\alpha O}(X^*) \)) whenever \( G \) is an open subset of \( (X, \tau, \leq) \).

It is evident that every \( x\)-\( \alpha\)-open map is an \( \alpha\)-open map for \( x = I, D, B \) and that every B-\( \alpha\)-open map is both I-\( \alpha\)-open and D-\( \alpha\)-open.

Following example shows that an \( \alpha\)-open map need not be \( x\)-\( \alpha\)-open for \( x = I, D, B \).
EXAMPLE 3.3.01. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} = \tau^*$ and $\leq = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. Let $f$ be the identity map from $(X, \tau, \leq)$ onto itself. Since $f$ is the identity map, every open set in $(X, \tau)$ is mapped into an open set and hence it is an $\alpha$-open set in $(X^*, \tau^*)$ ($\therefore$ every open set is an $\alpha$-open set). Hence $f$ is an $\alpha$-open map.

$\{b\}$ is an open set in $(X, \tau)$, $f(\{b\}) = \{b\}$, $i(\{b\}) = \{b, c\} \neq \{b\}$. $\Rightarrow f(\{b\}) \notin I\alpha O(X)$. $\Rightarrow f$ is not an $I$-$\alpha$-open map. $d(\{b\}) = \{a, b\} \neq \{b\}$. $\Rightarrow f(\{b\}) \notin D\alpha O(X)$. Therefore $f$ is not a $D$-$\alpha$-open map and consequently $f$ is not a $B$-$\alpha$-open map.

Thus an $\alpha$-open map need not be a $x$-$\alpha$-open map for $x = I, D, B$.

Following example shows that a $D$-$\alpha$-open map need not be a $B$-$\alpha$-open map. It needs reference from example 1.1.01.

EXAMPLE 3.3.02. Let $X = \{a, b, c\} = X^*$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} = \tau^*$, $\leq = \{(a, a), (b, b), (c, c), (a, c)\}$
and $\leq^* = \{(a, a), (b, b), (c, c), (a, c), (b, c)\}$. Let $f$ be the identity map from $(X, \tau, \leq)$ onto $(X^*, \tau^*, \leq^*)$. $\phi$ is the open set in $(X, \tau)$, $f(\phi) = \phi$ is an $\alpha$-open set in $(X^*, \tau^*)$, $d(\phi) = \phi$. $X$ is the open set in $(X^*, \tau^*)$, $f(X) = X^*$ is an $\alpha$-open set in $(X^*, \tau^*)$, $d(X^*) = X^*$. \{a\} is an open set in $(X, \tau)$, $f(\{a\}) = \{a\}$ is an $\alpha$-open set in $(X^*, \tau^*)$, $d(\{a\}) = \{a\}$. \{b\} is an open set, $f(\{b\}) = \{b\}$ is an $\alpha$-open set in $(X^*, \tau^*)$, $d(\{b\}) = \{b\}$. \{a, b\} be an open set in $(X, \tau)$, $f(\{a, b\}) = \{a, b\}$ is an $\alpha$-open set in $(X^*, \tau^*)$, $d(\{a, b\}) = \{a, b\}$. $\Rightarrow f(G) \in D\alpha O(X^*)$, whenever $G$ is an open set in $(X, \tau)$. $\Rightarrow f$ is a D-$\alpha$-open map.

\{a\} is an open set in $(X, \tau)$, $f(\{a\}) = \{a\}$, $i(\{a\}) = \{a, c\} \neq \{a\}$. $\Rightarrow f(\{a\}) \notin I\alpha O(X^*)$. Therefore $f$ is not an I-$\alpha$-open map and hence $f$ is not a B-$\alpha$-open map.

Thus a D-$\alpha$-open map need not be a B-$\alpha$-open map.

Following example shows that an I-$\alpha$-open map need not be a B-$\alpha$-open map. It needs reference from example 1.1.04.
EXAMPLE 3.3.03. Let $X = \{a, b, c\} = X^*$, $\tau = \{\varnothing, X, \{a\}, \{a, c\}\} = \tau^*$ and $\leq = \{(a, a), (b, b), (c, c), (c, a), (b, c), (b, a)\} = \leq^*$. Let $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be the identity map. $\varnothing$ is the open set in $(X, \tau)$, $f(\varnothing) = \varnothing$ is an $\alpha$-open set in $(X^*, \tau^*)$, $i(\varnothing) = \varnothing$. $X$ is the open set in $(X, \tau)$, $f(X) = X^*$ is an $\alpha$-open set in $(X^*, \tau^*)$, $i(X^*) = X^*$. $\{a\}$ is an open set in $(X, \tau)$, $f(\{a\}) = \{a\}$ is an $\alpha$-open set in $(X^*, \tau^*)$. $i(\{a\}) = \{a\}$. $\{a, c\}$ is an open set in $(X, \tau)$, $f(\{a, c\}) = \{a, c\}$ is an $\alpha$-open set in $(X^*, \tau^*)$, $i(\{a, c\}) = \{a, c\}$. $\Rightarrow f(G) \in I_\alpha O(X^*)$, whenever $G$ is an open set in $(X, \tau)$. Therefore $f$ is an I-$\alpha$-open map.

$G = \{a\}$ be an open set in $(X, \tau)$, $f(\{a\}) = \{a\}$, $d(\{a\}) = \{a, b, c\} \neq \{a\}$. $\Rightarrow f(\{a\}) \notin D_\alpha O(X^*)$. Therefore $f$ is not a D-$\alpha$-open map and consequently $f$ is not a B-$\alpha$-open map.

Thus an I-$\alpha$-open map need not be a B-$\alpha$-open map.
3.3.01 The above observations are given in the following diagram.

For a function \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \)

The following theorems are similar to that of pre-open maps.

**LEMMA 3.3.01.** Let \( A \) be any subset of a topological ordered space \((X, \tau, \leq)\). Then

1) \( C(d_{\alpha}cl(A)) = (C(A))^{i_{\alpha}o} \).

2) \( C(i_{\alpha}cl(A)) = (C(A))^{d_{\alpha}o} \).

3) \( C(b_{\alpha}cl(A)) = (C(A))^{b_{\alpha}o} \).

**THEOREM 3.3.01.** For any function \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \), the following statements are equivalent.

1) \( f \) is an I-\( \alpha \)-open map.
2) $f(A^0) \subseteq [f(A)]^{\text{dao}}$ for any $A \subseteq X$.

3) $[f^{-1}(B)]^0 = f^{-1}(B^{\text{dao}})$ for any $B \subseteq X^*$.

We obtain the following two theorems that give characterizations for D-α-open map and B-α-open maps trivially.

**THEOREM 3.3.02.** For any function $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$, the following statements are equivalent.

1) $f$ is a D-α-open map.

2) $f(A^0) \subseteq [f(A)]^{\text{dao}}$ for any $A \subseteq X$.

3) $[f^{-1}(B)]^0 \subseteq f^{-1}(B^{\text{dao}})$ for any $B \subseteq X^*$.

**THEOREM 3.3.03.** For any function $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$, the following statements are equivalent.

1) $f$ is a B-α-open map.

2) $[f(A^0)] \subseteq [f(A)]^{\text{bao}}$ for any $A \subseteq X$.

3) $[f^{-1}(B)]^0 \subseteq f^{-1}(B^{\text{bao}})$ for any $B \subseteq X^*$.

**THEOREM 3.3.04.** Let $f : (X, \tau, \leq_1) \to (Y, \sigma, \leq_2)$ and $g : (Y, \sigma, \leq_2) \to (Z, \eta, \leq_3)$ be any two mappings. Then $gof : (X, \tau, \leq_1) \to (Z, \eta, \leq_3)$ is $x$-α-open if $f$ is open and $g$ is $x$-α-open for $x = I, D, B$. 
**Proof.** Let $G$ be an open set in $X$. Since $f$ is an open map, $f(G)$ is an open set in $(Y, \sigma)$. Since $g$ is a $x$-$\alpha$-open map, $g(f(G))$ is a $x$-$\alpha$-open set in $(Z, \eta)$, for $x=I,D,B$. $\Rightarrow$ $gof(G)$ is a $x$-$\alpha$-open set in $(Z,\eta)$. Therefore $gof$ is a $x$-$\alpha$-open map for $x=I,D,B$.

**THEOREM 3.3.05 :** Let $(X, \tau, \leq)$ and $(X^{*}, \tau^{*}, \leq^{*})$ be two topological ordered spaces.

Let $f : (X, \tau, \leq) \rightarrow (X^{*}, \tau^{*}, \leq^{*})$ be a map. If $f$ is an $I$-$\alpha$-open map then it is an $I$-semi-open map and an $I$-pre-open map.

**Proof.** Let $f$ be an $I$-$\alpha$-open map. Let $G$ be an open set in $X$. Since $f$ is an $I$-$\alpha$-open map, $f(G)$ is an $I$-$\alpha$-open set in $X^{*}$. $\Rightarrow$ $f(G)$ is an $I$-semi-open set in $X^{*}$ and an $I$-pre-open set in $X^{*}$ (from Lemma 1.1.1). Therefore $f$ is an $I$-$\alpha$-open map and $f$ is an $I$-pre-open map.

Following example shows that an $I$-semi-open map need not be an $I$-$\alpha$-open map. It needs reference from example 1.1.01.
EXAMPLE 3.3.04. Let $X = \{a, b, c\} = X^*$, $\tau = \{\emptyset, X, \{a\}\}$, $\tau^* = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\leq = \{(a, a), (b, b), (c, c), (a, b), (c, b)\} = \leq^*$ Define $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ by $f(a) = b$, $f(b) = c$ and $f(c) = c$. $\emptyset$ is the open set in $(X, \tau)$, $f(\emptyset) = \emptyset$ is a semi-open set $(X^*, \tau^*)$, $i(\emptyset) = \emptyset$. $X$ is the open set in $(X, \tau)$, $f(X) = \{b, c\}$ is a semi-open set in $(X^*, \tau^*)$, $i(\{b, c\}) = \{b, c\}$. $\{a\}$ is an open set in $(X, \tau)$, $f(\{a\}) = \{b\}$ is a semi-open set in $(X^*, \tau^*)$, $i(\{b\}) = \{b\}$. $\Rightarrow f(G) \in \text{ISO}(X^*)$, where $G$ is an open set in $(X, \tau) \Rightarrow f$ is an I-semi-open map.

$X$ is an open set in $(X, \tau)$, $f(X) = \{b, c\}$ is not an $\alpha$-open set in $(X^*, \tau^*)$. Therefore $f(X) \notin \text{I} \alpha \text{O}(X^*)$ and consequently $f$ is not an I-$\alpha$-open map.

Thus an I-semi-open map need not be an I-$\alpha$-open map.

Following example shows that an I-pre-open map need not be an I-$\alpha$-open map. It needs reference from example 1.1.03.
EXAMPLE 3.3.05. Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$, $\tau^* = \{\phi, X, \{a\}, \{b, c\}\}$ and $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Let $f$ be the identity map from $(X, \tau, \leq)$ onto $(X^*, \tau^*, \leq^*)$. $\phi$ is the open set in $(X, \tau)$, $f(\phi) = \phi$ is a pre-open set in $(X^*, \tau^*)$, $i(\phi) = \phi$. $X$ is the open set in $(X, \tau)$, $f(X) = X^*$ is a pre-open set in $(X^*, \tau^*)$, $i(X^*) = X^*$. $\{a\}$ is an open set in $(X, \tau)$, $f(\{a\}) = \{a\}$ is a pre-open set in $(X^*, \tau^*)$, $i(\{a\}) = \{a\}$. $\{b\}$ is an open set in $(X, \tau)$, $f(\{b\}) = \{b\}$ is a pre-open set in $(X^*, \tau^*)$, $i(\{b\}) = \{b\}$. $\{a, b\}$ is an open set in $(X, \tau)$, $f(\{a, b\}) = \{a, b\}$ is a pre-open set, $i(\{a, b\}) = \{a, b\}$. \(\Rightarrow f(G) \in IPO(X^*)\), where $G$ is an open set in $(X, \tau)$. \(\Rightarrow f\) is an I-pre-open map.

$\{b\}$ is an open set in $(X, \tau)$, $f(\{b\}) = \{b\}$ is not an $\alpha$-open set. Therefore $f$ is not an $\alpha$-open map and consequently $f$ is not an I-$\alpha$-open map.

Thus an I-pre-open map need not be an I-$\alpha$-open map.

THEOREM 3.3.06. Let $(X, \tau, \leq)$ and $(X^*, \tau^*, \leq^*)$ be two topological ordered spaces.
Let \( f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*) \) be a map. If \( f \) is a D-\( \alpha \)-open map then it is a D-semi-open map and a D-pre-open map.

**Proof.** Let \( f \) be a D-\( \alpha \)-open map. Let \( G \) be an open set in \( X \). Since \( f \) is a D-\( \alpha \)-open map, \( f(G) \) is a D-\( \alpha \)-open set in \( X^* \). \( \Rightarrow \) \( f(G) \) is a D-semi-open set in \( X^* \) and a D-pre-open set in \( X^* \) (from Lemma 1.1.1). Therefore \( f \) is a D-\( \alpha \)-open map and \( f \) is a D-pre-open map.

Following example shows that a D-semi-open map need not be a D-\( \alpha \)-open map. It needs reference from example 1.1.01.

**EXAMPLE 3.3.06.** Let \( X = \{a, b, c\} = X^*, \tau = \{\emptyset, X, \{a\}\}, \tau^* = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \) and \( \leq = \{(a, a), (b, b), (c, c), (b, c), (c, a), (b, a)\} = \leq^* \).

Define \( f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*) \) by \( f(a) = b, f(b) = c \) and \( f(c) = c \). \( \emptyset \) is the open set in \( (X, \tau) \), \( f(\emptyset) = \emptyset \) is a semi-open set in \( (X^*, \tau^*) \), \( d(\emptyset) = \emptyset \). \( X \) is the open set in \( (X, \tau) \), \( f(X) = \{b, c\} \) is a semi-open set in \( (X^*, \tau^*) \), \( d(\{b, c\}) = \{b, c\} \). \( \{a\} \) is an open set in \( (X, \tau) \), \( f(\{a\}) \)
= {b} is a semi-open set in $(X^*, \tau^*)$, $d([b]) = \{b\}$. =>
$f(G) \in DSO(X^*)$, where $G$ is an open set in $(X, \tau)$. Therefore $f$ is an $D$-semi-open map.

$X$ is the open set in $(X, \tau)$, $f(X) = \{b, c\}$ is not an $\alpha$-open set in $(X^*, \tau^*)$. Therefore $f(X) \notin D\alpha O(X^*)$ and consequently $f$ is not a $D$-$\alpha$-open map.

Thus a $D$-semi-open map need not be a $D$-$\alpha$-open map.

Following example shows that a $D$-pre-open map need not be a $D$-$\alpha$-open map. It needs reference from example 1.1.03.

**EXAMPLE 3.3.07.** Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau^* = \{\phi, X, \{a\}, \{b, c\}\}$ and $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Let $f$ be the identity map from $(X, \tau, \leq)$ onto $(X^*, \tau^*, \leq^*)$. $\phi$ is the open set in $(X, \tau)$, $f(\phi) = \phi$ is a pre-open set in $(X^*, \tau^*)$, $d(\phi) = \phi$. $X$ is the open set in $(X, \tau)$, $f(X) = X^*$ is a pre-open set in $(X^*, \tau^*)$, $d(X^*) = X^*$. $\{a\}$ is an open set in $(X, \tau)$, $f(\{a\}) = \{a\}$ is a pre-open set in $(X^*, \tau^*)$, $d(\{a\}) = \{a\}$. $\{b\}$ is an
open set in \((X, \tau)\), \(f(\{b\}) = \{b\}\) is a pre-open set in \((X^*, \tau^*)\), \(d(\{b\}) = \{b\}\). \(\{a, b\}\) is an open set in \((X, \tau)\), \(f(\{a, b\}) = \{a, b\}\) is a pre-open set, \(d(\{a, b\}) = \{a, b\}\).

\[ \Rightarrow f(G) \in \text{DPO}(X^*), \] where \(G\) is an open set in \((X, \tau)\).

\[ \Rightarrow f\] is a D-pre-open map.

\(\{b\}\) is an open set in \((X, \tau)\), \(f(G) = f(\{b\}) = \{b\}\) is not an \(\alpha\)-open set. Therefore \(f\) is not an \(\alpha\)-open map and consequently \(f\) is not a D-\(\alpha\)-open map.

Thus a D-pre-open map need not be a D-\(\alpha\)-open map.

**Theorem 3.3.07.** Let \((X, \tau, \leq)\) and \((X^*, \tau^*, \leq^*)\) be two topological ordered spaces.

Let \(f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)\) be a map. If \(f\) is a B-\(\alpha\)-open map then it is a B-semi-open map and a B-pre-open map.

**Proof.** Let \(f\) be a B-\(\alpha\)-open map. Let \(G\) be an open set in \(X\). since \(f\) is a B-\(\alpha\)-open map, \(f(G)\) is a B-\(\alpha\)-open set in \(X^*\). \(\Rightarrow f(G)\) is a B-semi-open set in \(X^*\) and a B-pre-open set in \(X^*\) (from Lemma 1.1.1).
Therefore $f$ is a $B$-$\alpha$-open map and $f$ is a $B$-pre-open map.

Following example shows that a $B$-semi-open map need not be a $B$-$\alpha$-open map. It needs reference from example 1.1.01.

**EXAMPLE 3.3.08.** Let $X = \{a, b, c\} = X^*, \tau = \{\phi, X, \{a, b\}\}$, $\tau^* = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and

$\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Define $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. $\phi$ is the open set in $(X, \tau)$, $f(\phi) = \phi$ is a semi-open set in $(X^*, \tau^*)$, $i(\phi) = \phi$, $d(\phi) = \phi$. $X$ is the open set in $(X, \tau)$, $f(X) = X^*$ is a semi-open set in $(X^*, \tau^*)$, $i(X^*) = X^*$, $D(X^*) = X^*$. $\{a, b\}$ is an open set in $(X, \tau)$, $f(\{a, b\}) = \{b, c\}$ is a semi-open set in $(X^*, \tau^*)$, $i(\{b, c\}) = \{b, c\}$, $d(\{b, c\}) = \{b, c\}$.$\Rightarrow f(G) \in \text{BSO}(X^*)$, whenever $G$ is any open set in $(X, \tau)$. Therefore $f$ is a $B$-semi-open map.

$\{a, b\}$ is an open set in $(X, \tau)$, $f(\{a, b\}) = \{b, c\}$ is not an $\alpha$-open set in $(X^*, \tau^*)$. $\Rightarrow f(G) \notin \text{B}\alpha\text{O}(X^*)$. 
Therefore $f$ is not an $\alpha$-open map and hence $f$ is not a $B$-$\alpha$-open map.

Thus a $B$-semi-open map need not be a $B$-$\alpha$-open map.

Following example shows that a $B$-pre-open map need not be a $B$-$\alpha$-open map. It needs reference from example 1.1.03.

**EXAMPLE 3.3.09.** Let $X = \{a, b, c\} = X^*, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \tau^* = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Let $f$ be the identity map from $(X, \tau, \leq)$ onto $(X^*, \tau^*, \leq^*)$. From example 3.3.07 $f$ is a $D$-pre-open map. $i(\emptyset) = \emptyset, i(\{a\}) = \{a\}, i(\{b\}) = \{b\}, i(\{a, b\}) = \{a, b\}$. $\Rightarrow f(G) \in IPO(X^*)$, whenever $G$ is an open set in $(X, \tau)$. $\Rightarrow f(G) \in BPO(X^*)$, whenever $G$ is open set in $(X, \tau)$. Therefore $f$ is a $B$-pre-open map.

$\{b\}$ is an open set in $(X, \tau)$. $f(\{b\}) = \{b\}$ is not an $\alpha$-open set in $(X^*, \tau^*)$. Therefore $f$ is not an $\alpha$-open map and consequently $f$ is not a $B$-$\alpha$-open map.

Thus a $B$-pre-open map need not be a $B$-$\alpha$-open map.
3.4 I-β-OPEN, D-β-OPEN AND B-β-OPEN MAPS.

Introduction: We define the following for β-open sets.

\[ A^{iβo} = \bigcup \{G / G \text{ is an increasing } β\text{-open subset of } X \text{ contained in } A\}, \]

\[ A^{dβo} = \bigcup \{ G / G \text{ is a decreasing } β\text{-open subset of } X \text{ contained in } A\} \text{ and} \]

\[ A^{bβo} = \bigcup \{ G / G \text{ is a balanced } β\text{-open subset of } X \text{ contained in } A\}. \]

Clearly \( A^{iβo}\) (resp. \( A^{dβo}, A^{bβo}\)) is the largest increasing (resp. decreasing, balanced) β-open contained in A.

We introduce the following.

**DEFINITION 3.4.01.** A function \( f : (X, τ, ≤) \rightarrow (X^*, τ^*, ≤^*) \) is called an I-β-open map \([7]\) (resp. D-β-open map, B-β-open map) if \( f(G) \in IβO(X^*) \) (resp. \( f(G) \in DβO(X^*), f(G) \in BβO(X^*) \)) whenever \( G \) is an open subset of \((X, τ)\).
It is evident that every $x$-$\beta$-open map is an $\beta$-open map for $x= I,D,B$ and that every $B$-$\beta$-open map is both $I$-$\beta$-open and $D$-$\beta$-open.

Following example shows that a $\beta$-open map need not be a $x$-$\beta$-open map for $x= I,D,B$.

**EXAMPLE 3.4.01.** Let $X =\{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}$ and $\leq =\{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. Let $f$ be the identity map from $(X, \tau, \leq)$ onto itself. Since $f$ is the identity map, every open set in $X$ is mapped onto an open set and hence it is a $\beta$-open set in $X^*$ (Since every open set is a $\beta$-open set). Therefore $f$ is a $\beta$-open map.

$\{b\}$ is an open set in $(X, \tau)$, $f(\{b\}) = \{b\}$, $i(\{b\}) = \{b, c\} \neq \{b\}$. \implies f(\{b\}) \notin I\alpha O(X^*). \implies f$ is not an $I$-$\beta$-open map. $d(\{b\}) =\{a, b\} \neq \{b\}$. \implies f(\{b\}) \notin I\alpha O(X^*) \implies f$ is not an $I$-$\beta$-open map. $d(\{b\}) = \{a, b\} \neq \{b\}$. \implies f(\{b\}) \notin D\alpha O(X). Therefore $f$ is not a $D$-$\beta$-open map and consequently $f$ is not a $B$-$\beta$-open map.
Thus a $\beta$-open map need not be a $x$-$\beta$-open map for $x = I, D, B$.

The following example shows that a $D$-$\beta$-open map need not be a $B$-$\beta$-open map.

**EXAMPLE 3.4.02.** Let $X = \{a, b, c\} = X^*$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} = \tau^*$, $\leq = \{(a, a), (b, b), (c, c), (a, c)\}$ and $\leq^* = \{(a, a), (b, b), (c, c), (a, c), (b, c)\}$. Let $f$ be the identity map from $(X, \tau, \leq)$ onto $(X^*, \tau^*, \leq^*)$. Since $f$ is the identity map, every open set in $(X, \tau)$ is mapped onto an open set in $(X^*, \tau^*)$ and hence it is a $\beta$-open set in $(X^*, \tau^*)$ (Since every open set is a $\beta$-open set). Therefore $f$ is a $\beta$-continuous map. We have $d(\emptyset) = \emptyset$, $d(X) = X$, $d(\{a\}) = \{a\}$, $d(\{b\}) = \{b\}$, $d(\{a, b\}) = \{a, b\}$. $\Rightarrow f(G) \in D\beta O(X^*)$, for every open set $G$ in $(X, \tau)$. $\Rightarrow f$ is a $D$-$\beta$-open map.

$\{a\}$ is an open set, $f(\{a\}) = \{a\}$. $i(\{a\}) = \{a, c\} \neq \{a\}$. $\Rightarrow f(\{a\}) \notin I\beta O(X^*)$. $\Rightarrow f$ is not an $I$-$\beta$-open map and hence $f$ is not a $B$-$\beta$-open map.

Thus a $D$-$\beta$-open map need not be a $B$-$\beta$-open map.
EXAMPLE 3.4.03. Let \( X = \{a, b, c\} = X^*, \tau = \{\emptyset, X, \{a\}, \{a, c\}\} = \tau^* \) and \( \leq = \{(a, a), (b, b), (c, c), (c, a), (b, c), (b, a)\} = \leq^* \). Let \( f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*) \) be the identity map. Since \( f \) is the identity map, every open set in \((X, \tau)\) is mapped onto an open set in \((X^*, \tau^*)\) and hence it is a \( \beta \)-open set in \((X^*, \tau^*)\) (Since every open set is a \( \beta \)-open set). Therefore \( f \) is a \( \beta \)-continuous map. We have \( i(\emptyset) = \emptyset, i(X^*) = X^*, i(\{a\}) = \{a\}, i(\{a, c\}) = \{a, c\} \). \( \Rightarrow \) \( f(G) \in I\beta O(X^*) \), for every open set \( G \) in \((X, \tau)\). \( \Rightarrow \) \( f \) is an \( I-\beta \)-open map.

\( \{a\} \) is an open set in \((X, \tau)\), \( f(\{a\}) = \{a\}, d(\{a\}) = \{a, b, c\} \neq \{a\} \) is not a \( D-\beta \)-open set in \((X^*, \tau^*)\). \( \Rightarrow \) \( f(\{a\}) \notin D\beta O(X^*) \). \( \Rightarrow \) \( f \) is not a \( D-\beta \)-open map and hence \( f \) is not a \( B-\beta \)-open map.

Thus an \( I-\beta \)-open map need not be a \( B-\beta \)-open map.
3.4.01 The above observations are given in the following diagram.

For a function \( f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*) \)

![Diagram showing relationships between open, \(d\beta\)-open, \(i\beta\)-open, \(b\beta\)-open, and \(\beta\)-open functions.]

Following results are trivially obtained parallel to that of pre-open maps.

**LEMMA 3.4.01.** Let \( A \) be any subset of a topological ordered space \((X, \tau, \leq)\). Then

1) \( C(d\beta cl(A)) = (C(A))^{i\beta_0} \).
2) \( C(i\beta cl(A)) = (C(A))^{d\beta_0} \).
3) \( C(b\beta cl(A)) = (C(A))^{b\beta_0} \).

Following theorem characterizes \(I\)-\(\beta\)-open functions.
THEOREM 3.4.01. For any function \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \), the following statements are equivalent.

1) \( f \) is an I-\( \beta \)-open map.

2) \( f(A^0) \subseteq [f(A)]^{i\beta_0} \) for any \( A \subseteq X \).

3) \( [f^{-1}(B)]^0 = f^{-1}(B^{i\beta_0}) \) for any \( B \subseteq X^* \).

We can obtain the following two theorems that give characterizations for D-\( \beta \)-open map and B-\( \beta \)-open maps trivially.

THEOREM 3.4.02. For any function \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \), the following statements are equivalent.

1) \( f \) is D-\( \beta \)-open map.

2) \( f(A^0) \subseteq [f(A)]^{d\beta_0} \) for any \( A \subseteq X \).

3) \( [f^{-1}(B)]^0 \subseteq f^{-1}(B^{d\beta_0}) \) for any \( B \subseteq X^* \).

THEOREM 3.4.03. For any function \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \), the following statements are equivalent.

1) \( f \) is B-\( \beta \)-open map.

2) \( [f(A^0)] \subseteq [f(A)]^{b\beta_0} \) for any \( A \subseteq X \).

3) \( [f^{-1}(B)]^0 \subseteq f^{-1}(B^{b\beta_0}) \) for any \( B \subseteq X^* \).
THEOREM 3.4.04. Let $f : (X, \tau, \leq_1) \rightarrow (Y, \sigma, \leq_2)$ and $g : (Y, \sigma, \leq_2) \rightarrow (Z, \eta, \leq_3)$ be any two mappings. Then $gof : (X, \tau, \leq_1) \rightarrow (Z, \eta, \leq_3)$ is $x$-β-open if $f$ is open and $g$ is $x$-β-open for $x = I, D, B.$

THEOREM 3.4.05. Every I-semi-open map is an I-β-open map.

Proof. Let $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be an I-semi-open map. Let $G$ be an open set in $X.$

$f(G)$ is an I-semi-open set in $X^*.$ $=>$ $C(f(G))$ is a D-semi-closed set in $X^*.$ $=> C(f(G))$ is a D-β-closed set in $X^*\, (\text{from Lemma 1.1.2}).$ $=>$ $f(G)$ is I-β-open set in $X^*.$ Therefore $f$ is I-β-open map.

Following example shows that an I-β-open map need not be an I-semi-open map. It needs reference from example 1.1.03.

EXAMPLE 3.4.04. Let $X = \{a, b, c\} = X^*, \tau = \{\phi, X, \{a\}, \{b, c\}\} = \tau^*$ and $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*.$ Define a map $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ by $f(a) = b,$ $f(b) = c$ and $f(c) = a.$ $\phi$ is the open set in $(X, \tau),$ $f(\phi) = \phi$ is a β-open set in $(X^*, \tau^*),$ $i(\phi) = \phi.$ $X$ is the open
set in \((X, \tau), f(X) = X^*\) is a \(\beta\)-open set in \((X^*, \tau^*)\), \(i(X^*) = X^*\). \(\{a\}\) is an open set in \((X, \tau)\), 
f(\{a\}) = \{b\} is a \(\beta\)-open set in \((X^*, \tau^*)\), \(i(\{b\}) = \{b\}\). \(\{b, c\}\) is an open set in \((X, \tau)\), 
f(\{b, c\}) = \{c, a\} is a \(\beta\)-open set in \((X^*, \tau^*)\), \(i(\{a, c\}) = \{a, c\}\). \(\Rightarrow f(G) \in \mathbb{I}\beta O(X^*)\), for every open set \(G\) in \((X, \tau)\). \(\Rightarrow f\) is an \(I, \beta\)-open map.

\(\{a\}\) is an open set in \((X, \tau)\), \(f(\{a\}) = \{b\}\) is not a semi-open set in \((X^*, \tau^*)\). Therefore \(f\) is not a semi-open map and consequently \(f\) is not an \(I,\)semi-open map.

Thus an \(I,\)\(\beta\)-open map need not be an \(I,\)semi-open map

**THEOREM 3.4.06.** Every \(I\)-pre-open map is an \(I,\)\(\beta\)-open map.

**Proof.** Let \(f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)\) be an \(I\)-pre-open map. Let \(G\) be an open set in \(X\). \(\Rightarrow f(G)\) is an \(I\)-pre-open set in \(X^*\). \(\Rightarrow C(f(G))\) is a \(D\)-pre-closed set in \(X^*\). \(\Rightarrow C(f(G))\) is a \(D,\)\(\beta\)-closed set in \(X^*\) (from Lemma 1.1.3). \(\Rightarrow f(G)\) is \(I, \beta\)-open set in \(X^*\). Therefore \(f\) is an \(I,\)\(\beta\)-open map.
Following example shows that an I-\(\beta\)-open map need not be an I-pre-open map. It needs reference from example 1.1.01.

**EXAMPLE 3.4.05.** Let \(X = \{a, b, c\} = X^*, \tau = \{\emptyset, X, \{a\}, \{b, c\}\}, \tau^* = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \) and \(\leq = \{(a, a), (b, b), (c, c)\} = \leq^*\). Define a map \(f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)\) by \(f(a) = b, \ f(b) = c\) and \(f(c) = a\). \(\emptyset\) is the open set in \((X, \tau)\), \(f(\emptyset) = \emptyset\) is a \(\beta\)-open set in \((X^*, \tau^*)\), \(i(\emptyset) = \emptyset\). \(X\) is the open set in \((X, \tau)\), \(f(X) = X^*\) is a \(\beta\)-open map in \((X^*, \tau^*)\). \(i(X^*) = X^*\). \(\{a\}\) is an open set in \((X, \tau)\), \(f(\{a\}) = \{b\}\) is a \(\beta\)-open set in \((X^*, \tau^*)\), \(i(\{b\}) = \{b\}\). \(\{b, c\}\) is an open set in \((X, \tau)\), \(f(\{b, c\}) = \{a, c\}\) is a \(\beta\)-open set in \((X^*, \tau^*)\), \(i(\{a, c\} = \{a, c\}, \Rightarrow f(\{a, c\}) \in I\beta O(X^*), \) for every open set \(G\) in \((X, \tau)\). \(\Rightarrow f\) is an I-\(\beta\)-open map.

\(\{b, c\}\) is an open set in \((X, \tau)\), \(f(\{b, c\}) = \{a, c\}\) is not a pre-open set in \((X^*, \tau^*)\). Therefore \(f\) is not a pre-open map and consequently \(f\) is not an I-pre-open map.
Thus an I-β-open map need not be a I-pre-open map.

**Theorem 3.4.07.** Every I-α-open map is an I-β-open map.

**Proof.** Let \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) be an I-α-open map. Let \( G \) be an open set in \( X \). \( \Rightarrow \) \( f(G) \) is an I-α-open set in \( X^* \). \( \Rightarrow \) \( C(f(G)) \) is a D-α-closed set in \( X^* \).

\( \Rightarrow \) \( C(f(G)) \) is a D-β-closed set in \( X^* \) (from Lemma 1.1.4). \( \Rightarrow \) \( f(G) \) is an I-β-open set in \( X^* \). Therefore \( f \) is an I-β-open map.

Following example shows that an I-β-open map need not be an I-α-open map. It needs reference from example 1.1.03.

**Example 3.4.06.** Let \( X = \{a, b, c\} = X^* \), \( \tau = \{\phi, X, \{a\}, \{b, c\}\} = \tau^* \) and \( \leq = \{(a, a), (b, b), (c, c)\} = \leq^* \).
Define a map \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) by \( f(a) = b \), \( f(b) = c \) and \( f(c) = a \). \( \phi \) is the open set in \( (X, \tau) \), \( f(\phi) = \phi \) is a β-open set in \( (X^*, \tau^*) \), \( i(\phi) = \phi \). \( X \) is the open set in \( (X, \tau) \), \( f(X) = X^* \) is a β-open set in \( (X^*, \tau^*) \), \( i(X^*) = X^* \). \( \{a\} \) is an open set in \( (X, \tau) \), \( f(\{a\}) = \{b\} \) is a β-
open set in \((X^*, \tau^*)\), \(i(\{b\}) = \{b\}\). \(\{b, c\}\) is an open set in \((X, \tau)\), \(f(\{b, c\}) = \{a, c\}\) is a \(\beta\)-open set in \((X^*, \tau^*)\), \(i(\{a, c\}) = \{a, c\}\). \(\Rightarrow\) \(f(G) \in I\beta O(X^*)\), for every open set \(G\) in \((X, \tau)\). \(\Rightarrow\) \(f\) is an I-\(\beta\)-open map.

\(\{a\}\) is an open set in \((X, \tau)\), \(f(\{a\}) = \{b\}\) is not an \(\alpha\)-open set in \((X^*, \tau^*)\). \(\Rightarrow\) \(f(\{a\}) \notin I\alpha O(X^*)\). Therefore \(f\) is not an \(\alpha\)-open map and consequently \(f\) is not an I-\(\alpha\)-open map.

Thus an I-\(\beta\)-open map need not be an I-\(\alpha\)-open map.

**THEOREM 3.4.08.** Every \(D\)-semi-open map is a \(D\)-\(\beta\)-open map.

**Proof.** Let \(f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)\) be an \(D\)-semi-open map. Let \(G\) be an open set in \(X\). \(\Rightarrow\) \(f(G)\) is a \(D\)-semi-open set in \(X^*\). \(\Rightarrow\) \(C(f(G))\) is an I-semi-closed set in \(X^*\). \(\Rightarrow\) \(C(f(G))\) is an I-\(\beta\)-closed set in \(X^*\) (from Lemma 1.1.2). \(\Rightarrow\) \(f(G)\) is a \(D\)-\(\beta\)-open set in \(X^*\). Therefore \(f\) is a \(D\)-\(\beta\)-open map.
Following example shows that a D-β-open map need not be a D-semi-open map. It needs reference from example 1.1.03.

**EXAMPLE 3.4.07.** Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b, c\}\} = \tau^*$ and $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Define a map $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. From example 3.4.04 $f$ is a β-open map. We have $d(\phi) = \phi$, $d(X^*) = X^*$, $d(\{b\}) = \{b\}$, $d(\{a, c\}) = \{a, c\}$. $\Rightarrow f(G) \in D\beta O(X^*)$, for any open set $G$ in $(X, \tau)$. $\Rightarrow f$ is a D-β-open map.

$\{a\}$ is an open set in $(X, \tau)$, $f(\{a\}) = \{b\}$ is not a semi-open set in $(X^*, \tau^*)$. $\Rightarrow f(\{a\}) \notin DSO(X^*). \Rightarrow f$ is not a semi-open map and hence $f$ is not a D-semi-open map.

Thus a D-β-open map need not be a D-semi-open map.

**THEOREM 3.4.09.** Every D-pre-open map is a D-β-open map.

**Proof.** Let $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ be an D-pre-open map. Let $G$ be an open set in $X$. $\Rightarrow f(G)$ is a D-
pre-open set in $X^*$. $\Rightarrow C(f(G))$ is an I-pre-closed set in $X^*$. $\Rightarrow C(f(G))$ is an I- $\beta$-closed set in $X^*$ (from Lemma 1.1.3). $\Rightarrow f(G)$ is a D- $\beta$-open set in $X^*$. Therefore $f$ is a D- $\beta$-open map.

Following example shows that a D- $\beta$-open map need not be a D-pre-open map. It needs reference from example 1.1.01.

**EXAMPLE 3.4.08.** Let $X=\{a, b, c\}=X^*$, $\tau=\{\emptyset, X, \{a\}, \{b, c\}\}$, $\tau^*=\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Define a map $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. From example 3.4.05 $f$ is a $\beta$-open map. We have $d(\emptyset) = \emptyset$, $d(X^*) = X^*$, $d(\{b\}) = \{b\}$, $d(\{a, c\}) = \{a, c\}$. $\Rightarrow f(G) \in D\beta O(X^*)$, for every open set $G$ in $(X, \tau)$. Therefore $f$ is a D- $\beta$-open map.

$\{b, c\}$ is an open set in $(X, \tau)$, $f(\{b, c\}) = \{a, c\}$ is not a pre-open set in $(X^*, \tau^*)$. Therefore $f$ is not a pre-open map and consequently $f$ is not a D-pre-open map.
Thus a D-β-open map need not be a D-pre-open map.

**THEOREM 3.4.10.** Every D-α-open map is a D-β-open map.

**Proof.** Let $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ be a D-α-open map. Let $G$ be an open set in $X$. $f(G)$ is a D-α-open set in $X^*$. $C(f(G))$ is an I-α-closed set in $X^*$. $C(f(G))$ is an I-β-closed set in $X^*$ (from Lemma 1.1.4). Therefore $f$ is a D-β-open map.

Following example shows that a D-β-open map need not be a D-α-open map. It needs reference from example 1.1.03.

**EXAMPLE 3.4.09.** Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$, $\tau^* = \{(a, a), (b, b), (c, c)\}$, $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Define a map $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. From example 3.4.06 $f$ is a β-open map. We have $d(\phi) = \phi$, $d(X^*) = X^*$, $d(\{b\}) = \{b\}$.
\{b\}, d(\{a, c\}) = \{a, c\} f(G) \in D\beta O(X^*) for every open set G in (X, \tau). \Rightarrow f is a D-\beta-open map.

\{a\} is an open set, f(G) = f(\{a\}) = \{b\} is not an \alpha-open set in (X^*, \tau^*). Therefore f is not an \alpha-open map and consequently f is not a D-\alpha-open map.

Thus a D-\beta-open map need not be a D-\alpha-open map.

**THEOREM 3.4.11.** Every B-semi-open map is a B-\beta-open map.

**Proof.** Let f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) be a B-semi-open map. \Rightarrow f is I-semi-open map and \ D-semi-open map. Since f is I-semi-open map by theorem 3.4.05 f is I-\beta-open map. Since f is D-semi-open map by theorem 3.4.08 f is D-\beta-open map. Therefore f is \ B-\beta-open map.

Following example shows that a B-\beta-open map need not be a B-semi-open map. It needs reference from example 1.1.03.

**EXAMPLE 3.4.10.** Let X = \{a, b, c\} = X^*, \tau = \{\phi, X, \{a\}, \{b, c\}\} = \tau^* and \leq = \{(a, a), (b, b),
Define a map \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) by \( f(a) = b, f(b) = c \) and \( f(c) = a \). From example 3.4.04 \( f \) is an I-\( \beta \)-open map and from example 3.4.07 \( f \) is a D-\( \beta \)-open map. \( \Rightarrow \) \( f \) is a B-\( \beta \)-open map.

\{a\} is an open set in \((X, \tau)\), \( f(\{a\}) = \{b\} \) is not a semi-open set in \((X^*, \tau^*)\). Therefore \( f \) is not a semi-open map and consequently \( f \) is not a B-semi-open map.

Thus a B-\( \beta \)-open map need not be a B-semi-open map.

**THEOREM 3.4.12.** Every B-pre-open map is a B-\( \beta \)-open map.

**Proof.** Let \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) be a B-pre-open map. \( \Rightarrow \) \( f \) is I-pre-open map and D-pre-open map. Since \( f \) is an I-pre-open map by theorem 3.4.06 \( f \) is I-\( \beta \)-open map. Since \( f \) is D-pre-open map by theorem 3.4.09 \( f \) is D-\( \beta \)-open map. Therefore \( f \) is B-\( \beta \)-open map.
Following example shows that a $\text{B-}\beta$-open map need not be a $\text{B-pre-opening}$ map. It needs reference from example 1.1.01.

**EXAMPLE 3.4.11.** Let $X = \{a, b, c\} = X^*$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\tau^* = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Define a map $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. From example 3.4.05 $f$ is an $I$-$\beta$-open map and from example 3.4.08 $f$ is a $D$-$\beta$-open map. $\Rightarrow f$ is a $B$-$\beta$-open map.

$\{b, c\}$ is an open set in $(X, \tau)$, $f(\{b, c\} = \{a, c\}$ is not a pre-open set in $(X^*, \tau^*)$. Therefore $f$ is not a pre-open map and consequently $f$ is not a $\beta$-pre-open map.

Thus a $B$-$\beta$-open map need not be a $B$-pre-open map.

**THEOREM 3.4.13.** Every $\text{B-}\alpha$-open map is a $\text{B-}\beta$-open map.

**Proof.** Let $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ be a $\text{B-}\alpha$-open map. $\Rightarrow f$ is an $I$-$\alpha$-open map and a
D-α-open map. Since \( f \) is an I-α-open map by theorem 3.4.07 \( f \) is an I-β-open map. Since \( f \) is a D-α-open map by theorem 3.4.10 \( f \) is a D-β-open map. Therefore \( f \) is a B-β-open map.

Following example shows that a B-β-open map need not be a B-α-open map. It needs reference from example 1.1.03.

**EXAMPLE 3.4.12.** Let \( X = \{a, b, c\} = X^* \), \( \tau = \{\phi, X, \{a\}, \{b, c\}\} = \tau^* \) and \( \leq = \{(a, a), (b, b), (c, c)\} = \leq^* \). Define a map \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) by \( f(a) = b \), \( f(b) = c \) and \( f(c) = a \). From example 3.4.06 \( f \) is an I-β-open map and from example 3.4.09 \( f \) is a D-β-open map. \( \Rightarrow f \) is a B-β-open map.

\{a\} is an open set in \( (X, \tau) \), \( f(\{a\}) = \{b\} \) is not an α-open set in \( (X^*, \tau^*) \). Therefore \( f \) is not an α-open map and consequently \( f \) is not a B-α-open map.

Thus a B-β-open map need not be a B-α-open map.