Chapter 3

\textit{k-out-of-n} system with repair: \textit{T-policy}

3.1 Introduction

In this chapter, we consider a \textit{k-out-of-n} system. In \textit{k-out-of-n} system, the system functions \textit{iff} at least \(k(1 \leq k \leq n)\) of the \(n\) components function. Server is activated on the elapse of \(T\) time units where \(T\) is exponentially distributed with parameter \(\alpha\) from the epoch of it being inactivated previously. The activation time after switching on, is negligible. Thus server is brought to the system at the moment which is \(\min\{T,\ \text{epoch of failure of } n-k \text{ units}\}\) after his previous departure. He continues to remain in the system until all the failed units are repaired, once he arrives. The process continues in this fashion. Both the continuous time Markovian case and the embedded Markov chain case are considered. Embedded case is discussed in section 3.3. We consider three different situations (a) cold system (b) warm system and (c) hot system. These are defined in section 3.2.1. We aim at finding out optimal \(T\) to maximize the profit, that is, to minimize the running cost and maximize the system reliability.

\(N\)-policy for repair of the \textit{k-out-of-n} system has been studied extensively in Krishnamoorthy, Ushakumari and Lakshmi (1998). \textit{k-out-of-n} system with general repair under \(N\)-policy has been studied by Ushakumari and Krishnamoorthy (1998). In these, the authors obtain the optimal number of components to fail before repair facility is activated in order to minimize the running cost and maximize the system reliability.

Waiting until a large number of units (very close to \(n-k\)) fail inorder for the server
to be called may lead to the system being down for longer duration thereby decreasing its up time and hence the reliability. Activating the server frequently results in high fixed cost. Hence we go for $T$-policy.

The chapter is presented as follows. Section 3.2 deals with the analysis of the model and it gives some preliminaries, notations, modelling and analysis of the problem under investigation. We outline the system state distribution in the finite time and in the long run for all the three models. Section 3.3 is devoted to the study of some measures of performance and section 3.4 discusses a control problem. It also provides some numerical illustration. Section 3.5 gives the general case where $T$ is assumed to be arbitrarily distributed.

3.2 Analysis of the Model

Life times of units are assumed to have independent exponential distributions with parameter $\lambda_i$, when $i$ units are functioning. $T$ is exponentially distributed with parameter $\alpha$. Repair time is also assumed to be exponentially distributed with rate $\mu$.

**Definition 3.2.1.** The $k$-out-of-$n$ system is called a cold system if once the system is down (that is exactly $k - 1$ functional units) there is no further failure of units that are not in failed state, until system starts functioning.

**Definition 3.2.2.** The system is called a warm system if functional units continue to deteriorate and so fail even when the system is down, but now at a lesser rate.

**Definition 3.2.3.** A hot system is one in which components deteriorate at the same rate during the system down state as they deteriorate when the system is up.

We discuss these three situations separately. First, we introduce some notations.

$X(t)$ : number of functional components at time $t$.

$Y(t)$ : server state at time $t$.

Write

$$Y(t) = \begin{cases} 
1 & \text{if the server is available at time } t \\
0 & \text{otherwise}
\end{cases}$$
Under assumptions made on the distribution of repair time, life time of components and on $T$, we see that $\{(X(t), Y(t)), t \in \mathbb{R}_+\}$ is a Markov chain on $E_1 = \{(i, 0)|k + 1 \leq i \leq n\} \cup \{(i, 1)|i = k - 1, \ldots n\}$ for model a. (Definition 3.2.1) and $E_2 = \{(i, 0)|k + 1 \leq i \leq n\} \cup \{(i, 1)|0 \leq i \leq n\}$ for models b and c (Definition 3.2.2 and 3.2.3 respectively). Denote by $P_{ij}(t)$ the system state probability at time $t$ given $X(0) = n$, $Y(0) = 0$ that is $P_{ij}(t) = P((X(t), Y(t)) = (i, j)|(X(0), Y(0)) = (n, 0))$ for $(i, j) \in E_1(E_2))$.

### 3.2.1 Transient Solution

**Model a**

Here the functioning units do not deteriorate while the system is down. The Kolmogrov forward differential difference equations satisfied by $P_{ij}(t)$ are

\[
\begin{align*}
P_{m1}'(t) &= -(m\lambda_m + \mu(1 - \delta_{mn})P_{m1}(t) + (m + 1)\lambda_{m+1}(1 - \delta_{mn})P_{m+1,1}(t) \\
&\quad + \alpha(1 - \delta_{mk})P_{m0}(t) + (m + 1)\lambda_{m+1}\delta_{mk}P_{m+1,0}(t) + \mu P_{m-1,1}(t) , \ k \leq m \leq n \\

P_{m0}'(t) &= -(m\lambda_m + \alpha)P_{m0}(t) + (m + 1)\lambda_{m+1}(1 - \delta_{mn})P_{m+1,0}(t) \\
&\quad + \mu\delta_{mn}P_{m-1,1}(t) , \ k \leq m \leq n \\

P_{k-1,1}'(t) &= k\lambda_k P_{k1}(t) - \mu P_{k-1,1}(t)
\end{align*}
\]

where $\delta_{ij}$ is the Kronecker delta. The solution of equations 3.1 is given by $P(t) = e^{tA}P(0)$ where $P(0)$ is the initial probability vector which has 1 corresponding to state $(n, 0)$ and rest zeros. $A$ is the matrix of coefficients on the right side of the system of equations.

### 3.2.2 Steady State Probabilities

From the above equations, by setting $q_{ij} = \lim_{t \to \infty} P_{ij}(t), (i, j) \in E_1$, we get steady state probabilities

\[
q_{n1} = \frac{\alpha}{n\lambda_n} q_{no} \quad q_{n-1,1} = \frac{(n\lambda_n + \alpha)}{\mu} q_{no} \quad q_{rl} = \prod_{t=r}^{n-1} \frac{(l + 1)\lambda_{l+1}}{l\lambda_l} q_{no} , \ k + 1 \leq r \leq n - 1
\]
\[ q_{n-r,l} = \frac{(n - r + 1)}{\mu} q_{n-r+1,l} + \mu \frac{\alpha}{\mu} q_{n-r+1,0} - \frac{(n - r + 2)}{\mu} q_{n-r+2,l} \]

where \( l = 1, 2, \ldots, n - k \) and \( q_{n-l,0} \) for \( l = 1, 2, \ldots, n - k - 1 \) can be expressed in terms of \( q_{n0}, q_{k-1,1} = \frac{k}{\mu} q_{k1} q_{n0} \) can be determined from the relation \( \sum_{i=k+1}^{n} q_{i0} + \sum_{i=k-1}^{n} q_{i1} = 1 \). However the expressions for \( q_{n-l,1} \) for different \( i \) values is unwieldy and so we consider the particular case of \( \lambda_i = \frac{\lambda}{i} \) in further development. It is reasonable to make the assumption since deterioration rate increases with decreasing number of operational units.

3.2.3 Model b

In this model, when the number of functional components reduce to \( k - 1 \), the units that have not failed start deteriorating at a rate \( \delta < \lambda \). Then life time of functioning components are exponential with parameter \( \delta \). The Kolmogorov forward differential equations are

\[
P'_{m1}(t) = -(m \lambda_m + \mu(1 - \delta_{mn}))P_{m1}(t) + \alpha(1 - \delta_{mk})P_{m0}(t) + (m + 1)\lambda_{m+1}(1 - \delta_{mn})P_{m+1,1}(t) + (m + 1)\lambda_{m+1}\delta_{mk}P_{k+1,0}(t) + \mu(1 - \delta_{mn})P_{m-1,1}(t), \quad k \leq m \leq n
\]

\[
P'_{m0}(t) = -(m \lambda_m + \alpha)P_{m0}(t) + (m + 1)\lambda_{m+1}(1 - \delta_{mn})P_{m+1,0}(t) + \mu \delta_{mn}P_{m-1,1}(t).
\]

\[
P'_{m1}(t) = -(m \delta_m + \mu)P_{m1}(t) + (m + 1)\delta_{m+1}P_{m+1,1}(t) + \mu P_{m-1,1}(t) + (m + 1)\lambda_{m+1}\delta_{mk-1}P_{m+1,1}(t), \quad 0 < m \leq k - 1
\]

\[
P'_{01}(t) = -\mu P_{01}(t) + \delta P_{11}(t)
\]

These lead to the system state probabilities in the steady state with evolution of time

\[
q_{k-l,1} = \frac{(k - l + 1)}{\mu} \delta_{k-l+1,1} + \mu q_{k-l+1,1} - \frac{(k - l + 2)}{\mu} \delta_{k-l+2,1} q_{k-l+2,1} \leq l \leq k
\]

The rest of the system state probabilities are as in model a \( q_{k-l+1,1} \) and \( q_{k-l+2,1} \), \( l = 2, 3, \ldots, k \) are available in terms of \( q_{l0} \) which in turn can be obtained from the relation \( \sum_{i=k+1}^{n} q_{i0} + \sum_{i=k-1}^{n} q_{i1} = 1 \)
3.2.4 Model C

Here the functional components deteriorate at the same rate during down state of the system as that when the system is up. The time dependent system state distribution can be obtained as in model a. The long run system state probabilities are given by

$$ q_{k-l,1} = \frac{(k-l+1)\lambda_{k-l+1} + \mu}{\mu} q_{k-l+1,1} - \frac{(k-l+2)\lambda_{k-l+2}}{\mu} q_{k-l+2,1}, \quad 2 \leq l \leq k $$

and the rest of the system state probabilities are as in model a. The normalizing condition is $\sum_{(i,j) \in E_2} q_{ij} = 1$.

3.3 Some Performance Measures

We compute the optimal $\alpha$ for the three models. To do this, we need to compute the distribution of time during which the server is continuously available we assume $A_i = 1$ for $i = k, \ldots, n$ for model a, $A_i = \lambda_i$ for $i = k, \ldots, n$ and $\delta_i = \delta_i$ for $i = 1, 2, \ldots, k-1$ for model b, $\lambda_i = \lambda_i$, $i = 1, 2, \ldots, n$ for model c. This assumption states that failure rate decreases with increasing number of functioning units, which is quite reasonable.

3.3.1 Model a

Theorem 3.3.1. The system state probabilities in the long run are given by

$$ q_{n1} = \frac{\alpha}{\lambda} q_{n0}, \quad q_{n-1,1} = \frac{\lambda + \alpha}{\mu} q_{n0} $$

$$ q_{n-r,1} = \left[ \lambda^{-1}(\lambda + \alpha)^{n-r} + \lambda^{r}\mu(\lambda + \alpha)^{r-2} + \mu^{r-3}(\lambda + \alpha + \mu) + \ldots \right. $$

$$ + \mu(\lambda + \alpha)^{r-3} + \ldots + \mu^{r-4}(\lambda + \alpha)^2 \right] q_{n0} + q_{n0} \left[ \mu^{r}(\lambda + \alpha)^{r-1} \right] $$

$$ 2 \leq r \leq n-k $$

$$ q_{k-1,1} = \frac{\lambda}{\mu} q_{k1}, \quad q_{0} = \left( \frac{\lambda}{\lambda + \alpha} \right)^{n-r} q_{0}, \quad k+1 \leq r \leq n-1 $$
Proof. Consider the equation (3.1) from the equation

\[ P'_{n1}(t) = -n \lambda_n P_{n1}(t) + \alpha P_{n0}(t) \]

we can write the steady state equation as

\[ 0 = -n \lambda_n q_{n1} + \alpha q_{n0} \]

Hence, \( q_{n1} = \frac{\alpha}{n \lambda_n} q_{n0} \). Rest of the steady state equations are

\[ \begin{align*}
0 &= -(m \lambda_m + \mu (1 - \delta_{mn})) q_{m1} + (m + 1) \lambda_{m+1} (1 - \delta_{mn}) q_{m+1,1} + \alpha (1 - \delta_{mk}) q_{m0} \\
&\quad + (m + 1) \lambda_{m+1} \delta_{mk} q_{m+1,0} + \mu q_{m-1,1}, \quad k \leq m \leq n \\
0 &= -(m \lambda_m + \alpha) q_{m0} + (m + 1) \lambda_{m+1} q_{m+1,0}, \quad k + 1 \leq m < n. \\
0 &= k \lambda_k q_{k1} - \mu q_{k-1,1}
\end{align*} \]

Solving the above equations, we get steady state probabilities in terms of \( q_{n0} \).

\[ \text{Note: The system availability at any epoch is given by } 1 - q_{k-1,1}. \text{ Hence the fraction of time the system is not available is } q_{k-1,1}. \text{ Under the normalizing condition, we get } q_{n0}. \]

3.3.2 Distribution of the time server is continuously available

Consider the Markov chain on the state space \( \{(k - 1, 1), \ldots, (n, 0)\} \) with state \((n, 0)\) absorbing and the rest all transient. We have to compute the distribution of the time until reaching \((n, 0)\) starting from one of the transient states (corresponding to server arrival).

The infinitesimal generator of this chain is

\[
\begin{pmatrix}
-\mu & \mu & 0 \\
\lambda & -(\lambda + \mu) & \mu \\
0 & \lambda & \ddots \\
0 & 0 & \lambda & -(\lambda + \mu) & 0 \\
0 & 0 & \lambda & -\lambda & 0 \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
\end{pmatrix}
= 
\begin{pmatrix}
M_1 \\
\epsilon_\mu \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]
where \( M_1 \) is the matrix obtained by deleting the last row and last column of the generator and \( e_\mu \) is the column vector with last but one entry \( \mu \) and all other zero. \( \mathbf{q} \) is a row vector of zeros. The distribution of time till absorption is of phase type given by
\[
F_1(x) = 1 - \alpha_1 \exp(T \tau) e_\mu \text{ for } x \geq 0,
\]
where \( \alpha_1 \) is the row vector of initial probability with entries \( \alpha_k, \ldots, \alpha_n \) where \( \alpha_k = 0 \), \( \alpha_k = 1 - (\alpha_{k+1} + \ldots + \alpha_n) \) with \( \alpha_i = P(S_{n-i} < T < S_{n-i+1}) \) for \( i = k + 1, \ldots, n \) where the random variable \( S_i \) is the time till \( i \) failures take place starting from the instant at which all units function write \( S_0 = 0 \), then we have \( S_0 < S_1 < \ldots < S_{n-k} \) and \( e_\mu = (1, 1, \ldots, 1)^T \).

### 3.3.3 Expected duration of time the server is busy in a cycle is given by

\[
\sum_{i=k}^{n-2} \frac{1}{(\mu - \lambda)} ((\tau - i) - \frac{\lambda}{\mu})^{i-k+2} \frac{\mu(1 - (\frac{\lambda}{\mu})^{n-i})}{(\mu - \lambda)} P(S_{n-i-1} < T < S_{n-i}) + \frac{1}{(\mu - \lambda)} \frac{\alpha^{n-i}}{\alpha + \lambda}.
\]

where
\[
P(S_{n-i-1} < T < S_{n-i}) = \frac{\alpha^{n-i-1}}{\lambda + \alpha}^{n-i}, \quad k \leq i \leq n - 1.
\]

Let \( T_i \) denote the time to reach \((i + 1, 1)\) starting from \((i, 1)\), \(i \geq k - 1\). We can recursively compute \( E(T_i) \), \( i \geq k - 1 \) from the relation \( E(T_i) = \frac{1}{\mu} + \frac{\lambda}{\mu} E(T_{i-1}) \) starting from \( E(T_{k-1}) = \frac{1}{\mu} \).

From the state \((i, 1)\) both \((i + 1, 1)\) and \((i - 1, 1)\) can be reached

\[
(i, 1) \rightarrow (i + 1, 1) \quad (i, 1) \rightarrow (i - 1, 1) \rightarrow (i, 1) \rightarrow (i + 1, 1)
\]

(3.3)

\( T_i \) denote the time to reach \((i + 1, 1)\) from \((i, 1)\). Hence
\[
E(T_i) = \frac{1}{\lambda + \mu} \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} \left( \frac{1}{\lambda + \mu} + E(T_{i-1}) \right) + E(T_i)
\]

ie.
\[
E(T_i) \frac{1}{\lambda + \mu} = \frac{1}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} E(T_{i-1})
\]
Thus we get the relation

\[
E(T_i) = \frac{1}{\mu} + \frac{\lambda}{\mu} E(T_{i-1}), i \geq k - 1
\]

\[
= \frac{1}{\mu} + \frac{\lambda}{\mu} \left( \frac{1}{\mu} + \frac{\lambda}{\mu} E(T_{i-2}) \right)
\]

\[
= \frac{1}{\mu} + \frac{\lambda}{\mu} + \left( \frac{\lambda}{\mu} \right)^2 E(T_{i-2})
\]

\[
\ldots
\]

\[
= \frac{1}{\mu} \left( 1 - \left( \frac{\lambda}{\mu} \right)^{k-1} \right) = \frac{1 - \left( \frac{\lambda}{\mu} \right)^{k+2}}{(1 - \frac{\lambda}{\mu})}
\]

The expected time to reach \((n, 0)\) conditional on server getting activated between \((n - i)\)th and \((n - i + 1)\)th component failures is \(\sum_{j=i}^{n-1} E(T_j) \Pr(S_i < T < S_{i+1})\), where the random variable \(S_i\) is the time till \(i\) failures take place starting from the instant at which all units functions write \(S_0 = 0\), then we have \(S_0 < S_1 < \ldots < S_{n-k}\). With this the expected time to reach \((n, 0)\) is

\[
\sum_{i=k}^{n-2} \frac{1}{\mu - \lambda} \left( (n - i) - \left( \frac{\lambda}{\mu} \right)^{i} \frac{\mu((n - i) - \left( \frac{\lambda}{\mu} \right)^{i+2}}{(\mu - \lambda)} \Pr(S_{n-i-1} < T < S_{n-i})
\]

\[
+ \frac{1}{\mu - \lambda} \left( 1 - \left( \frac{\lambda}{\mu} \right)^{n-k+1} \right) \frac{\alpha}{\alpha + \lambda}
\]

where \(\Pr(S_{n-i-1} < T < S_{n-i}) = \frac{\alpha^n_{i-1}}{(\alpha + \lambda)^n}, k \leq i \leq n - 1\) and is obtained as follows.

\[
\Pr(S_{n-i-1} < T < S_{n-i}) = \int_0^\infty \int_0^\infty e^{-(\alpha + \lambda)(x-u)} \alpha e^{-\alpha x} e^{-\lambda(x-u)} du dx
\]

\[
= \frac{\alpha^n_{i-1}}{(\alpha + \lambda)^n}, k \leq i \leq n - 1
\]

3.3.4 **Expected time the server is not in the system in a cycle is given by**

\[
\frac{2}{\alpha} \left( 1 - \left( \frac{\lambda}{\lambda + \alpha} \right)^{n-k} \right)
\]

From the state \((n, 0)\) the system can move either to \((n, 1)\) or \((n - 1, 0)\). If it goes to \((n - 1, 0)\), then from this the system further moves to \((n - 2, 0)\) or \((n - 1, 1)\). This processing go on till the state \((k + 1, 0)\) is reached. From \((k + 1, 0)\) it can either go to \((k + 1, 1)\) or \((k, 1)\). At \((k + 1, 1)\) on failure of one unit the system goes to \((k, 1)\) by an assumption. Thus
the expected amount of time the server is not in the system in a cycle is

$$\frac{1}{\alpha} P(T < S_1) + \left(\frac{1}{\alpha} + \frac{1}{\lambda}\right) P(S_1 < T < S_2) + \cdots + \left(\frac{1}{\alpha} + \frac{n-k-1}{\lambda}\right) P(S_{n-k-1} < T < S_{n-k})$$

$$+ P(T > S_{n-k}) \frac{n-k}{\lambda}$$

$$= \frac{1}{\alpha} \frac{1}{\lambda + \alpha} + \left(\frac{1}{\alpha} + \frac{1}{\lambda}\right) \frac{\lambda}{\lambda + \alpha} \alpha + \left(\frac{1}{\alpha} + \frac{2}{\lambda}\right) \frac{\lambda^2}{\lambda + \alpha} + \cdots + \left(\frac{1}{\alpha} + \frac{n-k-1}{\lambda}\right) \frac{\lambda^{n-k-1}}{\lambda + \alpha} + \frac{n-k}{\lambda} \frac{\lambda^{n-k}}{\lambda + \alpha} = \frac{2}{\alpha} \left(1 - \left(\frac{1}{\lambda + \alpha}\right)^{n-k}\right)$$

3.3.5 Expected duration of time the system is down in a cycle is

$$\left(\frac{\lambda}{\mu}\right)^n \frac{1}{\mu^2} (\lambda + \alpha) + \left(\frac{\lambda}{\mu}\right)^n \frac{\mu}{\mu - 1} \frac{\lambda - n - k + 1}{\lambda - n - k + 1} + \left(\frac{\lambda}{\mu}\right)^n \frac{\lambda - n - k + 1}{\lambda - n - k + 1} \frac{(\lambda + \alpha + \mu)}{(\lambda + \alpha)^n - k} \frac{1}{\lambda - n - k + 1}$$

It is well known that $\frac{\theta_{k-1,1}}{\mu + 1}$ gives the expected number of visits to $(k - 1, 1)$ before first return to $(n, 0)$ (starting from $(n, 0)$) (see Tijms (1994)). Further $\frac{1}{\mu}$ is the expected amount of time system remains in $(k - 1, 1)$ during each visit to that state. Hence expected duration of time system is down is $\frac{\theta_{k-1,1}}{\mu + 1}$, which is equal to

$$\left(\frac{\lambda}{\mu}\right)^n \frac{1}{\mu^2} (\lambda + \alpha) + \left(\frac{\lambda}{\mu}\right)^n \frac{\mu}{\mu - 1} \frac{\lambda - n - k + 1}{\lambda - n - k + 1} + \left(\frac{\lambda}{\mu}\right)^n \frac{\lambda - n - k + 1}{\lambda - n - k + 1} \frac{(\lambda + \alpha + \mu)}{(\lambda + \alpha)^n - k} \frac{1}{\lambda - n - k + 1}$$

3.3.6 Model b

System state probabilities in the long run are the same as in model a for states $\{(k - 1, 1), \ldots, (n - 1, 1), (k + 1, 0), \ldots, (n, 0)\}$. Further since the functional units deteriorate even when the system is down, we have for $l = 2, \ldots, k, \theta_{k-1,1} = \frac{n}{\mu} \left(1 - q_{k-1,1}\right), 2 \leq l \leq k$. The system is down for the fraction of time $\sum_{i=0}^{k-1} q_{ii}$. So the system reliability is

$$1 - \sum_{i=0}^{k-1} q_{ii} = 1 - \frac{(1 - (1/\mu)^{n-k})}{(1 - (1/\mu))^{k-1}} q_{k-1,1}$$

3.3.7 Distribution of server availability

Consider the Markov chain on the state space $\{(0, 1), (1, 1), \ldots, (k, 1), \ldots (n - 1, 1), (n, 0)\}$ with state $(n, 0)$ absorbing. This distribution of phase type given by $F_2(x) = 1 - \alpha_2 \exp(M_2 x)\alpha_2$, where $M_2$ is the matrix.
\( \alpha_2 \) is the row vector of initial probabilities with first \( k \) entries zero, the rest of the entries are \( \alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n, \alpha_{n+1} \) where \( \alpha_{k+1} = 1 - (\alpha_{k+2} + \ldots + \alpha_{n+1}) \) and \( \alpha_i = P(S_{n-i+1} < T < S_{n-i+2}), i = k + 2, \ldots, n \). \( \alpha_{n+1} = P(T < S_1), e_2 = (1, 1, \ldots, 1)^T \).

### 3.3.8 Expected duration of time the server is continuously busy

As in the earlier model, \( T_i \) denote the time to enter state \((i + 1, 1)\) starting from \((i, 1)\). Here \( E(T_0) = \frac{1}{\mu} \).

\[
E(T_1) = \frac{1}{\mu} (1 + \lambda) \quad E(T_{k-1}) = \frac{1}{\mu} \left(1 - \frac{(\frac{\delta}{\mu})^k}{1 - \frac{\lambda}{\mu}} \right)
\]

\[
E(T_k) = \frac{1}{\mu} + \frac{\lambda}{\mu} \frac{1}{1 - \frac{\lambda}{\mu}} \left(1 - \frac{(\frac{\delta}{\mu})^k}{1 - \frac{\lambda}{\mu}} \right)
\]

We can recursively compute \( E(T_i), i \geq 0 \) from the relation \( E(T_i) = \frac{1}{\mu} + \frac{\lambda}{\mu} E(T_{i-1}) \) starting from \( E(T_{k-1}) = \frac{1}{\mu} (1 - (\frac{\delta}{\mu})^k) \). Thus

\[
E(T_j) = \frac{1}{\mu} \left[ \frac{1 - (\lambda/\mu)^{j+1-k}}{1 - (\lambda/\mu)} + (\lambda/\mu)^{j+1-k} \frac{1 - (\delta/\mu)^k}{1 - (\delta/\mu)} \right]
\]

The expected time to reach \((n, 0)\) conditional on server reaches between \((n - i)\)th and \((n - i + 1)\)th component failures is \( \sum_{j=i}^{n-1} E(T_j \mid S_i < T < S_{i+1}) P(S_i < T < S_{i+1}) \) which is equal to

\[
\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} \frac{1}{\mu} \left[ \frac{1 - (\lambda/\mu)^{j+1-k}}{1 - (\lambda/\mu)} + (\lambda/\mu)^{j+1-k} \frac{1 - (\delta/\mu)^k}{1 - (\delta/\mu)} \right] P(S_{n-i-1} < T < S_{n-i}),
\]

where \( P(S_{n-i-1} < T < S_{n-i}) = \frac{1}{(\lambda + \alpha)^{n-i}}, k \leq i \leq n - 1 \)

\[
P(T < S_1) = \frac{\alpha}{\lambda + \alpha}
\]

Expected time the server remains inactive during a cycle is same as in model a.
### 3.3.9 Expected duration of time the system is down in a cycle

To this end note that:

$$\frac{a_{k-1,1}}{q_0}$$

gives the expected number of visits to \((k-1, 1)\) before first return to \((n, 0)\). Consider the class \(\{(0, 1), (1, 1), \ldots, (k-1, 1)\}\). The process spends on the average \(\frac{1}{\mu} \left(1 - \left(\frac{\mu}{\lambda}\right)^k\right)\) amount of time in this class during each visit before returning to state \((k, 1)\). Hence expected duration of time the system is down in a cycle is

$$\frac{1}{\mu} \left(1 - \left(\frac{\mu}{\lambda}\right)^k\right) q_{k-1,1} = \frac{1}{\mu} \left(1 - \left(\frac{\mu}{\lambda}\right)^k\right) \left(\frac{\lambda}{\mu}\right)^{n-k} (\frac{\lambda + \alpha}{\mu})^k + \left(\frac{x_{k+1}}{\lambda + \alpha}\right)^{n-k-3}$$

$$\frac{(\lambda + \alpha + \mu)}{(\lambda + \alpha)^{n-k-1}} + \left(\frac{\lambda}{\mu}\right)^{n-k+1} \frac{1 - \left(\frac{x_{k+1}}{\lambda + \alpha}\right)^{n-k-3}}{(\lambda + \alpha - \mu)}$$

### 3.3.10 Model c

System state probabilities in the long run are the same as in model a for states \((k-1,1), \ldots, (n-1,1), (k+1,0), \ldots, (n,0)\). Further since the functional units deteriorate at the same rate even when the system is down as when it is up \(q_{k-1,1} = \left(\frac{\mu}{\lambda}\right)^{l-1} q_{k-1,1}\) for \(l = 2, 3, \ldots, k\) can be expressed in terms of \(q_{0}\). System reliability is computed as earlier. \(q_{0}\) can be obtained using the normalizing condition \(\sum_{ij \in E} q_{ij} = 1\).

The distribution of the duration of time the server continuously remains in the system is given by

$$F_3(x) = 1 - a_3 \exp(M_3 x) e_3$$

where \(a_3\) is a \((n+1)\) component row vector with first \(k\) entries zero the rest of the entries are \(a_{k+1}, a_{k+2}, \ldots, a_n\) and \(a_{n+1} = P(S_{n-1} < T < S_{n+i})\), \(i = k+2, \ldots, n\). \(a_{n+1} = P(T < S_1)\). \(e_3\) is also of the same dimension with all entries 1. \(M_3\) is a non-singular matrix of order \(n\) given by first \(n\) rows and \(n\) columns of the matrix \(I - P\) where \(I\) is of order \((n+1)\) and \(P\) is the transition probability matrix of the chain on the set \(\{(0,1), (1,1), \ldots, (n-1,1), (n,1), (n,0)\}\).
3.3.11 Expected amount of time the server is continuously busy

In this case \( E(T_j) = \frac{1}{\mu} \frac{(1-(\lambda/\mu)^j)}{1-(\frac{\lambda}{\mu})} \), \( j \geq 0 \) starting with \( E(T_0) = \frac{1}{\mu} \). As in model b, we get \( E(T_i) = \sum_{j=1}^{n-1} E(T_j) \setminus S_i < T < S_{i+1} \).

\[
P(S_i < T < S_{i+1}) = \sum_{i=0}^{n-1} \sum_{j=1}^{n-1} \frac{1}{\mu} \frac{1-(\lambda/\mu)^{j+1}}{1-\lambda/\mu} P(S_{n-i-1} < T < S_{n-i}),
\]

where \( P(S_{n-i-1} < T < S_{n-i}) = \frac{\alpha^{n-i-1}}{(\lambda+\alpha)^{n-i}} \), \( k \leq i \leq n - 1 \) Thus

\[
E(T_i) = \sum_{i=0}^{n-2} (n-i) - \left( \frac{\lambda}{\mu} \right)^{i+1} \frac{1-(\frac{\lambda}{\mu})^{n-i}}{(1-\frac{\lambda}{\mu})} \frac{\alpha^{n-i-1}}{(\lambda+\alpha)^{n-i}} + (1-(\frac{\lambda}{\mu})^n) \frac{\alpha}{\lambda+\alpha}
\]

Here also the expected time the server is not in the system is the same as in the above two models.

3.3.12 Expected amount of time the system is nonfunctional

The process spends on the average \( \frac{1}{\mu} \frac{(1-(\frac{\lambda}{\mu})^k)}{1-(\frac{\lambda}{\mu})} \) amount of time in the class \( \{(0,1), (1,1), \ldots, (k-1,1)\} \). Expected amount of time the system is non-functional in a cycle is

\[
\frac{1}{\mu} \frac{(1-(\frac{\lambda}{\mu})^k) q_{k-1,1}}{1-\frac{\lambda}{\mu} q_{n0}}
\]

3.4 A Control Problem

Here we attempt to find the optimal value of \( \alpha \) by maximizing the profit and the system reliability. The following costs are considered.

1. Cost \( C \) per unit time due to the machine remaining non-functional
2. Profit per unit time when the server is not activated in the system.

Let \( C \) denote the cost per unit time due to the machine remaining non-functional and \( w \) denote the wages given to the server.
3.4.1 Model a

Profit per unit time when the server is not activated $= w\left(\frac{2}{\alpha}(1 - \left(\frac{\lambda}{\lambda + \alpha}\right)^{n-k})\right)$.

Expected cost per unit time due to the system remaining non-functional $= C\left(\frac{1}{\mu}\right)^{q_k-1} = C(\lambda + \alpha)^{n-k+1} \left(\frac{1}{\lambda \mu^2}(\lambda + \alpha) + \frac{\mu^{n-k-2}(\lambda + \alpha + \mu)}{(\lambda + \alpha)^{n-k-1}} + \frac{(1 - (\frac{\mu}{\lambda + \alpha})^{n-k-3})}{(\lambda + \alpha - \mu)}\right)$

Therefore the total expected profit per unit time $(TEP)_a$ is

$$w\left(\frac{2}{\alpha}(1 - \left(\frac{\lambda}{\lambda + \alpha}\right)^{n-k})\right) - C\left(\frac{\lambda}{\mu}\right)^{n-k+1} \left(\frac{1}{\lambda \mu^2}(\lambda + \alpha) + \frac{\mu^{n-k-3}(\lambda + \alpha + \mu)}{(\lambda + \alpha)^{n-k-1}} + \frac{(1 - (\frac{\mu}{\lambda + \alpha})^{n-k-3})}{(\lambda + \alpha - \mu)}\right)$$

The above function is concave in $\alpha$ as it can be seen by differentiating it twice with respect to $\alpha$. However it is difficult to find optimal $\alpha$ value from the first derivative equated to zero.

3.4.2 Model b

In model b, the total expected profit per unit time $(TEP)_b$ is

$$(TEP)_b = w\left(\frac{2}{\alpha}(1 - \left(\frac{\lambda}{\lambda + \alpha}\right)^{n-k})\right) - C\left(\frac{\lambda}{\mu}\right)^{n-k+1} \left(\frac{1}{\mu^2}(\lambda + \alpha) + \frac{\mu^{n-k-3}(\lambda + \alpha + \mu)}{(\lambda + \alpha)^{n-k-1}} + \frac{(1 - (\frac{\mu}{\lambda + \alpha})^{n-k-3})}{(\lambda + \alpha - \mu)}\right) \frac{1 - (\frac{\mu}{\lambda + \alpha})^{n-k-3}}{\mu - \delta}$$

Here again $(TEP)_b$ is a concave function in $\alpha$ as can be seen by differentiating the profit function with respect to $\alpha$.

3.4.3 Model c

In this case the total expected profit per unit time $(TEP)_c$ is

$$(TEP)_c = w\left(\frac{2}{\alpha}(1 - \left(\frac{\lambda}{\lambda + \alpha}\right)^{n-k})\right) - C\left(\frac{\lambda}{\mu}\right)^{n-k+1} \left(\frac{1}{\mu^2}(\lambda + \alpha) + \frac{\mu^{n-k-3}(\lambda + \alpha + \mu)}{(\lambda + \alpha)^{n-k-1}} + \frac{(1 - (\frac{\mu}{\lambda + \alpha})^{n-k-3})}{(\lambda + \alpha - \mu)}\right) \frac{1 - (\frac{\mu}{\lambda + \alpha})^{n-k-3}}{\mu - \lambda}$$

which is concave in $\alpha$ and hence has global maximum.
3.4.4 Numerical illustration

For illustration, we calculate the total expected profit per unit time for given parameters for the three models and for various value of α. On comparing the three models for different set of parameters, we can see that total expected profit is maximum for model b.

Comparison of three models

\( n = 12, \lambda = 7.5, \mu = 13, k = 6, w = 70, C = 80, \delta = 5 \)

<table>
<thead>
<tr>
<th>α</th>
<th>((TEP)_a)</th>
<th>((TEP)_b)</th>
<th>((TEP)_c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>38.986</td>
<td>40.284</td>
<td>40.209</td>
</tr>
<tr>
<td>3.1</td>
<td>38.052</td>
<td>39.315</td>
<td>39.242</td>
</tr>
<tr>
<td>3.2</td>
<td>37.156</td>
<td>38.386</td>
<td>38.315</td>
</tr>
<tr>
<td>3.3</td>
<td>36.296</td>
<td>37.495</td>
<td>37.426</td>
</tr>
<tr>
<td>3.4</td>
<td>35.471</td>
<td>36.646</td>
<td>36.573</td>
</tr>
<tr>
<td>3.5</td>
<td>34.679</td>
<td>35.819</td>
<td>35.753</td>
</tr>
<tr>
<td>3.6</td>
<td>33.917</td>
<td>35.03</td>
<td>34.966</td>
</tr>
<tr>
<td>3.7</td>
<td>33.184</td>
<td>34.271</td>
<td>34.209</td>
</tr>
<tr>
<td>3.8</td>
<td>32.479</td>
<td>33.541</td>
<td>33.48</td>
</tr>
<tr>
<td>3.9</td>
<td>31.8</td>
<td>32.839</td>
<td>32.779</td>
</tr>
<tr>
<td>4</td>
<td>31.146</td>
<td>32.162</td>
<td>32.104</td>
</tr>
<tr>
<td>4.1</td>
<td>30.516</td>
<td>31.51</td>
<td>31.453</td>
</tr>
<tr>
<td>4.2</td>
<td>29.908</td>
<td>30.882</td>
<td>30.826</td>
</tr>
<tr>
<td>4.3</td>
<td>29.322</td>
<td>30.276</td>
<td>30.221</td>
</tr>
<tr>
<td>4.4</td>
<td>28.756</td>
<td>29.691</td>
<td>29.637</td>
</tr>
<tr>
<td>4.5</td>
<td>28.21</td>
<td>29.126</td>
<td>29.073</td>
</tr>
<tr>
<td>4.6</td>
<td>27.682</td>
<td>28.581</td>
<td>28.529</td>
</tr>
<tr>
<td>4.7</td>
<td>27.171</td>
<td>28.054</td>
<td>28.003</td>
</tr>
<tr>
<td>4.8</td>
<td>26.678</td>
<td>27.544</td>
<td>27.494</td>
</tr>
<tr>
<td>4.9</td>
<td>26.2</td>
<td>27.051</td>
<td>27.002</td>
</tr>
<tr>
<td>5</td>
<td>25.738</td>
<td>26.575</td>
<td>26.526</td>
</tr>
</tbody>
</table>

\( n = 18, k = 7, \lambda = 9.5, \mu = 14, w = 80, C = 110, \delta = 4 \)

<table>
<thead>
<tr>
<th>α</th>
<th>((TEP)_a)</th>
<th>((TEP)_b)</th>
<th>((TEP)_c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>49.157</td>
<td>50.57</td>
<td>50.402</td>
</tr>
<tr>
<td>2.1</td>
<td>47.805</td>
<td>49.153</td>
<td>48.992</td>
</tr>
<tr>
<td>2.2</td>
<td>46.518</td>
<td>47.805</td>
<td>47.652</td>
</tr>
<tr>
<td>2.3</td>
<td>45.292</td>
<td>46.523</td>
<td>46.376</td>
</tr>
</tbody>
</table>
### Total expected profit/unit time

<table>
<thead>
<tr>
<th>α</th>
<th>$(TEP)_a$</th>
<th>$(TEP)_b$</th>
<th>$(TEP)_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4</td>
<td>44.124</td>
<td>45.302</td>
<td>45.161</td>
</tr>
<tr>
<td>2.5</td>
<td>43.01</td>
<td>44.138</td>
<td>44.003</td>
</tr>
<tr>
<td>2.6</td>
<td>41.946</td>
<td>43.028</td>
<td>42.898</td>
</tr>
<tr>
<td>2.7</td>
<td>40.929</td>
<td>41.968</td>
<td>41.844</td>
</tr>
<tr>
<td>2.8</td>
<td>39.957</td>
<td>40.955</td>
<td>40.835</td>
</tr>
<tr>
<td>2.9</td>
<td>39.027</td>
<td>39.986</td>
<td>39.871</td>
</tr>
<tr>
<td>3</td>
<td>38.136</td>
<td>39.059</td>
<td>38.949</td>
</tr>
<tr>
<td>3.1</td>
<td>37.282</td>
<td>38.172</td>
<td>38.065</td>
</tr>
<tr>
<td>3.2</td>
<td>36.463</td>
<td>37.321</td>
<td>37.218</td>
</tr>
<tr>
<td>3.3</td>
<td>35.677</td>
<td>36.505</td>
<td>36.406</td>
</tr>
<tr>
<td>3.4</td>
<td>34.922</td>
<td>35.722</td>
<td>35.627</td>
</tr>
<tr>
<td>3.5</td>
<td>34.796</td>
<td>35.03</td>
<td>35.002</td>
</tr>
<tr>
<td>3.6</td>
<td>33.499</td>
<td>34.248</td>
<td>34.158</td>
</tr>
<tr>
<td>3.7</td>
<td>32.828</td>
<td>33.553</td>
<td>33.466</td>
</tr>
<tr>
<td>3.8</td>
<td>32.182</td>
<td>32.885</td>
<td>32.806</td>
</tr>
<tr>
<td>3.9</td>
<td>31.56</td>
<td>32.241</td>
<td>32.216</td>
</tr>
<tr>
<td>4</td>
<td>30.96</td>
<td>31.621</td>
<td>31.542</td>
</tr>
</tbody>
</table>

$n = 10, K = 5, \lambda = 5.5, \mu = 10, w = 50, C = 100, \delta = 3$

### Total expected profit/unit time

<table>
<thead>
<tr>
<th>α</th>
<th>$(TEP)_a$</th>
<th>$(TEP)_b$</th>
<th>$(TEP)_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>27.435</td>
<td>29.251</td>
<td>29.106</td>
</tr>
<tr>
<td>3.1</td>
<td>26.737</td>
<td>28.512</td>
<td>28.37</td>
</tr>
<tr>
<td>3.2</td>
<td>26.069</td>
<td>27.806</td>
<td>27.667</td>
</tr>
<tr>
<td>3.3</td>
<td>25.43</td>
<td>27.131</td>
<td>26.995</td>
</tr>
<tr>
<td>3.4</td>
<td>24.819</td>
<td>26.484</td>
<td>26.351</td>
</tr>
<tr>
<td>3.5</td>
<td>24.232</td>
<td>24.865</td>
<td>25.734</td>
</tr>
<tr>
<td>3.6</td>
<td>23.67</td>
<td>25.271</td>
<td>25.143</td>
</tr>
<tr>
<td>3.7</td>
<td>23.13</td>
<td>24.702</td>
<td>24.576</td>
</tr>
<tr>
<td>3.8</td>
<td>22.612</td>
<td>24.156</td>
<td>24.032</td>
</tr>
<tr>
<td>3.9</td>
<td>22.114</td>
<td>23.631</td>
<td>23.509</td>
</tr>
<tr>
<td>4</td>
<td>21.635</td>
<td>23.126</td>
<td>23.007</td>
</tr>
<tr>
<td>4.1</td>
<td>21.174</td>
<td>22.641</td>
<td>22.524</td>
</tr>
<tr>
<td>4.2</td>
<td>20.73</td>
<td>22.174</td>
<td>22.059</td>
</tr>
<tr>
<td>4.3</td>
<td>20.303</td>
<td>21.725</td>
<td>21.611</td>
</tr>
<tr>
<td>4.4</td>
<td>19.891</td>
<td>21.292</td>
<td>21.18</td>
</tr>
</tbody>
</table>
3.5 General case

Here we assume that the repair time is arbitrarily distributed with distribution function $G(.)$ having density $g(.)$. The server is activated after the elapse of $T$ time units since last inactivation after completion of most recent repair of all failed units or when the number of failed units accumulate to $n - k$ units, whichever occurs first. Life times of components are i.i.d random variables. $T$ is exponentially distributed with parameter $\alpha$.

3.5.1 Formulation and Analysis of the problem

Assume that at time $T_0 = 0$ the last of the failed units completed repair. That is we start the system at time zero with all units operational. $X(t)$ be the number of working components and $Y(t)$, the state of the repair man at time $t$. Write $X_n = X(T_n+)$ and $Y_n = Y(T_n+)$ for $n \in Z$.

We consider three cases (i) cold system (ii) warm system (iii) hot system where we designate the system as cold, warm or hot according as the functional components do not fail, fail at a slower rate or at the same rate during system down state as when the system functions, respectively.

We observe that $\{(X(t), Y(t)). t \in R_+\}$ is a semi-Markov process on $E_1 = \{(k - 1, 1), (k, 1), \ldots, (n - 1, 1), (k + 1, 0) \ldots (n, 0), (n, 1)\}$ in model 1 and on $E_2 = \{(0, 1), \ldots, (k - 1, 1), (k, 1) \ldots (n, 1), (k + 1, 0) \ldots (n, 0)\}$ in models 2 and 3 (the warm and hot systems).

Let time $T_0 = 0$, the system starts with all components operational. Thus $X(0+) = X_0 = n$ and $Y(0+) = Y_0 = 0$. Let $T_1, T_2, \ldots, T_n, \ldots$ be the successive repair completion epochs of failed units.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$(TEP)_a$</th>
<th>$(TEP)_b$</th>
<th>$(TEP)_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5</td>
<td>20.047</td>
<td>20.953</td>
<td>20.88</td>
</tr>
<tr>
<td>4.6</td>
<td>19.11</td>
<td>20.472</td>
<td>20.363</td>
</tr>
<tr>
<td>4.7</td>
<td>18.739</td>
<td>20.083</td>
<td>19.976</td>
</tr>
<tr>
<td>4.8</td>
<td>18.382</td>
<td>19.708</td>
<td>19.602</td>
</tr>
<tr>
<td>4.9</td>
<td>18.036</td>
<td>19.346</td>
<td>19.241</td>
</tr>
<tr>
<td>5</td>
<td>17.701</td>
<td>18.996</td>
<td>18.893</td>
</tr>
</tbody>
</table>
A server is activated after the elapse of $T$ time units since inactivation after completion of the most recent repair of all failed units or when the number of failed units accumulate to $n - k$, whichever occur first. Then, we have

**Theorem 3.5.1.** \{$(X_n, T_n)$, $n \in \mathbb{Z}^+$\} is a Markov renewal process with state space $E_3 = \{(k, 1), \ldots, (n-1, 1), (n, 0)\}$ for model 1 and $E_1 = \{(1, 1), \ldots, (n-1, 1), (n, 0)\}$ for model 2 and 3 with semi-Markov kernel $Q(i, j, t)$ define as $Q(i, j, t) = P((X_{n+1}, Y_{n+1}) = (j, l); T_{n+1} - T_n \leq t \mid (X_n, Y_n) = (i, l))$, $t \in \mathbb{R}_+$

**Proof.** For model 1. there are given by

$$Q((i, 1)(j, 1), t) = \int_0^t e^{-\lambda u} (\lambda u)^{i-j+1} g(u) du \quad j \leq i + 1, i \neq n - 1$$

$$Q((n-1, 1), (n, 0), t) = \int_0^t e^{-\lambda u} g(u) du$$

$$Q((n-1, 1), (j, 1), t) = \int_0^t e^{-\lambda u} (\lambda u)^{n-j-1} g(u) du \quad j \leq n - 1$$

$$Q((n, 0), (n, 0), t) = \int_0^t \sum_{n \geq 0} \int_{r = n}^t \int_{u = n}^t \frac{\alpha e^{-\alpha u} \lambda e^{-\lambda v} g(v - u) e^{-\lambda (w - v)}}{(n - j + 1)!} du dv$$

$$Q((n, 0), (i, 1), t) = \sum_{n \geq 0} \int_{r = n}^t \int_{u = n}^t \frac{\lambda e^{-\lambda u} (\lambda u)^j \alpha e^{-\alpha u} e^{-\lambda (v - u)} (\lambda (v - u)^{n-i-j+1})}{(n - i)!} du dv$$

For model 2, we have

$$Q((i, 1), (j, 1), t) = \int_0^t e^{-\lambda u} (\lambda u)^{i-j+1} g(u) du \quad i = k \ldots n - 1, k \leq j \leq i + 1$$

$$Q((i, 1), (j, 1), t) = \int_0^t e^{-\delta u} (\delta u)^{i-j+1} g(u) du \quad i = 1, 2 \ldots, k - 1; j \leq i + 1$$
\[ Q((i, 1), (j, 1), t) = \int_{u=0}^{t} \int_{v=u}^{t} \frac{e^{-\lambda u (\lambda u)^{i-k}} \lambda e^{-\delta(v-u)}(\delta(v-u)(\lambda(v-u))^{k-1}(v-u))^{-1}}{(i-k)!} g(v)dvdu \]

\[ i = k, k + 1, \ldots, n - 1; \quad j = 1, 2, \ldots, k \]

\[ Q((n - 1, 1), (n, 0), t) = \int_{r=0}^{t} e^{-\lambda u}g(u)du \]

\[ Q((n, 0), (i, 1), t) = \int_{u=0}^{t} \int_{w=u}^{t} \int_{x=w}^{t} \int_{r=x}^{t} \frac{\lambda e^{-\lambda u} (\lambda u)^{n-k-1}}{(n-k-1)!} e^{-\alpha u} g(v-u) \lambda e^{-\lambda(v-u)} e^{-\delta(v-w)} \]

\[ \frac{(\delta(v-w))^{k-1}}{(k-i)!} dvdu + \int_{u=0}^{t} \int_{w=u}^{t} \int_{x=w}^{t} \int_{r=x}^{t} \sum_{j=1}^{n-k-1} \frac{e^{-\lambda x_1 (\lambda x_1)^{i-1}} \lambda}{(j-1)!} \alpha e^{-\alpha u} e^{-\lambda x_2 (\lambda x_2 - x_1)} (\lambda(x_2 - x_1))^{n-k} g(v-u) \]

\[ \frac{(\delta(v-x_2))^{k-1}}{(k-i)!} dvdx_2du \]

\[ + \int_{u=0}^{t} \int_{w=u}^{t} \int_{x=w}^{t} \int_{r=x}^{t} \alpha e^{-\alpha u} \lambda e^{-\lambda w} g(v-u) e^{-\lambda(r-w)} \]

\[ \frac{(\lambda(r-w))^{n-k}}{(n-k)!} (\delta(v-x))^{k-1} \frac{(k-i)!}{(k-i)!} dvdx du. \quad i = 1, 2, \ldots, k \]

for model 3, we get \( Q(i, j, t) \) as,

\[ Q((i, 1), (j, 1), t) = \int_{u=0}^{t} e^{-\lambda u} (\lambda u)^{i-j+1} g(u)du \quad i = 1, 2, \ldots, k - 1; \quad j = i + 1 \]

\[ Q((i, 1), (j, 1), t) = \int_{u=0}^{t} e^{-\lambda u} (\lambda u)^{i-j} g(u)du, i = k, k + 1, \ldots, n - 1; \quad j = 1, 2, \ldots, k \]

\[ Q((n, 0), (i, 1), t) = \int_{u=0}^{t} \int_{v=u}^{t} \frac{\lambda e^{-\lambda u} (\lambda u)^{n-k-1}}{(n-k-1)!} e^{-\alpha u} g(v-u) e^{-\lambda(v-u)} (\lambda(v-u))^{k-1+1} dvdu \]

\[ + \int_{u=0}^{t} \int_{v=u}^{t} \int_{w=v}^{t} \sum_{j=1}^{n-k-1} \frac{\lambda e^{-\lambda u} (\lambda u)^{j}}{(j-1)!} \alpha e^{-\alpha u} e^{-\lambda(v-u)} (\lambda(v-u))^{n-j+1} g(v-u)dvdu \]

\[ + \int_{u=0}^{t} \int_{v=u}^{t} \int_{w=v}^{t} \alpha e^{-\alpha u} \lambda e^{-\lambda v} g(w-v) e^{-\lambda(v-w)} (\lambda(v-w))^{n-1} dvdu \]
3.5.2 Time dependent solution

Model 1

Let \( P_{(i_1,j_1),(i_2,j_2)}(t) = P((X(t) = i_2, Y(t) = j_2 \mid X(0) = i_1, Y(0) = j_1)) \) and \( P_{(i_1,j_1)}(t) = P((X(t) = i_1, Y(t) = j_1 \mid X(0) = i, Y(0) = 0)) \). These probabilities are obtained from the definition of \( Q(i,j,t) \), \( i,j \in E_3 \). Define

\[
U((i_1,1),(j_1,1),t) = P((X(t),Y(t)) = (j_1,1), T_1 > t \mid (X(0),Y(0)) = (i_1,1))
\]

Then \( U((i_1,1),(j_1,1),t) = (1 - G(t)) \frac{e^{-\lambda t} \mu^{n-1}}{(n-j_1)!} \) where \( k - 1 \leq j_1 \leq n - 1 \) and \( j_1 \leq i_1 \).

Further define

\[
H((n,0),(n,0),t) = \int_0^t \int_0^t e^{-\alpha u} \lambda e^{-\lambda u} \sum_{l \geq 1} Q^{(l)}((n-1,1),(n,0),t-v) dv du
\]

\[
+ \int_0^t \sum_{m=1}^{n-k} e^{-\lambda t} \frac{\mu^m}{m!} \alpha e^{-\alpha u} \sum_{l \geq m} Q^{(l)}((n-m,1),(n,0),t-u) du
\]

\[
+ \int_0^t e^{-\lambda t} \frac{\mu^{n-k-1}}{(n-k-1)!} \alpha e^{-\alpha u} \sum_{l \geq n-k} Q^{(l)}((k,1),(n,0),t-u) du
\]

where \( Q^{(l)}((i,1),(j,1),l) \) is the \( l \)-fold convolution of \( Q \) with itself and

\[
Q^{(0)} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}
\]

This represents the distribution of time of first return to \((n,0)\) starting from \((n,0)\)

\[
P_{(i,0)}(t) = \int_0^t \sum_{l=0}^{n-1} H^{(l)}((n,0),(n,0),du)e^{-\lambda t-u} \frac{\mu^{l-1}}{(l-1)!} du, \quad i = k + 1, \ldots, n
\]

\[
P_{(i,1)}(t) = \int_0^t \sum_{j=k}^{n-1} Q((n,0),(j,1),du)P_{(j,1),(1,1)}(t-u) du
\]

3.5.3 Limiting distribution

Let \( Q = (\lim_{t \to -\infty} Q((i_1,j_1),(i_2,j_2),t)), (i_1,j_1),(i_2,j_2) \in E_3 \) and \( \Pi = (\pi(k,1), \pi(k+1,1), \ldots, \pi(n-1,1), \pi(n,0)) \) is the stationary vector where \( \pi(i,j) = \lim_{n \to \infty} P(X_n = i, Y_n = j \mid X_0 = n, Y_0 = 0) \) where \( j = 0 \) if \( i = n \) and \( j = 1 \) when \( i = k, k+1, \ldots, n-1 \).

These probabilities can be computed from \( \Pi Q = \Pi \) and \( \sum_{i=1}^{n-1} \pi(i,1) + \pi(n,0) = 1 \). The long run system state distribution at arbitrary epoch can be derived as follows. Define

\[
q_{ij} = \lim_{t \to \infty} P((X(t) = i, Y(t) = j \mid X(0) = n, Y(0) = 0))
\]

for \( i = k+1, \ldots, n-1; j = 0, i = k-1, \ldots, n-1; j = 1 \).
Model 1

Let $\mu = \int_0^\infty (1 - G(t))dt$ which we assume to be finite. Then,

$$q_{00} = \frac{\pi(n, 0)}{\mu}$$

$$q_{n1} = \frac{\pi(n, 0)\alpha}{(\lambda + \alpha)\mu}$$

$$q_{i0} = \frac{\pi(n, 0)}{\mu} \frac{\lambda^{n-i}}{(\alpha + \lambda)^{n+1}}, \quad i = k + 1, \ldots, n - 1$$

$$q_{11} = \sum_{j \leq i} \frac{\pi(j, 1)}{\mu} \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^{j-i}}{(j-i)!} (1 - G(u))du$$

Next, we find $q_{(k-1, 1)}$. State $(k - 1, 1)$ can be reached from the states $(j, 1), j = k, \ldots, n - 1$ and $(n, 0)$. We have derived $\pi(j, 1), j = k, \ldots, n - 1$ and $\pi(n, 0)$. From the state $(j, 1)$ state $(k - 1, 1)$ can be reached by the failure of the $j - k$ units. From state $(n, 0)$ state $(k - 1, 1)$ can be reached due to failure of $n - k$ units. Here we consider three cases

(i) server arrives before any failure
(ii) server arrives between $l^{th}$ and $(l + 1)$th failure
(iii) server arrives only on the failure of $n - k$ units.

Therefore,

$$q_{(k-1, 1)} = \sum_{j=k}^{n-1} \frac{\pi(j, 1)}{\mu} \int_0^\infty e^{-\lambda u} \frac{(\lambda u)^j}{(j-k+1)!} (1 - G(u))du \cdot \frac{\lambda^{n-k-1} u^{n-k-1}}{(n-k)!} (1 - G(v-u)) \sum_{i=k}^{\infty} \frac{e^{-\lambda v} \lambda v^{n-i}}{(n-k)!} (1 - G(v-u))$$

Model 2

We have $Q = (\lim_{t \to \infty} Q((i_1, j_1), (i_2, j_2), (i_1, j_1), (i_2, j_2) \in E_1$ with $\pi(i, j) = \lim_{n \to \infty} P(X_n = i, Y_n = j), (i, j) \in E_1$. Let $I_1 = \{\pi(1, 1), \pi(2, 1), \ldots, \pi(n - 1, 1), \pi(n, 0)\}$. Then $I_1$ is given by $I_1 Q = I_1$ with $\sum_{(i,j) \in E_1} \pi(i, j) = 1$. Next, we find out $q_{ij}, (i, j) \in E_2$. $q_{ij} = \lim_{t \to \infty} P(X(t) = i, Y(t) = j)$. These have the same form as in model 1, except for $q_{i0}$ for $i = 0, 1, 2, \ldots, k - 2$. 

$$q_{11} = \sum_{m \geq k} \frac{\pi(m, 1)}{\mu} \int_{u=0}^{\infty} \int_{v=u}^{\infty} \left( \frac{e^{-\lambda u} (\lambda u)^{m-k} \lambda}{(m-k)!} \right) e^{-\alpha u} e^{-\delta(v-u)} (\delta(v-u))^k-1} (1 - G(u)) dv du$$

$$+ \frac{\pi(n, 0)}{\mu} \int_{u=0}^{\infty} \int_{v=u}^{\infty} \left( \frac{e^{-\lambda u} (\lambda u)^{n-k} \lambda}{(n-k)!} \right) e^{-\alpha u} e^{-\delta(v-u)} (\delta(v-u))^k-1} (1 - G(u)) dv du$$

$$+ \int_{u=0}^{\infty} \int_{v=u}^{\infty} \int_{w=v}^{\infty} \sum_{j=0}^{n-k-1} \left( \frac{e^{-\lambda u} (\lambda u)^j}{j!} e^{-\alpha u} e^{-\lambda(v-u)} (\lambda(v-u))^{n-k-j+1}} (n-k-j+1)! \right)$$

$$\int_{v=0}^{v=v} \int_{w=w}^{w=w} \frac{e^{-\lambda(w-v)} (\lambda(w-v))^{k-i}}{(k-i)!} g(w-v) dw dv$$

**Model 3**

In the case of model 3, we have \( Q = (\text{lim}_{t \to \infty} Q((i_1, j_1), (i_2, j_2), t)), (i_1, j_1), (i_2, j_2) \in E_4. \) Here the failure rate is \( \lambda \) even when the system is down. The limiting system state probabilities can be obtained. The expressions for \( q_{ij} \) remains identical except for the states \((i, 1), i = 0, 1, 2, \ldots, k - 2.\)

$$q_{i1} = \sum_{m \geq k} \frac{\pi(m, 1)}{\mu} \int_{u=0}^{\infty} \int_{v=u}^{\infty} \left( \frac{e^{-\lambda u} (\lambda u)^{m-k} \lambda}{(m-k)!} \right) e^{-\alpha u} e^{-\lambda(v-u)} (\lambda(v-u))^{k-1} (1 - G(u)) du$$

$$+ \frac{\pi(n, 0)}{\mu} \int_{u=0}^{\infty} \int_{v=u}^{\infty} \left( \frac{e^{-\lambda u} (\lambda u)^{n-k} \lambda}{(n-k)!} \right) e^{-\alpha u} e^{-\lambda(v-u)} (\lambda(v-u))^{k-1} (1 - G(u)) dv du$$

$$+ \int_{u=0}^{\infty} \int_{v=u}^{\infty} \int_{w=v}^{\infty} \sum_{j=0}^{n-k-1} \left( \frac{e^{-\lambda u} (\lambda u)^j}{j!} e^{-\alpha u} e^{-\lambda(v-u)} (\lambda(v-u))^{n-k-j+1}} (n-k-j+1)! \right)$$

$$\int_{v=0}^{v=v} \int_{w=w}^{w=w} \frac{e^{-\lambda(w-v)} (\lambda(w-v))^{k-i}}{(k-i)!} g(w-v) dw dv$$

### 3.6 Control problem

Here we derive optimal value of \( \alpha \) for a suitable cost function associated with the problem. For that, first we compute the distribution of time between two successive \((n, 0)\) to \((n, 0)\) transition that is the distribution of the time of the first return. The distribution of the time duration since the server arrival till all the failed units are repaired can be derived as follows.
3.6.1 Model 1

Suppose $B(t)$ is the distribution of the random variable. Let $u$ be the time at which $i$ units are repaired. During this time there may be none, one or more failures. Suppose there are $j$ failures. Thus busy period generated by $j$ failures has distribution $B_j(.)$. Thus

$$B(t) = B_i(t)P(T < S_1) + \sum_{i=1}^{n-k-1} P(S_i < T < S_{i+1})B_i(t) + P(T > S_{n-k})B_{n-k}(t) =$$

$$\frac{\alpha}{\alpha + \lambda} \int_0^t \sum_{j=0}^{n-k-1} g(u) \frac{e^{-\lambda u}}{j!} B_j(t-u)du + \sum_{i=1}^{n-k-1} P(S_i < T < S_{i+1}) \int_0^t \sum_{j=0}^{n-k-1} g^{*i}(u) \frac{e^{-\lambda u}}{j!} B_j(t-u)du$$

$$e^{-\lambda u} \frac{j!}{j!} B_j(t-u)du + P(T > S_{n-k}) \int_0^t \sum_{j=0}^{n-k} g^{*n-k}(u) \frac{e^{-\lambda u}}{j!} B_j(t-u)du$$

where $g(.)$ is the density of the service time of a single unit. When $n - k$ is large we have

$$B(t) = \frac{\alpha}{\alpha + \lambda} \int_0^t \sum_{j=0}^{\infty} g(u) \frac{e^{-\lambda u}}{j!} B_j(t-u)du$$

$$+ \sum_{i=1}^{n-k-1} P(S_i < T < S_{i+1}) \int_0^t \sum_{j=0}^{\infty} g^{*(i)}(u) \frac{e^{-\lambda u}}{j!} B_j(t-u)du + P(T > S_{n-k}) \int_0^t \sum_{j=0}^{\infty} g^{*n-k}(u) \frac{e^{-\lambda u}}{j!} B_j(t-u)du$$

(3.4)

$b(t)$ is the density function corresponding to $B(t)$. Differentiating (3.4) and taking Laplace transform, we get

$$L(b(t)) = \frac{\alpha}{\alpha + \lambda} L(g(u)) \mid s' = s + \lambda - \lambda L(b(t))$$

$$+ \sum_{i=1}^{n-k-1} \frac{\alpha^n}{(\lambda + \alpha)^{i+1}} L(g^{*(i)}(u)) \mid s' = s + \lambda - \lambda L(b(t))$$

$$+ (\frac{\lambda}{\lambda + \alpha})^{n-k} L(g^{*n-k}(u)) \mid s' = s + \lambda - \lambda L(b(t))$$

(3.5)

inverting this, we get

$$b(t) = \frac{\alpha}{\alpha + \lambda} \sum_{i=1}^{\infty} \frac{(\lambda t)^{i-1}}{i (i - 1)!} e^{-\lambda t} g^{*t+1}(t) + \sum_{i=1}^{n-k-1} \frac{\alpha^n}{(\lambda + \alpha)^{i+1}} \sum_{l=1}^{\infty} \frac{1 (\lambda t)^{l-1}}{l (l - 1)!} e^{-\lambda t} g^{*t+1}(t)$$

$$+ (\frac{\lambda}{\lambda + \alpha})^{n-k} \sum_{l=1}^{\infty} \frac{1 (\lambda t)^{l-1}}{l (l - 1)!} e^{-\lambda t} g^{*t+1}(t)$$

Let $b$ be the expected length of the busy period. Differentiating (3.4) and using

$$-\frac{d}{ds} L(g(u))/_{s=0} = \mu^{-1} \text{ and } -\frac{d}{ds} L(b(t))/_{s=0} = b,$$

we get $b = \frac{1}{\mu} (1 + \lambda b)$ which gives $b = \frac{1}{n-\lambda}$.
3.6.2 Expected length of time the server continuously remains inactive is given by

\[
\frac{1}{\alpha} P(T < S_1) + \left(\frac{1}{\alpha} + \frac{1}{\lambda}\right) P(S_1 < T < S_2) + \ldots \\
+ \left(\frac{1}{\alpha} + \frac{n - k - 1}{\lambda}\right) P(S_{n-k-1} < T < S_{n-k}) + \frac{n - k}{\lambda} P(T > S_{n-k}) \\
= \frac{1}{\alpha} \frac{\alpha}{\alpha + \lambda} + \left(\frac{1}{\alpha} + \frac{1}{\lambda}(\lambda + \alpha)^2\right) + \frac{2}{\lambda}(\lambda + \alpha)^2 + \ldots \\
+ \left(\frac{1}{\alpha} + \frac{n - k - 1}{\lambda}\right) \frac{\alpha\lambda^{n-k-1}}{(\lambda + \alpha)^{n-k}} + \frac{\lambda}{\lambda + \alpha} = \frac{2}{\alpha}(1 - \frac{\lambda}{\lambda + \alpha})^{n-k}
\]

3.6.3 Expected duration of a busy cycle (that is the length of the time of first return to \((n, 0)\) starting from \((n, 0)\)) is

\[
\frac{1}{(\mu - \lambda)} + \frac{2}{\alpha}(1 - \frac{\lambda}{\lambda + \alpha})^{n-k}
\]

The fraction of time the server remains continuously in the system is

\[
\frac{1}{(\mu - \lambda)} + \frac{\lambda}{\alpha}(1 - \frac{\lambda}{\lambda + \alpha})^{n-k}
\]

3.6.4 Total expected cost per unit time

Let \(C_1\) be the fixed cost of hiring the server and \(C_2\) the wage of the server per unit time. The total expected cost per unit time

\[
(TEC)_1 = C_1\left(\frac{1}{(\mu - \lambda)} + \frac{2}{\alpha}(1 - \frac{\lambda}{\lambda + \alpha})^{n-k}\right) + C_2\left(\frac{1}{(\mu - \lambda)} + \frac{2}{\alpha}(1 - \frac{\lambda}{\lambda + \alpha})^{n-k}\right)^{-1}
\]

It is seen that \((TEC)_1\) is convex in \(\alpha\). Hence global minimum value \(\alpha^*\) that minimizes \((TEC)_1\) exists.

3.6.5 Model 2

In order to compute the distribution of the time of first return to \((n, 0)\), note that, once the system is down, further failures take place at rate \(\delta\). The distribution of the time duration
the system is in the set of states \{(0, 1), (1, 1), \ldots (k - 2, 1)\} continuously is obtained by
considering a process that starts at \((k - 2, 1)\) and returns to \((k - 1, 1)\) for the first time. The
distribution is given by

\[
B(t) = \int_0^t \sum_{j=0}^{k-2} \frac{e^{-\delta u} \delta u^j}{j!} B_j(t-u) \, du
\]

For large \(k\), we get

\[
B(t) = \int_0^t \sum_{j=0}^{\infty} \frac{e^{-\delta u} \delta u^j}{j!} B_j(t-u) \, du
\]

This has mean \(b = \frac{1}{(\mu - \delta)}\). Thus the expected time between two successive visits to \((n, 0)\) is

\[
\frac{1}{(\mu - \delta)} + \frac{1}{(\mu + \delta)} + \frac{1}{(\mu - \lambda)} + \frac{2}{\alpha} (1 - \frac{\lambda}{\lambda + \alpha})^{n-k}.
\]

Therefore

\[
(TEC)_2 = C_1\left[\frac{1}{(\mu - \lambda)} + \frac{2\mu}{(\mu^2 - \delta^2)} + \frac{2}{\alpha} (1 - \frac{\lambda}{\lambda + \alpha})^{n-k}\right]^{-1} + C_2\left[\frac{1}{(\mu - \lambda)} + \frac{2\mu}{\mu^2 - \delta^2}\right]
\]

\[
\left[\frac{1}{(\mu - \lambda)} + \frac{2\mu}{(\mu^2 - \delta^2)} + \frac{2}{\alpha} (1 - \frac{\lambda}{\lambda + \alpha})^{n-k}\right]^{-1}
\]

This is convex in \(\alpha\) and so global minimum value \(\alpha^*\) of \(\alpha\) exists.

### 3.6.6 Model 3

In this case the first return to \((n, 0)\) starting from \((n, 0)\) has on the average the duration

\[
\left[\frac{1}{(\mu - \lambda)} + \frac{2\mu}{(\mu^2 - \lambda^2)} + \frac{2}{\alpha} (1 - \frac{\lambda}{\lambda + \alpha})^{n-k}\right].\]

Hence the expected cost per unit time is

\[
(TEC)_3 = C_1\left[\frac{1}{(\mu - \lambda)} + \frac{2\mu}{(\mu^2 - \lambda^2)} + \frac{2}{\alpha} (1 - \frac{\lambda}{\lambda + \alpha})^{n-k}\right]^{-1} + C_2\left[\frac{1}{(\mu - \lambda)} + \frac{2\mu}{\mu^2 - \lambda^2}\right]
\]

\[
\left[\frac{1}{(\mu - \lambda)} + \frac{2\mu}{(\mu^2 - \lambda^2)} + \frac{2}{\alpha} (1 - \frac{\lambda}{\lambda + \alpha})^{n-k}\right]^{-1}
\]

This is also convex in \(\alpha\). Hence optimal value of \(\alpha\) that minimize \((TEC)_3\) exists.
3.6.7 Numerical illustration

For illustration we calculate the total expected cost for given parameters for the three models and for values of $\alpha$. On comparing the three models for different sets of parameters, we can see that total expected cost is minimum for model 3.

**Table 1.** $C_1 = 100, C_2 = 80, n = 10, k = 5, \lambda = 5.5, \mu = 10, \delta = 3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$(TEC)_1$</th>
<th>$(TEC)_2$</th>
<th>$(TEC)_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>116.944</td>
<td>110.063</td>
<td>108.564</td>
</tr>
<tr>
<td>2.1</td>
<td>119.240</td>
<td>112.307</td>
<td>110.670</td>
</tr>
<tr>
<td>2.2</td>
<td>122.669</td>
<td>114.520</td>
<td>112.736</td>
</tr>
<tr>
<td>2.3</td>
<td>125.555</td>
<td>116.700</td>
<td>114.772</td>
</tr>
<tr>
<td>2.4</td>
<td>128.424</td>
<td>118.846</td>
<td>116.771</td>
</tr>
<tr>
<td>2.5</td>
<td>131.273</td>
<td>120.962</td>
<td>118.735</td>
</tr>
<tr>
<td>2.6</td>
<td>134.103</td>
<td>123.043</td>
<td>120.665</td>
</tr>
<tr>
<td>2.7</td>
<td>136.783</td>
<td>125.092</td>
<td>122.562</td>
</tr>
<tr>
<td>2.8</td>
<td>139.698</td>
<td>127.108</td>
<td>124.425</td>
</tr>
<tr>
<td>2.9</td>
<td>142.464</td>
<td>129.091</td>
<td>126.255</td>
</tr>
<tr>
<td>3.0</td>
<td>145.205</td>
<td>131.043</td>
<td>128.051</td>
</tr>
</tbody>
</table>

**Table 2.** $C_1 = 200, C_2 = 110, n = 12, k = 6, \lambda = 8.5, \mu = 15, \delta = 5$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$(TEC)_1$</th>
<th>$(TEC)_2$</th>
<th>$(TEC)_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>248.882</td>
<td>228.390</td>
<td>223.26</td>
</tr>
<tr>
<td>2.1</td>
<td>254.551</td>
<td>232.817</td>
<td>227.394</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$(TEC)_1$</th>
<th>$(TEC)_2$</th>
<th>$(TEC)_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.2</td>
<td>260.211</td>
<td>237.209</td>
<td>231.494</td>
</tr>
<tr>
<td>2.3</td>
<td>265.865</td>
<td>241.567</td>
<td>235.548</td>
</tr>
<tr>
<td>2.4</td>
<td>271.512</td>
<td>245.886</td>
<td>239.565</td>
</tr>
<tr>
<td>2.5</td>
<td>276.690</td>
<td>250.169</td>
<td>243.539</td>
</tr>
<tr>
<td>2.6</td>
<td>282.770</td>
<td>254.414</td>
<td>247.474</td>
</tr>
<tr>
<td>2.7</td>
<td>288.378</td>
<td>258.621</td>
<td>251.367</td>
</tr>
<tr>
<td>2.8</td>
<td>293.972</td>
<td>262.790</td>
<td>255.217</td>
</tr>
<tr>
<td>2.9</td>
<td>299.548</td>
<td>266.920</td>
<td>259.026</td>
</tr>
<tr>
<td>3.0</td>
<td>305.106</td>
<td>271.009</td>
<td>262.792</td>
</tr>
</tbody>
</table>