CHAPTER – 2

SEMI-CONTINUOUS, PRE-CONTINUOUS, α-CONTINUOUS, β-CONTINUOUS MAPPINGS
IN TOPOLOGICAL ORDERED SPACES.

In this chapter we introduce x-semi-continuous, x-pre-continuous x-α-continuous, x-β-continuous mappings for topological ordered spaces together with their characterizations for x=I,D,B where I,D,B represent increasing, decreasing, balanced respectively.

2.1 I-SEMI-CONTINUOUS, D-SEMI-CONTINUOUS AND B-SEMI-CONTINUOUS MAPS.

INTRODUCTION:

Levine introduced semi-open sets and semi-closed sets[8]. Note that the complement of a
semi-open set is a semi-closed set and vice versa. We denote the complement of a subset A of X by C(A). For a subset A of a topological ordered space (X, τ, ≤), we define
\[ \text{iscl}(A) = \cap \{F/F \text{ is an increasing semi-closed subset of } X \text{ containing } A\}, \]
\[ \text{dscl}(A) = \cap \{F/F \text{ is a decreasing semi-closed subset of } X \text{ containing } A\}, \]
\[ \text{bscl}(A) = \cap \{F/F \text{ is a balanced semi-closed subset of } X \text{ containing } A\}, \]
Clearly iscl(A) (resp. dscl(A), bscl(A)) is the smallest increasing (resp. decreasing, balanced) semi-closed set containing A.

ISO(X) (resp. DSO(X), BSO(X)) denotes the collection of all increasing (resp. decreasing, balanced) semi-open subsets of a topological ordered space (X, τ, ≤). ISC(X) (resp. DSC(X), BSC(X)) denotes the collection of all increasing (resp. decreasing, balanced) semi-closed subsets of a topological ordered spaced (X, τ, ≤).
We introduce the following.

**DEFINITION 2.1.01.** A function \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) is called I–semi-continuous [5] (resp. D-semi-continuous, B-semi-continuous) map if 
\[
1 - f(G) \in \text{ISO}(X) \quad \text{(resp.} \quad f^{-1}(G) \in \text{DSO}(X), 
\]
\[
f^{-1}(G) \in \text{BSO}(X)) \quad \text{whenever} \ G \text{ is an open set of } (X^*, \tau^*).
\]

It is evident that every \( x \)-semi-continuous map is semi-continuous for \( x = I, D, B \).

Every B–semi-continuous map is both I–semi-continuous and D–semi-continuous.

Following example shows that a semi-continuous map need not be a \( x \)-semi-continuous for \( x = I, D, B \).

**EXAMPLE 2.1.01.** Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \) and \( \leq = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\} \). Clearly \( (X, \tau, \leq) \) is a topological ordered space.

Let \( f \) be the identity map from \( (X, \tau, \leq) \) onto itself.

Since \( f \) is the identity map from \( (X, \tau, \leq) \) on to itself, the inverse image of an open set is an open set and
hence it is a semi-open set (Since every open set is a semi-open set). Thus \( f \) is semi-continuous map. \( \{b\} \) is an open set in \( X \). \( f^{-1}(\{b\}) = \{b\}, \ i(\{b\}) = \{b, c\} \neq \{b\} \). Therefore \( \{b\} \) is not an increasing semi-open set in \((X, \tau)\). Thus \( f \) is not an I-semi-continuous map. \( d(\{b\}) = \{b, a\} \neq \{b\}\). Therefore \( f \) is not a decreasing semi-open set in \((X, \tau)\). Thus \( f \) is not a D-semi-continuous map and hence \( f \) is not a B-semi-continuous map.

Thus a semi-continuous map need not be a \( x \)-semi-continuous map for \( x=I, D, B \).

**EXAMPLE 2.1.02.** Let \( X = \{a, b, c\} = X^* \), \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} = \tau^*, \leq = \{(a, a), (b, b), (c, c), (b, c), (a, c)\} \), \( \leq^* = \{(a, a), (b, b), (c, c), (a, c), (b, c)\} \). Let \( f \) be the identity map from \((X, \tau, \leq)\) on to \((X^*, \tau^*, \leq^*)\). \( \emptyset \) is an open set in \((X^*, \tau^*)\). \( f^{-1}(\emptyset) = \emptyset \) is a semi-open set in \((X, \tau)\). \( d(\emptyset) = \emptyset \). \( X^* \) is the open set in \((X^*, \tau^*)\). \( f^{-1}(X^*) = X \) is a semi-open set in \((X, \tau)\). \( d(X) = X \). \( \{a\} \) is an open set in \((X^*, \tau^*)\). \( f^{-1}(\{a\}) = \{a\} \) is a semi-open set
in \((X, \tau)\), \(d([a]) = \{a\}\). \([b]\) is an open set in \((X^*, \tau^*)\).

\(f^{-1}([b]) = \{b\}\) is a semi-open set in \((X, \tau)\), \([a, b]\) is an open set in \((X^*, \tau^*)\). \(f^{-1}([a, b]) = \{a, b\}\) is a semi-
open set in \((X, \tau)\), \(d([a, b]) = \{a, b\}\).

\(\Rightarrow f^{-1}(G) \in DSO(X)\), for every open subset \(G\) of
\((X^*, \tau^*)\). Therefore \(f\) is a \(D\)-semi-continuous map.

\([a]\) an open subset of \((X^*, \tau^*)\), \(f^{-1}([a]) = \{a\}\),
\(i([a]) = \{a, c\} \neq \{a\}\). \(\Rightarrow f^{-1}([a])\) is not an increasing
semi-open set of \((X, \tau)\). \(\Rightarrow f^{-1}([a]) \notin ISO(X)\).

Therefore \(f\) is not an I-semi-continuous map and hence \(f\) is not a \(B\)-semi-continuous map. Thus a \(D-
semi-continuous\) map need not be a \(B\)-semi-
continuous map.

**EXAMPLE 2.1.03.** Let \(X = \{a, b, c\} = X^*\), \(\tau = \{\emptyset, X, \{a\},
\{b\}, \{a, b\}\) and \(\tau^* = \{\emptyset, X^*, \{a\}\}\). Let \(\leq\ = \{(a, a), (b, b), (c,
c), (a, b), (c, b)\} = \leq^*\) Define \(f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)\)
by \(f(a) = b\), \(f(b) = a\) and \(f(c) = c\). \(\emptyset\) is an open set
in \((X^*, \tau^*)\). \(f^{-1}(\emptyset) = \emptyset\) is a semi-open set in
\((X, \tau)\), \(i(f^{-1}(\emptyset)) = i(\emptyset) = \emptyset\). \(X^*\) is an open set in \((X^*,
\tau^*)\).
\( f^{-1} \left( X^* \right) = \{a, b, c\} = X \) is a semi-open set in \( (X, \tau) \), \( i \left( f^{-1} \left( X^* \right) \right) = i(X) = \{a, b, c\} = X. \) \{a\} is an open set in \( (X^*, \tau^*) \), \( f^{-1} \left( \{a\} \right) = \{b\} \) is a semi-open set in \( (X, \tau) \), \( i(f^{-1}(\{a\})) = i(\{b\}) = \{b\}. \) Therefore \( f^{-1}(G) \in \text{ISO}(X) \), whenever \( G \) is an open set in \( (X^*, \tau^*) \). Therefore \( f \) is an I-semi-continuous map.

\( \{a\} \) is an open set in \( (X^*,\tau^*) \). \( f^{-1}(a) = \{b\} \), \( d(\{b\}) = \{a, b, c\} \neq \{b\}. \Rightarrow f^{-1}(\{a\}) \notin \text{DSO}(X). \) Therefore \( f \) is not a D-semi-continuous map and hence \( f \) is not a B-semi-continuous map. Thus an I-semi-continuous map need not be a B-semi-continuous map.

2.01 The above observations are given in the following diagram.

For a function \( f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \)
THEOREM 2.1.01. For a function $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$, the following statements are equivalent.

1. $f$ is I-semi-continuous.
2. $f(\text{dscl}(A)) \subseteq \text{cl}(f(A))$ for any $A \subseteq X$.
3. $\text{dscl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$ for any $B \subseteq X^*$.
4. For every closed subset $K$ of $(X^*, \tau^*, \leq^*)$ 
   \[ f^{-1}(K) \text{ is decreasing semi-closed subset of } (X, \tau, \leq). \]

Proof. (1) $\Rightarrow$ (2) \because $C(\text{cl}(f(A)))$ is open in $X^*$ and $f$ is I-semi-continuous then $f^{-1}(C(\text{cl}(f(A))))$ is an increasing semi-open set in $X$. Then $C(f^{-1}(C(\text{cl}(f(A))))))$ is a decreasing semi-closed subset in $X$. Since $C(f^{-1}(C(\text{cl}(f(A)))))) = f^{-1}(\text{cl}(f(A)))$ then $f^{-1}(\text{cl}(f(A)))$ is decreasing semi-closed subset
of $X$. Since $A \subseteq f^{-1}(\text{cl}(f(A)))$ and $\text{dscl}(A)$ is the smallest decreasing semi-closed set containing $A$, $\text{dscl}(A) \subseteq f^{-1}(\text{cl}(f(A)))$. Since $f(f^{-1}(\text{cl}(f(A)))) \subseteq \text{cl}(f(A))$, we have $f(\text{dscl}(A)) \subseteq \text{cl}(f(A))$.

**(2) ⇒ (3)** Let $A = f^{-1}(B)$ then $f(A) = f(f^{-1}(B)) \subseteq B$. This implies $\text{cl}(f(A)) \subseteq \text{cl}(B)$. Now $\text{dscl}(f^{-1}(B)) = \text{dscl}(A) \subseteq f^{-1}(f(\text{dscl}(A))) \subseteq f^{-1}(\text{cl}(f(A)))$ (by (2))

But $f^{-1}(\text{cl}(f(A))) \subseteq f^{-1}(\text{cl}(B))$. Thus $\text{dscl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$.

**(3) ⇒ (4)** By (3), we have $\text{dscl}(f^{-1}(K)) \subseteq f^{-1}(\text{cl}(K))$. 

$$\Rightarrow \text{dscl}(f^{-1}(K)) \subseteq f^{-1}(K).$$

But $f^{-1}(K) \subseteq \text{dscl}(f^{-1}(K))$. Thus $f^{-1}(K)$ is decreasing semi-closed set in $(X, \tau, \leq)$ whenever $K$ is closed in $(X^*, \tau^*, \leq^*)$.

**(4) ⇒ (1)** Let $G$ be an open set in $(X^*, \tau^*)$. Then $f^{-1}(C(G))$ is a decreasing semi-closed set in $(X, \tau)$.
since $C(G)$ is a closed set in $(X^*, \tau^*)$. But $C(f^{-1}(G)) = f^{-1}(C(G))$. Thus $C(f^{-1}(G))$ is a decreasing semi-closed set in $(X, \tau, \leq)$. So $f^{-1}(G)$ is an increasing semi-open set in $(X, \tau, \leq)$. Thus $f$ is I-semi-continuous.

In the light of above theorem the following characterizations of D-semi-continuous, B-semi-continuous maps are obtained trivially.

**THEOREM 2.1.02.** For a function $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$, the following statements are equivalent.

1). $f$ is D-semi-continuous.
2). $f(\text{iscl}(A)) \subseteq \text{cl}(f(A))$ for any $A \subseteq X$.
3). $\text{iscl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$ for any $B \subseteq X^*$.
4). For every closed subset $K$ of $(X^*, \tau^*, \leq^*)$, $f^{-1}(K)$ is an increasing semi-closed subset of $(X, \tau, \leq)$.

**THEOREM 2.1.03.** For a function $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ the following statements are equivalent.
1). $f$ is B–semi-continuous.
2). $f(bscl(A)) \subseteq cl(f(A))$ for any $A \subseteq X$.
3). $bscl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for any $B \subseteq X^*$. 
4). For every closed subset $K$ of $(X^*, \tau^*, \leq^*)$, $f^{-1}(K)$ is balanced semi-closed subset of $(X, \tau, \leq)$.

### 2.2 I-PRE-CONTINUOUS, D-PRE-CONTINUOUS AND B-PRE-CONTINUOUS MAPS

**INTRODUCTION.** A.S. Mashour [12] introduced pre-open sets and pre-closed sets. Note that the complement of pre-open set is pre-closed and vice versa. We denote the complement of $A$ by $C(A)$.

For a subset $A$ of a topological ordered space $(X, \tau, \leq)$ define

$$ipcl(A) = \bigcap\{F/F \text{ is an increasing pre-closed subset of } X \text{ containing } A\},$$

$$dpcl(A) = \bigcap\{F/F \text{ is a decreasing pre-closed subset of } X \text{ containing } A\},$$
bpcl(A) = ∩{F/F is a balanced pre-closed subset of X containing A},
Clearly ipcl(A) (resp. dpcl (A), bpcl (A)) is the smallest increasing (resp. decreasing, balanced) pre-closed set containing A. IPO(X) (resp. DPO(X), BPO(X)) denotes the collection of all increasing (resp. decreasing, balanced) pre-open subset of a topological ordered space (X, τ, ≤ ). IPC(X) (resp. DPC(X), BPC(X)) denotes the collection of all increasing (resp. decreasing, balanced) pre-closed subsets of a topological ordered space (X, τ, ≤ ).

We introduce the following .

**DEFINITION 2.2.01.** A function f: (X, τ, ≤) → (X*, τ*, ≤*) is called a I-pre-continuous [4] (resp. D-pre-continuous, B-pre-continuous) maps if 
\[ f^{-1}(V) ∈ IPC(X), \text{ (resp. } f^{-1}(V) ∈ DPC(X), f^{-1}(V) ∈ BPC(X) \text{) whenever } V \text{ is closed in } X. \]

It is evident that every x-pre-continuous map is pre-continuous for x = I, D,B and that every
B-pre-continuous map is both I-pre-continuous and D-pre-continuous.

Following example shows that a pre-continuous map need not be x-pre-continuous for x = I,D,B.

**EXAMPLE 2.2.01.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\leq = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. Let $f$ be the identity map from $(X, \tau, \leq)$ onto itself. Since $f$ is the identity mapping, the inverse image of a closed set is a closed set and hence it is a pre-closed set (since every closed set is a pre-closed set). Therefore $f$ is a pre-continuous mapping.

$\{a, c\}$ is a closed set in $(X, \tau)$, $f^{-1}(\{a, c\}) = \{a, c\}$, 
$i(f^{-1}\{a, c\}) = i(\{a, c\}) = \{a, b, c\} \neq \{a, c\} = f^{-1}(\{a, c\})$.

$\Rightarrow f^{-1}(\{a, c\})$ is not an increasing pre-closed set.

Therefore $f$ is not an I-pre continuous map.

$d(f^{-1}\{a, c\}) = d(\{a, c\}) = \{a, b, c\} \neq f^{-1}(\{a, c\})$. $\Rightarrow f^{-1}(\{a, c\})$ is not a decreasing pre-closed set.
Therefore $f$ is not a $D$-pre-continuous map and hence $f$ is not a $B$-pre-continuous map.

Thus a pre-continuous map need not be a $x$-pre-Continuous map for $x=I$, $D$, $B$. Following example shows that a $D$-pre-continuous map need not be a $B$-pre-continuous. It needs reference from example 1.1.09

**EXAMPLE 2.2.02.** Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{a, b\}\} = \tau^*$ and $\leq = \{(a, a), (b, b), (c, c), (b, a)\}$, $\leq^* = \{(a, b), (b, b), (c, c)\}$. Let $f$ be the identity map from $(X, \tau, \leq)$ onto $(X^*, \tau^*, \leq^*)$. $\phi$ is the closed set in $(X^*, \tau^*)$, $f^{-1}(\phi) = \phi$ is a pre-closed set in $(X, \tau)$, $d(\phi) = \phi$. $X^*$ is the closed set in $(X^*, \tau^*)$, $f^{-1}(X^*) = X$ is a pre-closed set in $(X, \tau)$, $d(X) = X$. $\{b, c\}$ is a closed set in $(X^*, \tau^*)$, $f^{-1}(\{b, c\}) = \{b, c\}$ is a pre-closed set in $(X, \tau)$, $d(\{b, c\}) = \{b, c\}$. $\{c\}$ is a closed set in $(X^*, \tau^*)$, $f^{-1}(\{c\}) = \{c\}$ is a pre-closed set in $(X, \tau)$, $d(\{c\}) = \{c\}$. Therefore $f$ is $D$-pre-continuous map. $i(\{b, c\}) = \{a, b, c\} \neq \{b, c\}$ is not an
increasing pre-closed set in \((X, \tau, \leq)\). Therefore \(f\) is not an I-pre-continuous map and hence \(f\) is not a B-pre-continuous map.

Thus a D-pre-continuous map need not be a B-pre-continuous map.

Following example supports that an I-pre-continuous map need not be B-pre continuous map. It needs reference from example 1.1.01.

**EXAMPLE 2.2.03.** Let \(X = \{a, b, c\} = X^*, \tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} = \tau^*\) and let \(\leq = \{(a, a), (b, b), (c, c), (a, c), (b, c)\} = \leq^*\). Define \(f: (X, \tau, \leq) \to (X^*, \tau^*, \leq^*) \) is an identity map. \(\phi\) is the closed set in \((X^*, \tau^*)\), \(f^{-1}(\phi) = \phi\) is a pre-closed set in \((X, \tau)\), \(i(\phi) = \phi\). \(X^*\) is the closed set in \((X^*, \tau^*)\), \(f^{-1}(X^*) = X\) is a pre-closed set in \((X, \tau)\), \(i(X) = X\). \(\{b, c\}\) is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{b, c\}) = \{b, c\}\) is a pre-closed set in \((X, \tau)\), \(i(\{b, c\}) = \{b, c\}\). \(\{c, a\}\) is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{c, a\}) = \{c, a\}\) is a pre-closed set in \((X, \tau)\), \(i(\{c, a\}) = \{c, a\}\). \(\{c\}\) is a closed set in \((X^*, \tau^*)\),
\( f^{-1}(\{c\}) = \{c\} \) is a pre-closed set in \((X, \tau)\), \(i(\{c\}) = \{c\}\). Therefore \( f \) is an I-pre-continuous map.

\( d(\{b, c,\}) = \{a, b, c\} \neq \{b, c\}, \{b, c\} \) is not a decreasing pre-closed set in \((X, \tau, \leq)\). Therefore \( f \) is not a D-pre-continuous map and hence \( f \) is not a B-pre-continuous map.

Thus a D-pre-continuous map need not be a pre-continuous map.

2.2.01 The above observations are given in the following diagram

For a function \( f:(X, \tau, \leq) \to (X^*, \tau^*, \leq^*)\),

\[\text{\textbullet \quad f is I-pre-continuous \quad \leftrightarrow \quad f is D-pre-continuous} \]

\[\text{\textbullet \quad f is pre-continuous} \]

\[\text{\textbullet \quad f is B-pre-continuous} \]
Following theorem characterizes I-pre-continuous maps.

**THEOREM 2.2.01.** For a function \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) the following statements are equivalent.

1) \( f \) is I-pre-continuous.

2) \( f(\text{ipcl}(A)) \subseteq \text{cl}(f(A)) \) for any \( A \subseteq X \).

3) \( \text{ipcl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B)) \) for any \( B \subseteq X^* \).

4) For any closed subset \( K \) of \( (X^*, \tau^*, \leq^*) \), \( f^{-1}(K) \) is an increasing pre-closed subset of \( (X, \tau, \leq) \).

**Proof (1)\Rightarrow(2)** Since \( \text{cl}(f(A)) \) is closed in \( X^* \) and \( f \) is I-pre-continuous we have that \( f^{-1}(\text{cl}(f(A))) \) is an increasing pre-closed set in \( X \). \( f(A) \subseteq \text{cl}(f(A)) \Rightarrow A \subseteq f^{-1}(\text{cl}(f(A))) \) and \( \text{ipcl}(A) \) is the smallest increasing pre-closed set containing \( A \). Therefore \( \text{ipcl}(A) \subseteq f^{-1}(\text{cl}(f(A))) \). Here \( f(\text{ipcl}(A)) \subseteq \text{cl}(f(A)) \).

**Proof (2)\Rightarrow(3)** Put \( A = f^{-1}(B) \). Then \( f(A) \subseteq B \) and \( \text{cl}(f(A)) \subseteq \text{cl}(B) \). Therefore \( \text{ipcl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B)) \).
(3) =>(4) Let K be any closed set in X*. From (3) ipcl($f^{-1}(K)) \subseteq f^{-1}(cl(K)) = f^{-1}(K)$. Clearly $f^{-1}(K) \subseteq ipcl(f^{-1}(K))$. Thus $f^{-1}(K)$ is increasing pre-closed set in $(X, \tau, \leq)$ whenever K is closed in $(X^*, \tau^*, \leq^*)$.

(4) =>(1) follows from definition.

In the light of above theorem the following two theorems 3.2.02, 3.2.03 are trivial.

**THEOREM 2.2.02.** For a function $f:(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$, the following statements are equivalent.

1) f is D-pre-continuous.

2) $f(dpcl(A)) \subseteq cl(f(A))$ for any $A \subseteq X$.

3) $dpcl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for any $B \subseteq X^*$.

4) for every closed subset K of $(X^*, \tau^*, \leq^*)$,
   
   $f^{-1}(K)$ is a decreasing pre-closed subset 
   
   of $(X, \tau, \leq)$.

**THEOREM 2.2.03.** For a function $f:(X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$, the following statements are equivalent.

1) f is B-pre-continuous.
2) $f(\text{bpcl}(A)) \subseteq \text{cl}(f(A))$ for any $A \subseteq X$.

3) $\text{bpcl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$ for any $B \subseteq X^*$.

4) For every closed subset $K$ of $(X^*, \tau^*, \leq^*)$, $f^{-1}(K)$ is both increasing and decreasing pre-closed subset of $(X, \tau, \leq)$.

2.3 I-$\alpha$-CONTINUOUS, D-$\alpha$-CONTINUOUS AND B-$\alpha$-CONTINUOUS MAPS

we introduce I-$\alpha$-continuous maps, D-$\alpha$-continuous maps and B-$\alpha$-continuous maps.


For a subset $A$ of a topological ordered space $(X, \tau, \leq)$, we define

\[
i_{\alpha}\text{cl}(A) = \bigcap\{F/F \text{ is an increasing } \alpha\text{-closed subset of } X \text{ containing } A\},
\]

\[
d_{\alpha}\text{cl}(A) = \bigcap\{F/F \text{ is a decreasing } \alpha\text{-closed subset of } X \text{ containing } A\},
\]
\[ \text{bcl}(A) = \bigcap \{ F / F \text{ is a balanced } \alpha\text{-closed subset of } X \text{ containing } A \} . \]

Clearly \( i\text{acl}(A) \) (resp. \( d\text{acl} \) (A), \( b\text{cl} \) (A)) is the smallest increasing (resp. decreasing, balanced) \( \alpha \)-closed set containing A. \( I\alpha O(X) \) (resp. \( D\alpha O(X) \), \( B\alpha O(X) \)) denotes the collection of all increasing (resp. decreasing, balanced) \( \alpha \)-open subsets of a topological ordered space \( (X, \tau, \leq) \). \( I\alpha C(X) \) (resp. \( D\alpha C(X) \), \( B\alpha C(X) \)) denotes the collection of all increasing (resp. decreasing, balanced) \( \alpha \)-closed subsets of a topological ordered space \( (X, \tau, \leq) \).

We introduce the following.

**DEFINITION 2.3.01.** A function \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) is called an \( I\)-\( \alpha \)-continuous [6] (resp. \( D\)-\( \alpha \) continuous, \( B\)-\( \alpha \)-continuous) map if \( f^{-1}(V) \in I\alpha C(X) \), (resp. \( f^{-1}(V) \in D\alpha C(X) \), \( f^{-1}(V) \in B\alpha C(X) \)) whenever \( V \) is closed in \( X \).
It is evident that every $x$-$\alpha$-continuous map is an $\alpha$-continuous for $x = I, D, B$ and that every $B$-$\alpha$-continuous map is both $I$-$\alpha$-continuous and $D$-$\alpha$-continuous.

Following example shows that an $\alpha$-continuous map need not be a $x$-$\alpha$-continuous for $x = I, D, B$.

**EXAMPLE 2.3.01.** Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} = \tau^*$ and $\leq = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. Let $f$ be the identity map from $(X, \tau, \leq)$ onto itself. Since $f$ is the identity mapping, the inverse image of a closed set is a closed set and hence it is an $\alpha$-closed set (Since every closed set is an $\alpha$-closed set). Therefore $f$ is an $\alpha$-continuous mapping.

$\{a, c\}$ is a closed set in $(X, \tau)$, $f^{-1}(\{a, c\}) = \{a, c\}$, $i((f^{-1}(\{a, c\})) = i(\{a, c\}) = \{a, b, c\} \neq \{a, c\} = f^{-1} \{a, c\}$.
({a, c}). \Rightarrow f^{-1}({a, c}) \text{ is not an increasing } \alpha-\text{closed set in (X, } \tau). \Rightarrow f \text{ is not an I-}\alpha-\text{continuous map.}

d( f^{-1}({a, c})) = d([{a, c})] = \{a, b, c\} \neq \{a, c\} \neq f^{-1}({a, c}). \Rightarrow f^{-1}({a, c}) \text{ is not a decreasing } \alpha-\text{closed set. Therefore } f \text{ is not a D-}\alpha-\text{continuous map and consequently } f \text{ is not a } B-\alpha-\text{continuous map.}

Thus an \alpha-\text{continuous map need not be a } x-\alpha-\text{continuous map for } x = I, D, B.

Following example shows that a D-\alpha-\text{continuous map need not be a } B-\alpha-\text{continuous. It needs reference from example 1.1.09.}

\textbf{EXAMPLE 2.3.02.} Let X = \{a, b, c\} = X^*, \tau = \{\phi, X, \{a\}, \{a, b\}\} = \tau^* \text{ and } \leq = \{(a, a), (b, b), (c, c), (b, a)\}, \text{ and } \leq^* = \{(a, b), (b, b), (c, c)\}. \text{ Let } f \text{ be the identity map from (X, } \tau, \leq) \text{ onto (X}^*, \tau^*, \leq^*). \phi \text{ is the closed set in (X}^*, \tau^*), f^{-1}(\phi) = \phi \text{ is an } \alpha-\text{closed set in (X, } \tau), d(\phi) = \phi. X^* \text{ is the closed set in (X}^*, \tau^*), f^{-1}(X^*) = X
is an \( \alpha \)-closed set in \((X, \tau)\), \(d(X) = X\). \(\{b, c\}\) is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{b, c\}) = \{b, c\}\). \(\{c\}\) is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{c\}) = \{c\}\) is an \(\alpha\)-closed set in \((X, \tau)\), \(d(\{c\}) = \{c\}\). \(\Rightarrow f\) is a \(D\)-\(\alpha\)-continuous map.

\(\{b, c\}\) is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{b, c\}) = \{b, c\}\), \(i(\{b, c\}) = \{a, b, c\} \neq \{b, c\}\). \(\Rightarrow \{b, c\} \notin I\alpha C(X)\). Therefore \(f\) is not an \(I\)-\(\alpha\)-continuous map and consequently \(f\) is not a \(B\)-\(\alpha\)-continuous map.

Thus a \(D\)-\(\alpha\)-continuous map need not be a \(B\)-\(\alpha\)-continuous map.

Following example supports that an \(I\)-\(\alpha\)-continuous map need not be a \(B\)-\(\alpha\)-continuous map. It needs reference from example 1.1.01.

**Example 2.3.03.** Let \(X = \{a, b, c\} = X^*, \tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} = \tau^*\) and \(\leq = \{(a, a), (b, b), (c, c), (a, c), (b, c)\} = \leq^*\). Let \(f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)\) be the identity map. \(\phi\) is the closed set in \((X^*, \tau^*, \leq^*)\).
\( \tau^* \), \( f^{-1}(\phi) = \phi \) is an \( \alpha \)-closed set in \( (X, \tau) \), \( i(\phi) = \phi \).

\( X^* \) is the closed set in \( (X^*, \tau^*) \), \( f^{-1}(X^*) = X \) is an \( \alpha \)-closed set in \( (X, \tau) \), \( i(X) = X \). \{b, c\} is a closed set in \( (X^*, \tau^*) \), \( f^{-1}(\{b, c\}) = \{b, c\} \) is an \( \alpha \)-closed set in \( (X, \tau) \), \( i(\{b, c\}) = \{b, c\} \). \{c, a\} is a closed in \( (X^*, \tau^*) \), \( f^{-1}(\{c, a\}) = \{c, a\} \) is an \( \alpha \)-closed set in \( (X, \tau) \), \( i(\{c, a\}) = \{c, a\} \). \{c\} is a closed set in \( (X^*, \tau^*) \), \( f^{-1}(\{c\}) = \{c\} \) is an \( \alpha \)-closed set in \( (X, \tau) \), \( i(\{c\}) = \{c\} \) \( \Rightarrow \) \( f \) is an \( I-\alpha \)-continuous map.

\{b, c\} is a closed set in \( (X^*, \tau^*) \), \( f^{-1}(\{b, c\}) = \{b, c\} \), \( d(\{b, c\}) = \{a, b, c\} \neq \{b, c\} \). \( \Rightarrow \) \{b, c\} \( \notin \) \( D\alpha C(X) \). Therefore \( f \) is not a \( D-\alpha \)-continuous map and consequently \( f \) is not a \( B-\alpha \)-continuous map.

Thus an \( I-\alpha \)-continuous map need not be a \( B-\alpha \)-continuous map.

2.3.01 The above observations are given in the following diagram.

For a function \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \)
Following theorem characterizes I-α-continuous maps, in the light of semi, pre continuous maps proofs are trivial.

**THEOREM 2.3.01.** For a function $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$, the following statements are equivalent.

1) $f$ is I-α-continuous.

2) $f (\text{iαcl}(A)) \subseteq \text{cl}(f(A))$ for any $A \subseteq X$.

3) $\text{iαcl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$ for any $B \subseteq X^*$.

4) For any closed subset $K$ of $(X^*, \tau^*, \leq^*)$, $f^{-1}(K)$ is an increasing α-closed subset of $(X, \tau, \leq)$.

**THEOREM 2.3.02.** For a function $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$, the following statements are equivalent.

1) $f$ is D-α-continuous.

2) $f (\text{dαcl}(A)) \subseteq \text{cl}(f(A))$ for any $A \subseteq X$. 
3) \( \text{dcl}( f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B)) \) for any \( B \subseteq X^* \).

4) For every closed subset \( K \) of \( (X^*, \tau^*, \leq^*) \),
\[ f^{-1}(K) \] is a decreasing \( \alpha \)-closed subset of
\( (X, \tau, \leq) \).

**THEOREM 2.3.03.** For a function \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \), the following statements are equivalent.

1) \( f \) is \( B \)-\( \alpha \)-continuous.

2) \( f(\text{bcl}(A)) \subseteq \text{cl}(f(A)) \) for any \( A \subseteq X \).

3) \( \text{bcl}( f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B)) \) for any \( B \subseteq X^* \).

4) For every closed subset \( K \) of \( (X^*, \tau^*, \leq^*) \),
\[ f^{-1}(K) \] is a balanced \( \alpha \)-closed subset of
\( (X, \tau, \leq) \).

**THEOREM 2.3.04.** Let \( (X, \tau, \leq) \) and \( (X^*, \tau^*, \leq^*) \) be two topological ordered spaces.

Let \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) be a map. If \( f \) is \( I \)-\( \alpha \)-continuous then it is \( D \)-semi-continuous and \( I \)-pre-continuous.

**Proof.** Let \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) be an \( I \)-\( \alpha \)-continuous map. Let \( G \) be an open set in \( X \).
C(G) is a closed set in X. 

\[ \Rightarrow f^{-1}(C(G)) \text{ is an I-}\alpha\text{-closed set in } X. \]

\[ \Rightarrow C(f^{-1}(G)) \text{ is an I-}\alpha\text{-closed set in } X. \]

\[ \Rightarrow f^{-1}(G) \text{ is a D-}\alpha\text{-open set in } X. \]

\[ \Rightarrow f^{-1}(G) \text{ is a D-semi-open set in } X \text{ (by Lemma 1.1.1).} \]

Therefore f is a D-semi-continuous map. Let F be a closed set in X. Since f is an I-\alpha-continuous map, \( f^{-1}(F) \) is an I-\alpha-closed set in X. 

\[ \Rightarrow C(f^{-1}(F)) \text{ is a D-}\alpha\text{-open set in } X. \]

\[ \Rightarrow C(f^{-1}(F)) \text{ is a D-pre-open set in } X \text{ (from Lemma 1.1.1).} \]

\[ \Rightarrow f^{-1}(F) \text{ is an I-pre-closed set in } X. \]

Therefore f is an I-pre-continuous map. Thus if f is an I-\alpha-continuous map then f is a D-semi-continuous map and an I-pre-continuous map.

Following example shows that a D-semi-continuous map need not be an I-\alpha-continuous. It needs reference from example 1.1.01.

**EXAMPLE 2.3.04.** Let \( X = \{a, b, c\} = X^* \), \( \tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau^* = \{\phi, X, \{a, b\}\} \) and \( \leq = \{(a, a), (b, b), (c, c)\} = \leq^* \). Define \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \)
by \( f(a) = c, f(b) = a \) and \( f(c) = b \). \( \phi \) is the open set in \((X^*, \tau^*)\), \( f^{-1}(\phi) = \phi \) is a semi-open set in \((X, \tau)\), \( d(\phi) = \phi \). \( X^* \) is the open set in \((X^*, \tau^*)\), \( f^{-1}(X^*) = X \) is a semi-open set in \((X, \tau)\), \( d(X) = X \). \{a, b\} is an open set in \((X^*, \tau^*)\), \( f^{-1}(\{a, b\}) = \{b, c\} \) is a semi-open set in \((X, \tau)\). \( d(\{b, c\}) = \{b, c\} \). \( \Rightarrow f^{-1}(G) \in DSO(X) \), for every open set \( G \) in \((X^*, \tau^*)\). \( \Rightarrow f \) is a D-semi-continuous map.

\( F = \{c\} \) is a closed set in \((X^*, \tau^*)\), \( f^{-1}(\{c\}) = \{a\} \) is not an \( \alpha \)-closed set. Therefore \( f \) is not an \( \alpha \)-continuous map and consequently \( f \) is not a I-\( \alpha \)-continuous map.

Thus a D-semi-continuous map need not be an I-\( \alpha \)-continuous map.

Following example shows that an I-pre-continuous map need not be an I-\( \alpha \)-continuous. It needs reference from example 1.1.08.

**EXAMPLE 2.3.05.** Let \( X = \{a, b, c\} = X^*, \tau = \{\phi, X, \{a, b\}\} \), \( \tau^* = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}, \)
\{b, c\} \text{ and } \leq = \{(a, a), (b, b), (c, c)\} = \leq^*. \text{ Define } f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \text{ by } f(a) = b, f(b) = a \text{ and } f(c) = c. \phi \text{ is the closed set in } (X^*, \tau^*), f^{-1}(\phi) = \phi \text{ is a pre-closed set in } (X, \tau), i(\phi) = \phi. X^* \text{ is the closed set in } (X^*, \tau^*), f^{-1}(X^*) = X \text{ is a pre-closed set in } (X, \tau), i(X) = X. \{a\} \text{ is a closed set in } (X^*, \tau^*), f^{-1}(\{a\}) = \{b\} \text{ is a pre-closed set in } (X, \tau), i(\{b\}) = \{b\}. \{c\} \text{ is a closed set in } (X^*, \tau^*), f^{-1}(\{c\}) = \{a\} \text{ is a pre-closed set in } (X, \tau), i(\{a\}) = \{a\}. \{b, c\} \text{ is a closed set in } (X^*, \tau^*), f^{-1}(\{b, c\}) = \{a, c\} \text{ is a pre-closed set in } (X, \tau), i(\{a, c\}) = \{a, c\}. \{c, a\} \text{ is a closed set in } (X^*, \tau^*), f^{-1}(\{c, a\}) = \{b, c\} \text{ is a pre-closed set in } (X, \tau), i(\{b, c\}) = \{b, c\}. \Rightarrow f^{-1}(F) \in \text{IPC}(X), \text{ whenever } F \text{ is a closed set in } (X^*, \tau^*). \Rightarrow f \text{ is an I-pre-continuous map.} \\
\{c\} \text{ is a closed set in } (X^*, \tau^*), f^{-1}(\{c\}) = \{a\} \text{ is not an } \alpha \text{-closed set in } (X, \tau). \text{ Therefore } f \text{ is not an } \alpha \text{-continuous map and consequently } f \text{ is not an I-}\alpha \text{-continuous map.}
Thus an I-pre-continuous map need not be an I-α-continuous map.

**THEOREM 2.3.05.** Let \((X, \tau, \leq)\) and \((X^*, \tau^*, \leq^*)\) be two topological ordered spaces. Let \(f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)\) be a map. If \(f\) is D-α-continuous then it is I-semi-continuous and D-pre-continuous.

**Proof.** Let \(f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)\) be an D-α-continuous map. Let \(G\) be an open set in \(X\). => \(C(G)\) is a closed set in \(X\). => \(f^{-1}(C(G))\) is an D-α-closed set in \(X\). => \(C(f^{-1}(G))\) is an D-α-closed set in \(X\). => \(f^{-1}(G)\) is an I-α-open set in \(X\). => \(f^{-1}(G)\) is an I-semi-open set in \(X\) (by Lemma 1.1.1). Therefore \(f\) is an I-semi-continuous map. Let \(F\) be a closed set in \(X\). Since \(f\) is a D-α-continuous map, \(f^{-1}(F)\) is a D-α-closed set in \(X\). => \(C(f^{-1}(F))\) is an I-α-open set in \(X\). => \(C(f^{-1}(F))\) is an I-pre-open set in \(X\) (from Lemma 1.1.1). => \(f^{-1}(F)\) is a D-pre-closed set in \(X\). Therefore \(f\) is a D-pre-continuous map.
Thus if $f$ is a D-$\alpha$-continuous map then $f$ is an I-semi-continuous map and a D-pre-continuous map.

Following example shows an I-semi-continuous map need not be a D-$\alpha$-continuous map. It needs reference from example 1.1.01.

**EXAMPLE 2.3.06:** Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^* = \{\phi, X, \{a, b\}\}$ and $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Define $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. $\phi$ is the open set in $(X^*, \tau^*)$, $f^{-1}(\phi) = \phi$ is a semi-open set in $(X, \tau)$, $i(\phi) = \phi$. $X^*$ is the open set in $(X^*, \tau^*)$, $f^{-1}(X^*) = X$ is a semi-open set in $(X, \tau)$, $i(X) = X$. $\{a, b\}$ is an open set in $(X^*, \tau^*)$, $f^{-1}(\{a, b\}) = \{b, c\}$ is a semi-open set in $(X, \tau)$. $i(\{b, c\}) = \{b, c\}$. $f^{-1}(G) \in \text{ISO}(X)$, for every open set $G$ in $(X^*, \tau^*)$. Therefore $f$ is a I-semi-continuous map.
{c} is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{c\}) = \{a\}\) is not an \(\alpha\)-closed set. Therefore \(f\) is not an \(\alpha\)-continuous map and consequently \(f\) is not a \(I\)-\(\alpha\)-continuous map.

Thus an \(I\)-semi-continuous map need not be a \(D\)-\(\alpha\)-continuous map.

Following example shows that a \(D\)-pre-continuous map need not be \(a\)\(D\)-\(\alpha\)-continuous. It needs reference from example 1.1.08.

**Example 2.3.07.** Let \(X = \{a, b, c\} = X^*\), \(\tau = \{\phi, X, \{a, b\}\}\), \(\tau^* = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}\), and \(\leq = \{(a, a), (b, b), (c, c)\} = \leq^*\). Define a map \(f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)\) by \(f(a) = b, f(b) = a\) and \(f(c) = c\). \(\phi\) is the closed set \((X^*, \tau^*)\), \(f^{-1}(\phi) = \phi\) is a pre-closed set in \((X, \tau)\), \(d(\phi) = \phi\). \(X^*\) is the closed set in \((X^*, \tau^*)\), \(f^{-1}(X^*) = X\) is the pre-closed set in \((X, \tau)\), \(d(X) = X\). \(\{a\}\) is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{a\}) = \{b\}\) is a pre-closed set in \((X, \tau)\), \(d(\{b\}) = \)
\{b\}. \{c\} is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{c\}) = \{a\}\) is a pre-closed set in \((X, \tau)\), \(d(\{a\}) = \{a\}\). \{b, c\} is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{b, c\}) = \{a, c\}\) is a pre-closed set in \((X, \tau)\), \(d(\{a, c\}) = \{a, c\}\). \{c, a\} is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{c, a\}) = \{b, c\}\) is a pre-closed set in \((X, \tau)\), \(d(\{b, c\}) = \{b, c\}\). \(\Rightarrow f^{-1}(F) \in \text{DPC}(X)\), whenever \(F\) is a closed set in \((X^*, \tau^*)\).

Therefore \(f\) is a D-pre-continuous map.

\{c\} is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{c\}) = \{a\}\) is not an \(\alpha\)-closed set in \((X, \tau)\).

Therefore \(f\) is not an \(\alpha\)-continuous map and consequently \(f\) is not a D-\(\alpha\)-continuous map.

Thus a D-pre-continuous map need not be a D-\(\alpha\)-continuous map.

**THEOREM 2.3.06.** Let \((X, \tau, \leq)\) and \((X^*, \tau^*, \leq^*)\) be two topological ordered spaces.

Let \(f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)\) be a map. If \(f\) is a B-\(\alpha\)-continuous map then it is a B-semi-continuous map and a B-pre-continuous map.
Proof. Let $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be a B-$\alpha$-continuous map. \(\Rightarrow\) $f$ is an I-$\alpha$-continuous map and $f$ is a D-$\alpha$-continuous map. Since $f$ is an I-$\alpha$-continuous map, by theorem 2.3.04 $f$ is a D-semi continuous map and an I-pre-continuous map. Since $f$ is a D-$\alpha$-continuous map by theorem 2.3.05 $f$ is an I-semi-continuous map and a D-pre-continuous map. Therefore if $f$ is a B-$\alpha$-continuous map then $f$ is a B-semi-continuous map and a B-pre-continuous map.

Following example shows that a B-semi-continuous map need not be a B-$\alpha$-continuous map. It needs reference from example 1.1.01.

**Example 2.3.08.** Let $X = \{a, b, c\} = X^*$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau^* = \{\emptyset, X, \{a, b\}\}$ and $\leq = \{(a, a), (b, b), (c, c)\}$ = $\leq^*$. Define $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. $\emptyset$ is the open set in $(X^*, \tau^*)$, $f^{-1}(\emptyset) = \emptyset$ is a semi-open set in $(X, \tau)$, $i(\emptyset) = \emptyset$, $d(\emptyset) = \emptyset$. $X^*$ is the open set in $(X^*, \tau^*)$, $f^{-1}(X^*) = X$ is a semi-open set in $(X, \tau)$, $i(X) = X$, $d(X) = X$. {a,
b} is an open set in \((X^*, \tau^*)\), \(f^{-1}([a, b]) = \{b, c\}\) is a semi-open set in \((X, \tau)\). \(i([b, c]) = \{b, c\}, d([b, c]) = \{b, c\}\). \(\Rightarrow\) \(f^{-1}(G) \in \text{BSO}(X)\), for every open set \(G\) in \((X^*, \tau^*)\). \(\Rightarrow f\) is a B-semi-continuous map.

\{c\} is a closed set in \((X^*, \tau^*)\), \(f^{-1}([c]) = \{a\}\) is not an \(\alpha\)-closed set. \(\Rightarrow f\) is not an \(\alpha\)-continuous map and hence \(f\) is not a B-\(\alpha\)-continuous map.

Thus a B-semi-continuous map need not be a B-\(\alpha\)-continuous map.

Following example shows that a B-pre-continuous map need not be a B-\(\alpha\)-continuous. It needs reference from example 1.1.08.

**EXAMPLE 2.3.09.** Let \(X = \{a, b, c\} = X^*, \tau = \{\phi, X, \{a, b\}\}\) and \(\tau^* = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}\) and \(\leq = \{(a, a), (b, b), (c, c)\} = \leq^*\). Define a map \(f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)\) by \(f(a) = b, f(b) = a\) and \(f(c) = c\). \(\phi\) is the closed set in \((X^*, \tau^*)\), \(f^{-1}(\phi) = \phi\) is a pre-closed
set in \((X, \tau)\), \(i(\phi) = \phi, d(\phi) = \phi. X^*\) is the closed set in 
\((X^*, \tau^*)\), \(f^{-1}(X^*) = X\) is a pre-closed set in \((X, \tau)\), \(i(X) = X, d(X) = X. \{a\}\) is a closed set in \((X^*, \tau^*)\), 
\(f^{-1}(\{a\}) = \{b\}\) is a pre-closed set in \((X, \tau)\), \(i(\{b\}) = \{b\}, d(\{b\}) = \{b\}. \{c\}\) is a closed set in \((X^*, \tau^*)\), 
\(f^{-1}(\{c\}) = \{a\}\) is a pre-closed set in \((X, \tau)\), \(i(\{a\}) = \{a\}, d(\{a\}) = \{a\}. \{b, c\}\) is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{b, c\}) = \{a, c\}\) is a pre-closed set \(i(\{a, c\}) = \{a, c\}, d(\{a, c\}) = \{a, c\}. \{a, c\}\) is closed set in \((X^*, \tau^*)\), \(f^{-1}(\{c, a\}) = \{b, c\}\) is a pre-closed set \(i(\{b, c\}) = \{b, c\}, d(\{b, c\}) = \{b, c\}. \Rightarrow f^{-1}(F) \in \text{BPC}(X), \) where \(F\) is a closed set in \((X^*, \tau^*). \Rightarrow f\) is a B-pre-continuous map. 

\(\{c\}\) is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{c\}) = \{a\}\) is not an \(\alpha\)-closed set in \((X, \tau)\). Therefore \(f\) is not an \(\alpha\)-continuous map and consequently \(f\) is not a B-\(\alpha\)-continuous map.

Thus a B-pre-continuous map need not be a B-\(\alpha\)-continuous map.
2.4 I-β-continuous, D-β-continuous and B-β-continuous maps.

In this chapter we introduce I-β-continuous, D-β-continuous and B-β-continuous maps for topological ordered together with their characterizations.

**INTRODUCTION.** Abd-El-Monsef et al [1] introduced β-open sets and β-closed sets. Note that the complement of β-open set is β-closed and vice versa. We denote the complement of A by \( C(A) \).

For a subset A of a topological ordered space \((X, \tau, \leq)\), we define

\[
i\beta\text{cl}(A) = \cap \{ F/F \text{ is an increasing } \beta\text{-closed subset of } X \text{ containing } A \},
\]

\[
d\beta\text{cl}(A) = \cap \{ F/F \text{ is decreasing } \beta\text{-closed subset of } X \text{ containing } A \},
\]

\[
 b\beta\text{cl}(A) = \cap \{ F/F \text{ is a balanced } \beta\text{-closed subset of } X \text{ containing } A \},
\]
Clearly $i\beta cl(A)$ (resp. $d\beta cl(A)$, $b\beta cl(A)$) is the smallest increasing (resp. decreasing, balanced) $\beta$-closed set containing $A$.

$I\beta O(X)$ (resp. $D\beta O(X)$, $B\beta O(X)$) denotes the collection of all increasing (resp. decreasing, balanced) $\beta$-open subsets of a topological ordered space $(X, \tau, \leq)$. $I\beta C(X)$ (resp. $D\beta C(X)$, $B\beta C(X)$) denotes the collection of all increasing (resp. decreasing, balanced) $\beta$-closed subsets of a topological ordered space $(X, \tau, \leq)$.

We introduce the following.

**DEFINITION 2.4.01.** A function $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ is called an $I$-$\beta$-continuous [7] (resp. $D$-$\beta$-continuous, $B$-$\beta$-continuous) map if $f^{-1}(V) \in I\beta C(X)$, (resp. $f^{-1}(V) \in D\beta C(X)$, $f^{-1}(V) \in B\beta C(X)$) whenever $V$ is closed in $X$.

It is evident that every $x$-$\beta$-continuous map is $\beta$-continuous for $x = I, D, B$ and that every
B-β-continuous map is both I-β-continuous and D-β-continuous.

Following example shows that a β-continuous map need not be x-β-continuous for x = I, D, B.

EXAMPLE 2.4.01. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\leq \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$. Clearly $(X, \tau, \leq)$ is a topological ordered space. Let $f$ be the identity map from $(X, \tau, \leq)$ onto itself. Since $f$ is the identity map, $f^{-1}(F)$ is a closed set in $(X, \tau)$ for every closed set $F$ in $(X, \tau)$ and it is a β-closed set in $(X, \tau)$ (Since every closed set is a β-closed set). Therefore $f$ is a β-continuous map.

\{a, c\} is a closed set in $(X, \tau)$, $f^{-1}(\{a, c\}) = \{a, c\}$, $i(\{a, c\}) = \{a, b, c\} \neq \{a, c\}$ => $f^{-1}(\{a, c\})$ is not an increasing β-closed set in $(X, \tau)$. => $f$ is not an I-α-continuous map. $d(\cup(f^{-1}(G)) = d(\{a, c\}) = \{a, b, c\} \neq \{a, c\} = f^{-1}(G)$. => $f^{-1}(G)$ is not a decreasing
\( \beta \)-closed set in \((X, \tau)\). \(\Rightarrow\) \(f\) is not a \(D\beta\)-continuous map and hence \(f\) is not a \(B\beta\)-continuous map.

Thus a \(\beta\)-continuous map need not be a \(x\)-\(\beta\)-continuous map for \(x = I, D, B\).

Following example shows that a \(D\beta\)-continuous map need not be a \(B\beta\)-continuous.

**EXAMPLE 2.4.02.** Let \(X = \{a, b, c\} = X^*, \tau = \{\phi, X, \{a\}, \{a, b\}\} \) and \(\leq = \{(a, a), (b, b), (c, c), (b, a)\}\) and \(\leq^* = \{(a, b), (b, b), (c, c)\}\). Let \(g\) be the identity map from \((X, \tau, \leq)\) onto \((X^*, \tau^*, \leq^*)\).

Since \(f\) is the identity map, \(f^{-1}(F)\) is a closed set in \((X, \tau)\), for every closed set \(F\) in \((X^*, \tau^*)\) and it is a \(\beta\)-closed set in \((X, \tau)\) (Since every closed set is a \(\beta\)-closed set). \(\Rightarrow\) \(f\) is a \(\beta\)-continuous map. \(d(\phi) = \phi, d(X) = X, d(\{b, c\}) = \{b, c\}\) and \(d(\{c\}) = \{c\}\). Therefore \(f\) is \(D\beta\)-continuous map.

\(\{b, c\}\) is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{b, c\}) = \{b, c\}, i\{b, c\} = \{a, b, c\} \neq \{b, c\}\). \(\Rightarrow\) \(f^{-1}(\{b, c\}) \notin I\beta C(X)\).

\(\Rightarrow\) \(f\) is not an \(I\beta\)-continuous map and hence \(f\) is not a \(B\beta\)-continuous map.
Following example supports that an I-β-continuous map need not be a B-β continuous map.

**EXAMPLE 2.4.03.** Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} = \tau^*$ and $\leq = \{(a, a), (b, b), (c, c), (a, c), (b, c)\} = \leq^*$. Let $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ be the identity map. Since $f$ is the identity map, $f^{-1}(F)$ is a closed set in $(X, \tau)$, for every closed set $F$ in $(X^*, \tau^*)$ and it is a $\beta$-closed set in $(X, \tau)$ (Since every closed set is a $\beta$-closed set). $\Rightarrow f$ is a $\beta$-continuous map.

$$i(\phi) = \phi, \quad i(X) = X, \quad i(\{b, c\}) = \{b, c\}, \quad i(\{c, a\}) = \{c, a\}, \quad i(\{c\}) = \{c\}.$$

Therefore $f$ is an I-β-continuous map.

$$\{b, c\} \text{ is a closed set in } (X^*, \tau^*), \quad f^{-1}(\{b, c\}) = \{b, c\}, \quad d(\{b, c\}) = \{a, b, c\} \neq \{b, c\}. \quad \Rightarrow f^{-1}(\{b, c\}) \notin D\beta C(X). \quad \Rightarrow f$ is not a D-β-continuous map and hence $f$ is not a B-β-continuous map.

Thus an I-β-continuous map need not be a B-β-continuous map.

2.4.01 The above observations are given in the following diagram.

For a function $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$
The following theorem characterizes I-β-continuous maps whose proof are similar to that of pre-continuous maps.

**THEOREM 2.4.01.** For a function $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ the following statements are equivalent.

1) $f$ is I-β-continuous.
2) $f(i\beta \text{cl}(A)) \subseteq \text{cl}(f(A))$ for any $A \subseteq X$.
3) $i\beta \text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$ for any $B \subseteq X^*$.
4) For any closed subset $K$ of $(X^*, \tau^*, \leq^*)$, $f^{-1}(K)$ is an increasing β-closed subset of $(X, \tau, \leq)$.

**THEOREM 2.4.02.** For a function $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$, the following statements are equivalent.
1) \( f \) is D-\( \beta \)-continuous.

2) \( f \left( \text{d} \beta \text{cl}(A) \right) \subseteq \text{cl}(f(A)) \) for any \( A \subseteq X \).

3) \( \text{d} \beta \text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B)) \) for any \( B \subseteq X^* \).

4) For every closed subset \( K \) of \((X^*, \tau^*, \leq^*)\), \( f^{-1}(K) \) is a decreasing \( \beta \)-closed subset of \((X, \tau, \leq)\).

**THEOREM 2.4.03.** For a function \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \), the following statements are equivalent.

1) \( f \) is B-\( \beta \)-continuous.

2) \( f \left( b \beta \text{cl}(A) \right) \subseteq \text{cl}(f(A)) \) for any \( A \subseteq X \).

3) \( b \beta \text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B)) \) for any \( B \subseteq X^* \).

4) For every closed subset \( K \) of \((X^*, \tau^*, \leq^*)\), \( f^{-1}(K) \) is a decreasing \( \beta \)-closed subset of \((X, \tau, \leq)\).

**THEOREM 2.4.04.** Every I-semi-continuous map is a D-\( \beta \)-continuous.

**Proof.** Let \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) be an I-semi-continuous map. Let \( F \) be a closed set in \( X^* \). \( \Rightarrow \) \( C(F) \) is an open set in \( X \). Since \( f \) is I-semi-continuous, \( f^{-1}(C(F)) \) is a balanced \( \beta \)-closed subset of \((X, \tau, \leq)\).
continuous map, \( f^{-1}(C(F)) \) is an I-semi-open set.

\[ \Rightarrow C(f^{-1}(F)) \text{ is an I-semi-open set} \Rightarrow f^{-1}(F) \text{ is a D-semi-closed set} \text{. Therefore } f^{-1}(F) \text{ is a D-β-closed set (from Lemma 1.1.2). Therefore } f \text{ is D-β-continuous map.} \]

Following example shows that a D-β-continuous map need not be an I-semi-continuous map. It needs reference from example 1.1.03.

**EXAMPLE 2.4.04.** Let \( X = \{a, b, c\} = X^*, \tau = \{\emptyset, X, \{a\}, \{b, c\}\} = \tau^* \text{ and } \leq = \{(a, a), (b, b), (c, c)\} = \leq^*. \)

Define a map \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) by \( f(a) = b, f(b) = a \text{ and } f(c) = c. \)

\( \phi \) is the closed set in \( (X^*, \tau^*) \), \( f^{-1}(\phi) = \phi \) is a β-closed set in \( (X, \tau) \), \( d(\phi) = \phi \). \( X^* \) is the closed set in \( (X^*, \tau^*) \), \( f^{-1}(X^*) = X \) is a β-closed set in \( (X, \tau) \), \( d(X) = X \). \( \{b, c\} \) is a closed set in \( (X^*, \tau^*) \), \( f^{-1}(\{b, c\}) = \{a, c\} \) is a β-closed set in \( (X, \tau) \), \( d(\{a, c\}) = \{a, c\} \). \( \{a\} \) is a closed set in \( (X^*, \tau^*) \), \( f^{-1}(\{a\}) = \{b\} \) is a β-closed set in \( (X, \tau) \), \( d(\{b\}) = \{b\} \Rightarrow \)
\( f^{-1}(F) \in D\beta C(X) \), for any closed set \( F \) in \((X^*, \tau^*)\).

Therefore \( f \) is an \( D\beta \)-continuous map.

\( \{a\} \) is an open set in \((X^*, \tau^*)\), \( f^{-1}(\{a\}) = \{b\} \) is not a semi-open set in \((X, \tau)\). \( \Rightarrow \) \( f^{-1}(\{a\}) \notin DSO(X) \).

Therefore \( f \) is not a semi-continuous map and consequently \( f \) is not an I-semi-continuous map.

Thus an \( D\beta \)-continuous map need not be an I-semi-continuous map.

**THEOREM 2.4.05.** Every I-pre-continuous map is an I-\( \beta \)-continuous.

**Proof.** Let \( f: (X, \tau, \leq) \to (X^*, \tau^*, \leq^*) \) be an I-pre-continuous map. Let \( F \) be a closed set in \( X^* \). \( \Rightarrow f^{-1}(F) \) is an I-pre-closed set in \( X \). \( \Rightarrow f^{-1}(F) \) is an I-\( \beta \)-closed set (from Lemma 1.1.3). Therefore \( f \) is I-\( \beta \)-continuous map.

Following example shows that an I-\( \beta \)-continuous map need not be an I-pre-continuous map. It needs reference from example 1.1.01.
EXAMPLE 2.4.05. Let $X = \{a, b, c\} = X^*$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau^* = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Define a map $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. $\emptyset$ is a closed set in $(X^*, \tau^*)$, $f^{-1}(\emptyset) = \emptyset$ is a $\beta$-closed set in $(X, \tau)$, $i(\emptyset) = \emptyset$. $X^*$ is a closed set in $(X^*, \tau^*)$, $f^{-1}(X^*) = X$ is a $\beta$-closed set in $(X, \tau)$, $i(X) = X$. $\{a\}$ is a closed set in $(X^*, \tau^*)$, $f^{-1}(\{a\}) = \{b\}$ is a $\beta$-closed set in $(X, \tau)$, $i(\{b\}) = \{b\}$. $\{b, c\}$ is a closed set in $(X^*, \tau^*)$, $f^{-1}(\{b, c\}) = \{a, c\}$ is a $\beta$-closed set in $(X, \tau)$, $i(\{a, c\}) = \{a, c\}$. $f^{-1}(\{b, c\}) \in I\beta C(X)$ for every closed set $F$ in $(X^*, \tau^*)$. $\Rightarrow$ $f$ is an I-$\beta$-continuous map.

$\{a\}$ is a closed set in $(X^*, \tau^*)$, $f^{-1}(\{a\}) = \{b\}$ is not a pre-closed set in $(X, \tau)$. Therefore $f$ is not a pre-continuous map and consequently $f$ is not an I-pre-continuous map.

Thus an I-$\beta$-continuous map need not be an I-pre-continuous map.
**THEOREM 2.4.06** Every I-α-continuous map is an I-β-continuous map.

**Proof.** Let $f: (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ be an I-α-continuous map. Let $F$ be a closed set in $X^*$. $f^{-1}(F)$ is an I-α-closed set. $f^{-1}(F)$ is I-β-closed set (from Lemma 1.1.4). Therefore $f$ is I-β-continuous map.

Following example shows that an I-β-continuous map need not be an I-α-continuous map. It needs reference from example 1.1.01.

**EXAMPLE 2.4.06.** Let $X = \{a, b, c\} = X^*, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \tau^* = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Define a map $f : (X, \tau, \leq) \to (X^*, \tau^*, \leq^*)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. $\phi$ is a closed set in $(X^*, \tau^*)$, $f^{-1}(\phi) = \phi$ is a β-closed set in $(X, \tau)$, $i(\phi) = \phi$. $X^*$ is a closed set in $(X^*, \tau^*)$, $f^{-1}(X^*) = X$ is a β-closed set in $(X, \tau)$, $i(X) = X$. $\{a\}$ is a closed set in $(X^*, \tau^*)$, $f^{-1}(\{a\}) = \{b\}$ is a β-closed set
in \((X, \tau), i(\{b\}) = \{b\}\). \{b, c\} is a closed set in \((X^*, \tau^*)\),
\[f^{-1}(\{b, c\}) = \{a, c\}\] is a \(\beta\)-closed set in \((X, \tau)\),
i(\{a, c\}) = \{a, c\}. \Rightarrow f^{-1}(F) \in I\beta C(X), for every closed set \(F\) in \((X^*, \tau^*)\). \Rightarrow f is an I-\(\beta\)-continuous map.

{a} is a closed set in \((X^*, \tau^*)\), \[f^{-1}(\{a\}) = \{b\}\] is not an \(\alpha\)-closed set in \((X, \tau)\). Therefore \(f\) is not an \(\alpha\)-continuous map and consequently \(f\) is not an I-\(\alpha\)-continuous map.

Thus an I-\(\beta\)-continuous map need not be an I-\(\alpha\)-continuous map.

**THEOREM 2.4.07.** Every D-semi-continuous map is a I-\(\beta\)-continuous.

**Proof.** Let \(f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)\) be D-semi-continuous map. Let \(F\) be a closed set in \(X^*\). \(\Rightarrow\) \(C(F)\) is an open set in \(X^*\). Since \(f\) is D-semi-continuous map, \(f^{-1}(C(F))\) is a D-semi-open set in \(X\). \(\Rightarrow\) \(C(f^{-1}(F))\) is a D-semi-open set in \(X\). \(\Rightarrow\) \(f^{-1}(F)\) is an I-semi-closed set in \(X\). \(\Rightarrow\) \(f^{-1}(F)\) is an
I-β-closed set in $X$ (from Lemma 1.1.4). Therefore $f$ is I-β-continuous map.

Following example shows that an I-β-continuous map need not be a D-semi-continuous map. It needs reference from example 1.1.03.

**EXAMPLE 2.4.07.** Let $X = \{a, b, c\} = X^*$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ = $\tau^*$ and $\leq = \{(a, a), (b, b), (c, c)\}$ = $\leq^*$. Define a map $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$ From example 2.4.04 $f$ is β-continuous map. We have $i(\phi) = \phi$, $i(X) = X$, $i(\{a, c\}) = \{a, c\}$, $i([b]) = [b]$. $\Rightarrow f^{-1}(F) \in I\beta C(X)$, for every closed set $F$ in $(X^*, \tau^*)$. $\Rightarrow f$ is a I-β-continuous map.

$\{a\}$ is an open set in $(X^*, \tau^*)$, $f^{-1}(\{a\}) = \{b\}$ is not a semi-open set in $(X, \tau)$. Therefore $f$ is not a semi-continuous map and consequently $f$ is not a D-semi-continuous map.

Thus an I-β-continuous map need not be a D-semi-continuous map.
THEOREM 2.4.08. Every D-pre-continuous map is a D-β-continuous.

Proof. Let \( f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) be an D-pre-continuous map. Let \( F \) be a closed set in \( X^* \). \( f^{-1}(F) \) is a D-pre-closed set. \( f^{-1}(F) \) D-β-closed set (from Lemma 1.1.3). Therefore \( f \) is D-β-continuous map.

Following example shows that a D-β-continuous map need not be a D-pre-continuous map. It needs reference from example 1.1.01.

EXAMPLE 2.4.08. Let \( X = \{a, b, c\} = X^* \), \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \), \( \tau^* = \{\emptyset, X, \{a\}, \{b, c\}\} \) and \( \leq = \{(a, a), (b, b), (c, c)\} = \leq^* \). Define a map \( f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) by \( f(a) = b \), \( f(b) = a \) and \( f(c) = c \). From example 2.4.05 \( f \) is a β-continuous map. \( d(\emptyset) = \emptyset \), \( d(X) = X \), \( d(\{b\}) = \{b\} \), \( d(\{a, c\}) = \{a, c\} \). \( f^{-1}(F) \in D\betaC(X) \), for every closed set \( F \) in \( (X^*, \tau^*) \). \( f^{-1}(\{a\}) = \{b\} \) is a closed set in \( (X^*, \tau^*) \). Therefore \( f \) is not a D-β-continuous map.
pre-continuous map and consequently f is not a D-pre-continuous map.

Thus a D-β-continuous map need not be a D-pre-continuous map.

**THEOREM 2.4.09.** Every D-α-continuous map is a D-β-continuous.

**Proof.** Let \( f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) be an D-α-continuous map. Let \( F \) be a closed set in \( X^* \). \( \Rightarrow \) \( f^{-1}(F) \) is a D-α-closed set. \( \Rightarrow \) \( f^{-1}(F) \) is a D-β-closed set (from Lemma 1.1.4). Therefore \( f \) is D-β-continuous map.

Following example shows that a D-β-continuous map need not be a D-α-continuous map. It needs reference from example 1.1.03.

**EXAMPLE 2.4.09.** Let \( X = \{a, b, c\} = X^* \), \( \tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} \), \( \tau^* = \{\phi, X, \{a\}, \{b, c\}\} \) and \( \leq = \{(a, a), (b, b), (c, c)\} = \leq^* \). Define a map \( f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) by \( f(a) = b \), \( f(b) = a \) and \( f(c) = c \). From example 2.4.06 \( f \) is β-continuous map. We have
\[ d(\phi) = \phi, \quad d(X) = X, \quad d([b]) = \{b\}, \quad d([a, c]) = \{a, c\}. \]

\[ \Rightarrow f(F) \in D\beta C(X), \text{ for every closed set } F \text{ in } (X^*, \tau^*). \]

\[ \Rightarrow f \text{ is } D\beta\text{-continuous map.} \]

\{a\} is a closed set in \((X^*, \tau^*)\). \[ f^{-1}\{a\} = \{b\} \]

is not an \(\alpha\)-closed set in \((X, \tau)\). Therefore \(f\) is not an \(\alpha\)-continuous map and consequently \(f\) is not a \(D\alpha\)-continuous map.

Thus a \(D\beta\)–continuous map need not be a \(D\alpha\)-continuous map.

**THEOREM 2.4.10.** Every B-semi-continuous map is a B-\(\beta\)-continuous.

**Proof.** Let \(f: (X, \tau, \leq) \rightarrow (X^*, \tau^* \leq^*)\) be a B-semi-continuous map. \(\Rightarrow f\) is I-semi-continuous map and \(f\) is D-semi-continuous map. Since \(f\) is I-semi-continuous map by theorem 2.4.04 \(f\) is D-\(\beta\)-continuous map. Since \(f\) is D-semi-continuous map by theorem 2.4.07 \(f\) is I-\(\beta\)-continuous map. Therefore \(f\) is \(-\beta\)-continuous map.
Following example shows that a B-β-continuous map need not be a B-semi-continuous map. It needs reference from example 1.1.03.

**EXAMPLE 2.4.10** Let $X = \{a, b, c\} = X^*$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\} = \tau^*$ and let $\leq = \{(a, a), (b, b), (c, c)\} = \leq^*$. Define a map $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. From example 2.4.07 $f$ is an I-β-continuous map and from example 2.4.04 $f$ is a D-β-continuous map. Thus $f$ is a B-β-continuous map.

$\{a\}$ is an open set in $(X^*, \tau^*)$. $\{a\} = \{b\}$ is not a semi-open set in $(X, \tau)$. Therefore $f$ is not a semi-continuous map and consequently $f$ is not a B-semi-continuous map.

Thus a B-β-continuous map need not be a B-semi-continuous map.

**THEOREM 2.4.11** Every B-pre-continuous map is a B-β-continuous.

**Proof.** Let $f : (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*)$ be a B-pre-continuous map. $\Rightarrow f$ is an I-pre-continuous map and $f$ is a D-pre-continuous map. Since $f$ is I-pre-
continuous map, by theorem 2.4.05 \( f \) is a \( I-\beta \)-continuous map. Since \( f \) is \( D \)-pre-continuous map, by theorem 2.4.08 \( f \) is a \( D-\beta \)-continuous map. Therefore \( f \) is a \( B-\beta \)-continuous map.

Following example shows that a \( B-\beta \)-continuous map need not be a \( B \)-pre-continuous map. It needs reference from example 1.1.01.

**EXAMPLE 2.4.11.** Let \( X = \{a, b, c\} = X^* \), \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \), \( \tau^* = \{\emptyset, X, \{a\}, \{b, c\}\} \) and \( \preceq = \{(a, a), (b, b), (c, c)\} = \preceq^* \). Define a map \( f : (X, \tau, \preceq) \rightarrow (X^*, \tau^*, \preceq^*) \) by \( f(a) = b \), \( f(b) = a \) and \( f(c) = c \). From example 2.4.05 \( f \) is an \( I-\beta \)-continuous map and from example 2.4.08 \( f \) is a \( D-\beta \)-continuous map. Thus \( f \) is a \( B-\beta \)-continuous map.

\( \{a\} \) is a closed set in \( (X^*, \tau^*) \), \( f^{-1}(\{a\}) = \{b\} \) is not a pre-closed set in \( (X, \tau) \). Therefore \( f \) is not a \( \preceq \)-continuous map and consequently \( f \) is not a \( B \)-pre-continuous map.

Thus a \( B-\beta \)-continuous map need not be a \( B \)-pre-continuous map.
THEOREM 2.4.12. Every B-α-continuous map is a B-β-continuous.

Proof. Let \( f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) be a B-α-continuous map. => \( f \) is I-α-continuous map and \( f \) is D-α-continuous map. Since \( f \) is I-α-continuous map, by theorem 2.4.06 \( f \) is I-β-continuous map. Since \( f \) is D-α-continuous map, by theorem 2.4.09 \( f \) is D-β-continuous map. Therefore \( f \) is B-β-continuous map.

Following example shows that a B-β-continuous map need not be a B-α-continuous map. It needs reference from example 1.1.03.

EXAMPLE 2.4.12. Let \( X = \{a, b, c\} = X^* \), \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \), \( \tau^* = \{\emptyset, X, \{a\}, \{b, c\}\} \) and \( \leq = \{(a, a), (b, b), (c, c)\} = \leq^* \). Define a map \( f: (X, \tau, \leq) \rightarrow (X^*, \tau^*, \leq^*) \) by \( f(a) = b, f(b) = a \) and \( f(c) = c \). From example 2.4.06 \( f \) is an I-β-continuous map and from example 2.4.09 \( f \) is a D-β-continuous map and hence \( f \) is a B-β-continuous map.
{a} is a closed set in \((X^*, \tau^*)\), \(f^{-1}(\{a\}) = \{b\}\) is not an \(\alpha\)-closed set in \((X, \tau)\). Therefore \(f\) is not an \(\alpha\)-continuous map and consequently \(f\) is not a B-\(\alpha\)-continuous map.

Thus a B-\(\beta\)-continuous map need not be a B-\(\alpha\)-continuous map.

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