Chapter 6

A General Representation

6.1 Introduction

From the study so far we saw that properties enjoyed by the Freund bivariate exponential distribution can be translated to other distributions transformed from the Freund bivariate exponential distribution. In fact all its properties can be translated into a property of an arbitrary bivariate continuous distribution. With the aim of doing so in this chapter, we explore the representation of the bivariate Pareto distributions in terms of uniform random variables. This idea is heavily borrowed from the idea of copulas. A copula is a function $C(u, v)$ from $I^2$ to $I$ where $I^2 = \{(x_1, x_2) \mid 0 < x_i < 1, i = 1, 2\}$ with the following properties.

1. $C(u, v)$ is a grounded function. That means for all $u, v$ in $I$, $C(u, 0) = 0 = C(0, v)$.

2. $C(u, 1) = u$ and $C(1, v) = v$ \hspace{1cm} (6.1)

3. For every $u_1, u_2, v_1, v_2$ in $I$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$
Sklar (1959) proposed a theorem which is central to the theory of copulas and is the foundation of many, if not most of the applications of that theory to statistics. This theorem elucidates the role that copulas play in the relationship between multivariate distribution functions and their univariate marginals. The statement is as follows.

Let \( H(x_1, x_2) \) be a joint distribution function with marginals \( F_1(x_1) \) and \( F_2(x_2) \), then there exists a copula \( C \) such that for any \( x_1, x_2 \in \overline{R} \), \( \overline{R} \) is the extended real line \([-\infty, \infty]\),

\[
H(x_1, x_2) = C\left(F_1(x_1), F_2(x_2)\right).
\] (6.2)

If \( F_1(x_1) \) and \( F_2(x_2) \) are continuous then \( C \) is unique otherwise \( C \) is uniquely determined on \( \text{Range } F \times \text{Range } G \). Conversely if \( C \) is the copula and \( F_1 \) and \( F_2 \) are marginals then the function \( H \) defined by (6.2) is the joint distribution function. If the function \( C(u, v) \) satisfies only the properties 1 and 3 in (6.1) then it is called a pseudo copula. This was introduced by Fermanian and Wegkamp (2004) to study the dynamic dependence structure. It was also shown by the same authors that Sklar’s theorem can be extended to pseudo copulas too.

A natural question that arises is that is there a relationship between univariate and joint survival functions analogous to the one between univariate and joint distribution functions as embodied in Sklar’s theorem. This paved way to the concept of survival copulas.

If \( \overline{H}(x_1, x_2) = P[X_1 \geq x_1, X_2 \geq x_2] \), the marginals of \( \overline{H}(x_1, x_2) \) are \( \overline{H}(x_1, -\infty) \) and \( \overline{H}(-\infty, x_2) \) which are univariate survival functions \( \overline{F}_1(x_1) \) and \( \overline{F}_2(x_2) \) respectively, then

\[
\overline{H}(x_1, x_2) = 1 - F_1(x_1) - F_2(x_2) + H (x_1, x_2)
\]
\[= F_1(x_1) + \overline{F}_2(x_2) - 1 + C\left(F_1(x_1), F_2(x_2)\right)\]
\[= F_1(x_1) + \overline{F}_2(x_2) - 1 + C\left(1 - F_1(x_1), 1 - F_2(x_2)\right)\).  

Denoting \( \hat{C} \) as a function from \( I^2 \) to \( I \) by

\[
\hat{C}(u, v) = u + v - 1 + C\left(1 - u, 1 - v\right),
\]
we have
\[ H(x_1, x_2) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2)) \]
where \( \hat{C}(u, v) \) is referred to as the survival copula of \( X_1 \) and \( X_2 \). Secondly \( \hat{C}(u, v) \) couples the joint survival functions to its univariate marginals in a manner completely analogous to the way in which a copula connects the joint distribution to its marginals. If \( \bar{C}(u, v) \) is the joint survival function for two uniform \((0,1)\) random variables whose joint distribution function is copula \( C(u, v) \), then
\[ \hat{C}(1-u, 1-v) = \bar{C}(u, v) = 1 - u - v + C(u, v). \]
It is easy to prove that a survival copula \( \hat{C}(u, v) \) is a copula since
(i) \( \hat{C}(u, v) \) is a grounded function, that is
\[ \hat{C}(0,0) = \hat{C}(0,v) = v \]
\[ = u - 1 + C(1, 1 - v) \]
\[ = u - 1 + 1 - u \]
\[ = 0 \]
(ii) For every \( u, v \) in \( I \),
\[ \hat{C}(u, 1) = u + C(1-u, 0) \]
\[ = u \]
and
\[ \hat{C}(1, v) = v. \]
(iii) For every \( u_1, u_2, v_1, v_2 \) in \( I \) such that \( u_1 \leq u_2 \) and \( v_1 \leq v_2 \),
\[ \hat{C}(u_2, v_2) - \hat{C}(u_2, v_1) - \hat{C}(u_1, v_2) + \hat{C}(u_1, v_1) \geq 0. \]
Barnett (1980) gave the survival copula called Gumbel-Barnett copula
which generalizes the dependence in the Gumbel’s bivariate exponential distribution specified by \( F(x_1, x_2) = e^{-(x_1 + x_2 + \theta x_1 x_2)}; x_1, x_2 > 0 \). The corresponding survival copula is given by
\[ \hat{C}(u, v) = uv e^{-\theta \ln(u) \ln(v)}. \]
The second Gumbel’s exponential distribution specified by
\[ F(x_1, x_2) = \left(1 - e^{-x_1}\right)\left(1 - e^{-x_2}\right)\left(1 + \theta e^{-(x_1 + x_2)}\right); x_1, x_2 > 0 \]

corresponds to the Farlie-Gumbel-Morgenstern copula is given by
\[ C(u, v) = uv + \theta uv(1 - u)(1 - v); 0 < u < 1, 0 < v < 1. \]

The Marshall-Olkin (1967) bivariate exponential distribution specified by the survival function
\[ F(x_1, x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max(x_1, x_2)}, x_1, x_2 > 0, \lambda_1, \lambda_2, \lambda_3 > 0 \]
has the survival copula given by
\[ \hat{C}(u, v) = \min\left(u^{1-\alpha}, uv^{1-\beta}\right). \quad (6.4) \]

This family is known both as the Marshall-Olkin family and the generalized Caudras-Auge family. Other copulas related to exponential distribution are detailed in Genest and Mackay (1986), Joe and Hu (1996), Joe (1997), Nelsen (1999) and Joe and Ma (2000). This now enables a study of the Marshall-Olkin type Pareto distributions (Veenus and Nair (1994), Hanagal (1996), Yeh (2004 a,b)) and other transformed distribution by studying these copulas.

But in the case of the Freund bivariate Pareto distribution discussed in this thesis, there does not exist an analytical expression of the copula though they could be evaluated numerically. This turns out to be a great drawback as an analytical expression for the copula helps in providing a very convenient model for studying the properties with tools that are scale free.

The major advantage of studying a general representation in terms of uniform variants is that certain properties are same for all distributions in a particular equivalence class. If \( H(u, v) \) denotes the uniform representation of a distribution \( F(x_1, x_2) \) and \( G(x_1, x_2) \) belongs to the same equivalence class if \( H_F(u, v) = H_G(u, v) \).

This motivates us to give a representation in terms of uniform variates which would enable us to study properties of the Freund bivariate exponential distribution and its transformation like the \( BP I(\sigma, \alpha_1, \alpha_2, \alpha_1', \alpha_2') \) distributions.
A General Representation

and \( BP \ II(\mu, \sigma, \alpha_1, \alpha_2, \alpha_1', \alpha_2') \) distribution discussed in this thesis under a
unified framework. Accordingly in the next section we give a representation and
call it the uniform representation. We also give examples of the distributions that
are derived using this representation. These distributions are the bivariate
distributions obtained by transforming the Freund bivariate exponential
distribution by suitable transformations. They include bivariate Weibull
distribution \( (Lu \ (1989)) \), \( BP \ I(\sigma, \alpha_1, \alpha_2, \alpha_1', \alpha_2') \) and
\( BP \ II(\mu, \sigma, \alpha_1, \alpha_2, \alpha_1', \alpha_2') \) distributions \( (Asha \ and \ Jagathnath \ (2008)) \). In the
third section, the reliability property particularly, the total failure rate of the
general representation is given. This expression enables us to directly compute
the failure rate of the distributions having this representation once we know the
uniform translate. This is illustrated in Table 6.1. A general property analogous to
the dullness property is defined for the uniform representation and a
characterization of this class is discussed. With this chapter we conclude the
thesis by briefly stating the direction of future course of study.

6.2 The General Representation

As in the previous section we adopt the representation \( U \) and \( V \) for
uniform variates. Obviously \( (U, V) \) ranges over the unit square. Consider the
uniform representation

\[
h(u, v) = \begin{cases} \frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_1'} & u^\alpha_1 + \alpha_2 - \alpha_1' v^{\alpha_2'} + \frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2'} v^{\alpha_1 + \alpha_2'}; 0 \leq u \leq v \leq 1 \\ \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1'} & v^{\alpha_1 + \alpha_2 - \alpha_1'} u^{\alpha_2'} + \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_1'} u^{\alpha_1 + \alpha_2'}; 0 \leq v \leq u \leq 1 
\end{cases} \)  \quad (6.5)
\]

It should be observed that (6.5) is a finite mixture of two pseudo-survival
copulas.
A General Representation

\[ h_1(u,v) = \begin{cases} 
  u^{\alpha_1 + \alpha_2 - \alpha_2'} v^{\alpha_2'}; u \leq v \\
  v^{\alpha_1 + \alpha_2 - \alpha_1'} u^{\alpha_1'}; v \leq u 
\end{cases} 
\]  
(6.6)

and

\[ h_2(u,v) = \max \{u,v\}^{\alpha_1 + \alpha_2}. \]  
(6.7)

Hence it follows that \( h(u,v) \) is also a pseudo-survival copula as (i) and (iii) of (6.3) are satisfied. The survival function associated with the pseudo-survival copula (6.5) is given by

\[ \overline{h}(u,v) = h(1-u,1-v) \]

i.e.,

\[ \overline{h}(u,v) = \begin{cases} 
  \frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2'} (1-u)^{\alpha_1 + \alpha_2 - \alpha_2'} (1-v)^{\alpha_2'} + \left( \frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2'} \right) (1-v)^{\alpha_1 + \alpha_2}; \\
  \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1'} (1-v)^{\alpha_1 + \alpha_2 - \alpha_1'} (1-u)^{\alpha_1'} + \left( \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_1'} \right) (1-u)^{\alpha_1 + \alpha_2}; \\
\end{cases} \]

0 \leq u \leq v \leq 1

\[ 0 \leq v \leq u \leq 1 \]

(6.8)

Since \( h(u,v) \) is a pseudo-survival copula, evidently not a bivariate uniform distribution. Some members belonging to this class are worked out below.


For, \( 1-u = e^{-x_1} \) and \( 1-v = e^{-x_2} \),

\[ \overline{F}(x_1,x_2) = \begin{cases} 
  \frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2'} e^{-(\alpha_1 + \alpha_2 - \alpha_2')x_1 - \alpha_2' x_2} + \left( \frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2'} \right) e^{-(\alpha_1 + \alpha_2)x_2}; \\
  \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1'} e^{-(\alpha_1 + \alpha_2 - \alpha_1')x_2 - \alpha_1' x_1} + \left( \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_1'} \right) e^{-(\alpha_1 + \alpha_2)x_1}; \\
\end{cases} \]

0 \leq x_1 \leq x_2

0 \leq x_2 \leq x_1
2. Bivariate Weibull distribution (Lu (1989)).

For, \( 1 - u = e^{-(x_1)^{\beta_1}} \) and \( 1 - v = e^{-(x_2)^{\beta_2}} \),

\[
\overline{F}(x_1, x_2) = \begin{cases} 
\frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha'_2} e^{-(\alpha_1 + \alpha_2 - \alpha'_2)x_1^{\beta_1} - \alpha'_2 x_2^{\beta_2}} + \\
\frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha'_1} e^{-(\alpha_1 + \alpha_2 - \alpha'_1)x_2^{\beta_2} - \alpha'_1 x_1^{\beta_1}} + \\
\frac{\alpha_2 - \alpha'_2}{\alpha_1 + \alpha_2 - \alpha'_2} \frac{e^{-(\alpha_1 + \alpha_2)x_2^{\beta_2}^{\prime}} \cdot 0 \leq x_2^{\beta_2} \leq x_1^{\beta_1} & ; \alpha_1 \leq x_2 \leq x_1 \\
\frac{\alpha_1 - \alpha'_1}{\alpha_1 + \alpha_2 - \alpha'_1} \frac{e^{-(\alpha_1 + \alpha_2)x_1^{\beta_1}^{\prime}} \cdot 0 \leq x_1^{\beta_1} \leq x_2^{\beta_2} & ; \alpha_2 \leq x_1 \leq x_2 \\
\end{cases}
\]

3. Bivariate Pareto I \((BP I(\sigma, \alpha_1, \alpha_2, \alpha'_1, \alpha'_2))\) distribution (Table 4.3).

For, \( 1 - u = \left( \frac{x_1}{\sigma} \right)^{\alpha_1} \) and \( 1 - v = \left( \frac{x_2}{\sigma} \right)^{\alpha_2} \),

\[
\overline{F}(x_1, x_2) = \begin{cases} 
\frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha'_2} \left( \frac{x_1}{\sigma} \right)^{-(\alpha_1 + \alpha_2 - \alpha'_2)} \left( \frac{x_2}{\sigma} \right)^{-(\alpha'_2)} + \\
\frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha'_1} \left( \frac{x_2}{\sigma} \right)^{-(\alpha_1 + \alpha_2 - \alpha'_1)} \left( \frac{x_1}{\sigma} \right)^{-(\alpha'_1)} + \\
\frac{\alpha_2 - \alpha'_2}{\alpha_1 + \alpha_2 - \alpha'_2} \left( \frac{x_2}{\sigma} \right)^{-(\alpha_1 + \alpha_2)} \cdot \sigma \leq x_1 \leq x_2 \\
\frac{\alpha_1 - \alpha'_1}{\alpha_1 + \alpha_2 - \alpha'_1} \left( \frac{x_1}{\sigma} \right)^{-(\alpha_1 + \alpha_2)} \cdot \sigma \leq x_2 \leq x_1 \\
\end{cases}
\]
4. Bivariate Pareto II \( \left( BP \ II \left( \mu, \sigma, \alpha_1, \alpha_2, \alpha_1', \alpha_2' \right) \right) \) distribution (Table 4.3).

For, \( 1 - u = \left( 1 + \frac{x_1 - \mu}{\sigma} \right)^{-c} \) and \( 1 - v = \left( 1 + \frac{x_2 - \mu}{\sigma} \right)^{-c} \),

\[
\overline{F}(x_1, x_2) = \begin{cases} 
\frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2'} \left( 1 + \frac{x_1 - \mu}{\sigma} \right)^{-c(\alpha_1 + \alpha_2 - \alpha_2')} \left( 1 + \frac{x_2 - \mu}{\sigma} \right)^{-c \alpha_2'} + \\
\frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2'} \left( 1 + \frac{x_2 - \mu}{\sigma} \right)^{-c(\alpha_1 + \alpha_2 - \alpha_2')} \left( 1 + \frac{x_1 - \mu}{\sigma} \right)^{-c \alpha_2'} \end{cases}
\]

\( ; \mu \leq x_1 \leq x_2 \)

\[
\overline{F}(x_1, x_2) = \begin{cases} 
\frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2'} \left( 1 + \frac{x_1 - \mu}{\sigma} \right)^{-c(\alpha_1 + \alpha_2 - \alpha_2')} \left( 1 + \frac{x_2 - \mu}{\sigma} \right)^{-c \alpha_2'} + \\
\frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2'} \left( 1 + \frac{x_2 - \mu}{\sigma} \right)^{-c(\alpha_1 + \alpha_2 - \alpha_2')} \left( 1 + \frac{x_1 - \mu}{\sigma} \right)^{-c \alpha_2'} \end{cases}
\]

\( ; \mu \leq x_2 \leq x_1 \)

5. Bivariate finite range distribution (Table 4.3)

For, \( 1 - u = \left( 1 - \frac{x_1}{R} \right)^c \) and \( 1 - v = \left( 1 - \frac{x_2}{R} \right)^c \),

\[
\overline{F}(x_1, x_2) = \begin{cases} 
\frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2'} \left( 1 - \frac{x_1}{R} \right)^{c(\alpha_1 + \alpha_2 - \alpha_2')} \left( 1 - \frac{x_2}{R} \right)^{c \alpha_2'} + \\
\frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2'} \left( 1 - \frac{x_2}{R} \right)^{c(\alpha_1 + \alpha_2 - \alpha_2')} \left( 1 - \frac{x_1}{R} \right)^{c \alpha_2'} \end{cases}
\]

\( ; 0 \leq x_2 \leq x_1 \leq R \)

\[
\overline{F}(x_1, x_2) = \begin{cases} 
\frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2'} \left( 1 - \frac{x_1}{R} \right)^{c(\alpha_1 + \alpha_2 - \alpha_2')} \left( 1 - \frac{x_2}{R} \right)^{c \alpha_2'} + \\
\frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2'} \left( 1 - \frac{x_2}{R} \right)^{c(\alpha_1 + \alpha_2 - \alpha_2')} \left( 1 - \frac{x_1}{R} \right)^{c \alpha_2'} \end{cases}
\]

\( ; 0 \leq x_1 \leq x_2 \leq R \)

For, \(1-u=(1+(\lambda x_1)^{\alpha_1})^{-1}\) and \(1-v=(1+(\lambda x_2)^{\alpha_2})^{-1}\),

\[
\begin{align*}
F(x_1, x_2) &= \begin{cases}
\frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_1'} \left(1+(\lambda x_1)^{\alpha_1'}\right)^{-(\alpha_1 + \alpha_2 - \alpha_1')} \left(1+(\lambda x_2)^{\alpha_2}\right)^{-\alpha_2'} + \\
\frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1'} \left(1+(\lambda x_2)^{\alpha_2}\right)^{-(\alpha_2 - \alpha_2')} \left(1+(\lambda x_1)^{\alpha_1'}\right)^{-\alpha_1'} + \\
\frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_1'} \left(1+(\lambda x_1)^{\alpha_1'}\right)^{-(\alpha_2 + \alpha_1)} \left(1+(\lambda x_1)^{\alpha_1'}\right)^{-\alpha_2'}
\end{cases}
\end{align*}
\]

6.3 Failure Rate of the General Class

In this section we consider how total failure rate can be related to the properties of the general representation given in (6.5) in terms of general representation of the total failure rate (Cox (1972)) is defined as

\[
\lambda(x) = -\frac{d}{du_z} \left[\log h(1-u_z, 1-u_z)\right] \frac{du_z}{dx}, \text{ where } Z = \min(X_1, X_2)
\]

\[
\lambda_{12}(x_1, x_2) = \frac{\partial}{\partial u} \left[\log \frac{\partial}{\partial u} h(1-u, 1-v)\right] \frac{du}{dx_1}, x_1 > x_2
\]

\[
\lambda_{21}(x_2, x_1) = \frac{\partial}{\partial u} \left[\log \frac{\partial}{\partial v} h(1-u, 1-v)\right] \frac{dv}{dx_2}, x_1 < x_2
\]

So that for (6.5), \(\lambda(x)\) is now

\[
\lambda(x) = \frac{-(\alpha_1 + \alpha_2) \frac{d(1-u_z)}{dx}}{1-u_z} - \alpha_1' \frac{\partial(1-u)}{\partial x_1} - \alpha_2' \frac{\partial(1-v)}{\partial x_2}.
\]

Table 6.1 lists the failure rate for distributions defined by representation (6.5).
Table 6.1 Total Failure Rate for the Uniform Representation

<table>
<thead>
<tr>
<th>Distributions</th>
<th>$1-u$</th>
<th>$1-v$</th>
<th>$\lambda(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freund’s bivariate exponential distribution (1961)</td>
<td>$e^{-x_1}$</td>
<td>$e^{-x_2}$</td>
<td>$(\alpha_1 + \alpha_2), (\alpha_1', \alpha_2')$</td>
</tr>
<tr>
<td>Bivariate Weibull distribution (Lu (1989))</td>
<td>$e^{-\frac{x_1}{\sigma}}$</td>
<td>$e^{-\frac{x_2}{\sigma}}$</td>
<td>$\left(\frac{c(\alpha_1 + \alpha_2)}{\sigma}\left(\frac{x}{\sigma}\right)^{c-1}, \frac{c\alpha_1'}{\sigma}, \frac{c\alpha_2'}{\sigma}\right)$</td>
</tr>
<tr>
<td>BP I $\left(\sigma, \alpha_1, \alpha_2, \alpha_1', \alpha_2'\right)$ distribution</td>
<td>$\left(\frac{x_1}{\sigma}\right)^{c}$</td>
<td>$\left(\frac{x_2}{\sigma}\right)^{c}$</td>
<td>$\left(\frac{c(\alpha_1 + \alpha_2)}{x}, \frac{c\alpha_1'}{x_1}, \frac{c\alpha_2'}{x_2}\right)$</td>
</tr>
<tr>
<td>BP II $\left(\mu, \sigma, \alpha_1, \alpha_2, \alpha_1', \alpha_2'\right)$ distribution</td>
<td>$\left(1 + \frac{x_1 - \mu}{\sigma}\right)^{c}$</td>
<td>$\left(1 + \frac{x_2 - \mu}{\sigma}\right)^{c}$</td>
<td>$\left(\frac{c(\alpha_1 + \alpha_2)}{\sigma + x - \mu}, \frac{c\alpha_1'}{\sigma + x - \mu}, \frac{c\alpha_2'}{\sigma + x - \mu}\right)$</td>
</tr>
<tr>
<td>Bivariate finite range distribution (Table 4.3)</td>
<td>$\left(1 - \frac{x_1}{R}\right)^{c}$</td>
<td>$\left(1 - \frac{x_2}{R}\right)^{c}$</td>
<td>$\left(\frac{c(\alpha_1 + \alpha_2)}{(R-x)}, \frac{c\alpha_1'}{(R-x)}, \frac{c\alpha_2'}{(R-x_2)}\right)$</td>
</tr>
<tr>
<td>Bivariate log-logistic distribution</td>
<td>$\frac{1}{1 + (\lambda x_1)^a}$</td>
<td>$\frac{1}{1 + (\lambda x_2)^a}$</td>
<td>$\left(\frac{(\alpha_1 + \alpha_2)a\lambda^a x_{1}^{a-1}}{1 + (\lambda x_1)^a}, \frac{\alpha_1' a\lambda^a x_{1}^{a-1}}{1 + (\lambda x_1)^a}, \frac{\alpha_2' a\lambda^a x_{2}^{a-1}}{1 + (\lambda x_2)^a}\right)$</td>
</tr>
</tbody>
</table>

Another interesting property that has gained vast attention of the researchers is the no-ageing property and its variants like the dullness property. As with other reliability concepts there are various extensions to the bivariate
A General Representation

We consider the extension in (3.3). A uniform representation is said to have a bivariate dullness property if it verifies

\[ h((1-t)(1-u),(1-t)(1-v)) = h((1-t),(1-t)) h((1-u),(1-v)) \]  \hspace{1cm} (6.10)

for all \( 0 \leq t, u, v \leq 1 \).

The representations that belong to the class (6.10) are characterized in the following theorem.

**Theorem 6.1** The uniform representation \( h(u, v) \) where \( U \) and \( V \) are uniform variates satisfies (6.10), if and only if \( h(u, v) \) can be written as

\[
h(1-u,1-v) = \begin{cases} 
(1-u)^c h\left(1, \frac{1-v}{1-u}\right); u \leq v \\
(1-v)^c h\left(1-u, 1\right); v \leq u
\end{cases}, c > 0. \hspace{1cm} (6.11)
\]

**Proof**

Let (6.10) be satisfied. Then for \( 1-v = 1-u \),

\[ h((1-t)(1-u),(1-t)(1-u)) = h((1-t),(1-t)) h((1-u),(1-u)). \]

From Aczel (1966 p.41) and the fact that \( h((1-u),(1-v)) \) is a survival function it follows that

\[ h((1-t),(1-t)) = (1-t)^c; c > 0 \]

so that

\[ h((1-t)(1-u),(1-t)(1-v)) = (1-t)^c h((1-u),(1-v)). \]  \hspace{1cm} (6.12)

Now let \( 1-v \leq 1-u \), then once again from (6.10) it follows that

\[
h((1-u),(1-v)) = h(1-u,1-u) h\left(1, \frac{1-v}{1-u}\right) \\
= (1-u)^c h\left(1, \frac{1-v}{1-u}\right), 0 \leq u \leq v \leq 1
\]

Similarly,

\[
h((1-u),(1-v)) = (1-v)^c h\left(1-u, 1\right), 0 \leq v \leq u \leq 1.
\]

To prove the converse, if \( h((1-u),(1-v)) \) is as given in (6.12) then
A General Representation

\[ h((1-t)(1-u),(1-t)(1-v)) = (1-t)^\varepsilon h((1-u),(1-v)) \] for all \( 0 \leq t, u, v \leq 1 \).

That is

\[ h((1-t)(1-u),(1-t)(1-v)) = h((1-t), (1-t))(1-u),(1-v)) \]

for all \( 0 \leq t, u, v \leq 1 \).

Hence the theorem.

**Corollary 6.1** The uniform representation given in (6.5) belongs to the class of distributions verifying (6.12).

**Proof**

Observe that the uniform representation (6.5) can be written as

\[
(1-u)^{\alpha_1 + \alpha_2} \left( \frac{1}{1-u} \right)^{\alpha_2} \left( \frac{1}{1-v} \right)^{\alpha_2} + \alpha_2 \frac{1}{1-u} \left( \frac{1}{1-v} \right)^{\alpha_2} \]

for all \( 0 \leq u, v \leq 1 \).

The proof now follows from Theorem 6.1.

**Corollary 6.2** When \( 1-u = e^{-x_1} \) and \( 1-v = e^{-x_2} \), it follows from Corollary 6.1 that for \( 0 \leq x_1 \leq x_2 \),

\[
\frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2} e^{-(\alpha_1 + \alpha_2)x_1 - \alpha_1 x_1} + \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_2} e^{-(\alpha_1 + \alpha_2)x_2 - \alpha_2 x_2}
\]

\[
e^{-\alpha_1 x_1} \left( \frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2} e^{-\alpha_1 (x_2 - x_1)} + \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_2} e^{-\alpha_2 (x_2 - x_1)} \right)
\]

\[
e^{-\alpha_1 x_1} f_1(x_2 - x_1),
\]
where
\[ F_1(x_2 - x_1) = \left( \frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2} e^{-\alpha_2(x_2 - x_1)} + \frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2} e^{-(\alpha_1 + \alpha_2)(x_2 - x_1)} \right). \]

Similarly for \( 0 \leq x_2 \leq x_1 \),
\[ \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1} e^{-(\alpha_1 + \alpha_2 - \alpha_1) x_2 - \alpha_1 x_1} + \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_1} e^{-(\alpha_1 + \alpha_2) x_1} \]
\[ = e^{-(\alpha_1 + \alpha_2) x_2} \left( \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1} e^{-\alpha_1(x_1 - x_2)} + \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_1} e^{-(\alpha_1 + \alpha_2)(x_1 - x_2)} \right) \]
\[ = e^{-(\alpha_1 + \alpha_2) x_2} F_2(x_1 - x_2), \]

where
\[ F_2(x_1-x_2) = \left( \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1} e^{-\alpha_1(x_1-x_2)} + \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_1} e^{-(\alpha_1 + \alpha_2)(x_1-x_2)} \right). \]

In general by writing
\[ F(x_1, x_2) = \begin{cases} e^{-(\alpha_1 + \alpha_2) x_1} F_1(x_2 - x_1); & 0 \leq x_1 \leq x_2, \\ e^{-(\alpha_1 + \alpha_2) x_2} F_2(x_1 - x_2); & 0 \leq x_2 \leq x_1. \end{cases} \]

This is the class of exponential minima given in equation (3.5) characterized by the bivariate lack of memory property,
\[ \overline{F}(x_1 + t, x_2 + t) = \overline{F}(x_1, x_2) \overline{F}(t, t) \text{ for } x_1, x_2, t > 0. \]

**Corollary 6.3** When \( 1 - u = (x_1)^{-c} \) and \( 1 - v = (x_2)^{-c} \), it follows from Corollary 6.1,
\[ \frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2} x_1^{-c(\alpha_1 + \alpha_2 - \alpha_2)} x_2^{-c(\alpha_1 + \alpha_2')} + \left( \frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2} \right) x_2^{-c(\alpha_1 + \alpha_2)} \]
\[ = x_1^{-c(\alpha_1 + \alpha_2)} \left( \frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2} \left( x_2^{-c(\alpha_2') x_1^{-c}} \right) + \left( \frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2} \right) \left( x_2^{-c(\alpha_1 + \alpha_2')} x_1^{-c} \right) \right), \]
\[ = x_1^{-c(\alpha_1 + \alpha_2)} F_2 \left( \frac{x_2}{x_1} \right); \ 1 \leq x_1 \leq x_2, \]
where
\[ F_2 \left( \frac{x_2}{x_1} \right) = \frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2'} \left( \frac{x_2}{x_1} \right)^{\alpha_1'} + \left( \frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2'} \right) \left( \frac{x_1}{x_2} \right)^{\alpha_1 + \alpha_2} \].

Similarly for \( 1 \leq x_2 \leq x_1 \)
\[ \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_2'} x_2^{-c(\alpha_1 + \alpha_2)} x_1^{-c\alpha_1'} + \left( \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_2'} \right) x_1^{-c(\alpha_1 + \alpha_2)} \]
\[ = x_2^{-c(\alpha_1 + \alpha_2)} \left( \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_2'} \left( \frac{x_1}{x_2} \right)^{\alpha_1'} + \left( \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_2'} \right) \left( \frac{x_1}{x_2} \right)^{\alpha_1 + \alpha_2} \right) \]
\[ = x_2^{-c(\alpha_1 + \alpha_2)} \hat{F}_1 \left( \frac{x_1}{x_2} \right); 1 \leq x_2 \leq x_1, \]

where
\[ \hat{F}_1 \left( \frac{x_1}{x_2} \right) = \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_2'} \left( \frac{x_1}{x_2} \right)^{\alpha_1'} + \left( \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_2'} \right) \left( \frac{x_1}{x_2} \right)^{\alpha_1 + \alpha_2} \].

In general
\[ \hat{F}(x_1, x_2) = \begin{cases} 
  x_2^{-c(\alpha_1 + \alpha_2)} \hat{F}_1 \left( \frac{x_1}{x_2} \right); 1 \leq x_2 \leq x_1 \\
  x_1^{-c(\alpha_1 + \alpha_2)} \hat{F}_2 \left( \frac{x_2}{x_1} \right); 1 \leq x_1 \leq x_2 
\end{cases} \]

This is class of Pareto minima given in (3.6) characterized by the bivariate
dullness property,
\[ \hat{F}(x_1, x_2) = \hat{F}(x_1, x_2) \hat{F}(t, t) \text{ for } x_1, x_2, t > 0. \]

**Corollary 6.4** When \( 1 - u = e^{c(\alpha_1)} \) and \( 1 - v = e^{c(\alpha_2)} \), it follows from Corollary 6.1,
\[ \frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2'} e^{c(\alpha_1 + \alpha_2 - \alpha_2')x_1' - \alpha_1'} x_2' + \left( \frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2'} \right) e^{c(\alpha_1 + \alpha_2)x_2'}. \]
\[
e^{-\left(\alpha_1 + \alpha_2\right)x_1^\epsilon} \left\{ \frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2} e^{-\alpha_1' \left(x_2^\epsilon - x_1^\epsilon\right)} + \frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2} e^{-\left(\alpha_1 + \alpha_2\right)(x_2^\epsilon - x_1^\epsilon)} \right\};
\]
\[0 \leq x_1 \leq x_2.\]

Similarly for \(0 \leq x_2 \leq x_1\),
\[
e^{-\left(\alpha_1 + \alpha_2\right)x_2^\epsilon} \left\{ \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1} e^{-\alpha_1' \left(x_2^\epsilon - x_1^\epsilon\right)} + \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_2} e^{-\left(\alpha_1 + \alpha_2\right)(x_2^\epsilon - x_1^\epsilon)} \right\}
\]

Thus we have,
\[
F(x_1, x_2) =
\left\{
\begin{array}{c}
\frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2} e^{-\left(\alpha_1 + \alpha_2\right)x_1^\epsilon} + \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_2} e^{-\left(\alpha_1 + \alpha_2\right)x_1^\epsilon}\quad; 0 \leq x_1 \leq x_2 \\
\frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1} e^{-\left(\alpha_1 + \alpha_2\right)x_2^\epsilon} + \frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2} e^{-\left(\alpha_1 + \alpha_2\right)x_2^\epsilon}\quad; 0 \leq x_2 \leq x_1
\end{array}
\right.
\]
i.e.,
\[
F \left(\left(x_1^\epsilon + t^\epsilon\right)^{\frac{1}{\epsilon}}, \left(x_2^\epsilon + t^\epsilon\right)^{\frac{1}{\epsilon}}\right) =
\left\{
\begin{array}{c}
e^{-\left(\alpha_1 + \alpha_2\right)t^\epsilon} \left\{ \frac{\alpha_1}{\alpha_1 + \alpha_2 - \alpha_2} e^{-\left(\alpha_1 + \alpha_2\right)x_1^\epsilon} + \frac{\alpha_1 - \alpha_1'}{\alpha_1 + \alpha_2 - \alpha_2} e^{-\left(\alpha_1 + \alpha_2\right)x_1^\epsilon}\right\}\quad; 0 \leq x_1 \leq x_2 \\
e^{-\left(\alpha_1 + \alpha_2\right)t^\epsilon} \left\{ \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1} e^{-\left(\alpha_1 + \alpha_2\right)x_2^\epsilon} + \frac{\alpha_2 - \alpha_2'}{\alpha_1 + \alpha_2 - \alpha_2} e^{-\left(\alpha_1 + \alpha_2\right)x_2^\epsilon}\right\}\quad; 0 \leq x_2 \leq x_1
\end{array}
\right.
\]

Hence,
\[
F \left(\left(x_1 + t\right)^{\frac{1}{\epsilon}}, \left(x_2 + t\right)^{\frac{1}{\epsilon}}\right) = F(t, t) F \left(x_1, x_2\right),
\]
which is the extension of the characterization of a univariate Weibull distribution.
Corollary 6.5 The Caudras-Auge copula given in (6.4) verifies the property (6.10).

Proof

From (6.4) we have

\[
\min\left(u^{1-\alpha}v, uv^{1-\beta}\right) = \begin{cases} 
    u^{1-\alpha}v; & v^\beta \leq u^\alpha \\
    uv^{1-\beta}; & u^\alpha \leq v^\beta 
\end{cases}
\]

i.e.,

\[
\min\left(u^{1-\alpha}v, uv^{1-\beta}\right) = \begin{cases} 
    u^{2-\alpha}\left(\frac{v}{u}\right); & v^\beta \leq u^\alpha \\
    v^{2-\beta}\left(\frac{u}{v}\right); & u^\alpha \leq v^\beta 
\end{cases}
\]

Hence the result.

6.4 Conclusion

In this chapter we have given a uniform representation of the Freund (1961) bivariate exponential distribution and its transformation. We showed how this representation can be used to infer on the total failure rate of each distribution having this representation, once we know their uniform translates. We also characterize this uniform representation by what we define as general dullness property (6.10). It is further shown that this property implies the bivariate lack of memory property (Marshall and Olkin (1967)) for the Freund’s bivariate exponential distribution. It implies the bivariate dullness property (Veenus and Nair (1994), Hanagal (1996), Yeh (2004 a,b)) for the $BP\ I\left(1,\alpha_1,\alpha_2,\alpha_1',\alpha_2'\right)$ distribution. It also implies a characterization for the bivariate Weibull distribution. With this we conclude the present thesis, after discussing below the future course of work.
6.5 Future Work

In this thesis the bivariate Pareto distributions, $BP I\left(1, \alpha_1, \alpha_2, \alpha'_1, \alpha'_2\right)$ and $BP II\left(\mu, \sigma, \alpha_1, \alpha_2, \alpha'_1, \alpha'_2\right)$ were obtained by transforming the Freund bivariate exponential distribution. Many distributional and reliability properties that find application in Reliability and Economic studies were considered. As mentioned earlier these models do not have a straightforward multivariate extension as in the case of Marshall-Olkin (1967) bivariate exponential distribution. An extension to this distribution has been discussed in Weimann (1966) under certain restrictive conditions. It remains to develop a multivariate distribution and study the properties of the same. One generalization is envisaged as follows. Let there be $n$ component parallel system, where the components work independently with lifetimes $X_i, i = 1, \ldots, n$. Let $f_1\left(x_i^1\right), i = 1, \ldots, n$ denote the probability density function of $X_i^1$ in the first stage. Assume that there occurs no simultaneous failure or failure occurs at stages and failed items are not replaced.

Once a failure occurs the distribution of the remaining lifetime of the other component has same distribution but with possibly changed parameter values. Let $X_i^j$ denote the remaining lifetime of the $i^{th}$ component in the $j^{th}$ stage (if it survived). If $Y_1, Y_2, \ldots, Y_n$ denotes the component lifetime, that is

$Y_h = X_h^1$

$Y_{i_2} = Y_h + X_{i_2}^2$

$Y_{i_3} = Y_{i_2} + X_{i_3}^3$

$\vdots$

$Y_{i_n} = Y_{i_{n-1}} + X_{i_n}^n$

Then the distribution of the Freund extension is derived as

$$f\left(y_1, y_2, \ldots, y_n\right) = \prod_{j=1}^{n} f_j^1\left(y_j\right) \int_{y_{j+1}}^{\infty} \prod_{k=j+1}^{n} f_k^j\left(y_k\right) dy_k ; \; y_1 < y_2 < \ldots < y_n.$$
It remains to study the properties of this distribution and this will be taken up as a continuation of this research.

The bivariate residual entropy function which is applicable to a two component parallel system is considered in this thesis. Recently the study on the past entropy has received much attention among the researchers. Hence the study on the bivariate past entropy function for a load sharing dependent models, its properties, characterizations and multivariate extensions can be consider for the future research.

The bivariate inequality measures are introduced in this thesis, which are applicable to data that shows a load sharing dependence. Also the income gap ratio for the rich in the bivariate situation is discussed here. The extension of these inequality measures in the multivariate case as well as the development of income gap ratio for the poor is also to be explored. The properties of the bivariate uniform representation and the study on its association measures are yet to be discussed.

These problems will be taken up as a future course of work plan.