Chapter 5

Inequality Measures for Bivariate Distributions with Load Sharing Dependence

5.1 Introduction

In the univariate case inequality measures related to or identical to the Lorenz order have gained general acceptance. Another measure of inequality for univariate populations which stands out from the rest in terms of acceptance and applicability is the Gini index.

It is only natural to seek appropriate extensions of the Lorenz curves and Gini index to higher dimensions. Some early works suggesting these extensions can be found in Taguchi (1972 a), Lunetta (1972), De Simoni (1979). Taguchi (1972 a) defined the concentration surface of a two dimensional random vector and extended the notions of concentration surface to complete surface called the Lorenz manifold. In Arnold (1983) a parametric representation of the bivariate Lorenz curve is given as

\[
L(u,v) = \frac{\int\int_{0}^{x} \xi \eta f_{12}(\xi, \eta) d\xi d\eta}{E[XY]} \tag{5.1}
\]
where \( f_{12} \) denotes the joint income density and \( f_i, i = 1, 2 \) denotes the marginals corresponding to the non-negative random variables \( X \) and \( Y \) respectively. Here \( u = \int_0^x f_i(\xi) d\xi \) and \( v = \int_0^y f_2(\eta) d\eta \). Koshevoy (1995) provides a definition in higher dimensions in terms of the Lorenz zonoids and the inequality measures for multivariate distributions are given in Arnold (2005).

Let \( X \) denote a k-dimensional non-negative vector with positive finite expectations. Let \( \psi^k(\underline{x}) \) be a measurable mapping from \( R^k \rightarrow [0,1] \). Then the Lorenz zonoid of \( X \) denoted by \( L(X) \) is defined by

\[
L(X) = \left\{ \left[ \psi(\underline{x}) dF(\underline{x}), \frac{x_1\psi(x)}{E[X_1]} dF(\underline{x}), ..., \frac{x_k\psi(x)}{E[X_k]} dF(\underline{x}) \right] : \psi(k) \in \psi^k \right\}.
\]

This definition is consistent with the univariate definition and higher order extensions are straightforward.

The Gini index is another popular inequality measure defined in terms of geometric features of the Lorenz curve. It represents twice the area between Lorenz curve of \( X \) and the line of equality. The Gini index has also been extended to higher dimensions. Mosler (2002) extended it as the volume of the Lorenz zonoid and calls it the Gini zonoid index, Weymark (2004) describes parameterized family of multivariate generalized Gini indices. A Gini index for truncated multivariate distributions was proposed by Sathar et al. (2007) which was consistent with that of Ord et al. (1983) for the univariate case.

However when studying distributions in higher dimensions there is a need to choose an appropriate measure that reflect appropriate aggregation aspects of the inequality when comparisons are to be made. This choice is very much related to the dependency enjoyed by the multivariate distributions. In this chapter we consider the load sharing dependence. This dependence finds exclusive use in socio-economic problems, where income from multiple sources is considered. To take a very simple situation one could envisage a unit with two sources of independent income. For example, it could be the income from farming two
different crops or income of a couple to the household. Both the source work independently till one of the source is unable to generate an income. Then the other source has either generates more income or perhaps be affected adversely. Thus the income distribution of the surviving source undergoes a parameter change and hence a load-sharing model is now apt to model the dependence. The measures (5.1) and (5.2) fail to reflect the inequality aspect of the data. A simple and effective way to formulate a measure to reflect the aggregate aspects of the inequalities in the data is to consider the two different states of the income generating source. Firstly, both the sources are generating an income and secondly, only one source is functional. Hence it is now more adequate to study the related one dimensional distribution

(i) \( P[Z \geq x] = \bar{F}_Z(x) \), where \( Z = \min(X_1, X_2) \)

(ii) \( P[X_2 \geq x \mid X_1 = x_1] = \left[ -\frac{\partial}{\partial u} F(u, x) \right]_{u=x_1} \)

(iii) \( P[X_1 \geq x \mid X_2 = x_2] = \left[ -\frac{\partial}{\partial u} F(x, u) \right]_{u=x_2} \)

together than the bivariate distribution \( F(x_1, x_2) \). In this chapter we study the inequality measures specifically, Lorenz function and Gini index of these distributions and characterize the original bivariate distribution using these measures.

The proposed study is organized into three sections. In the second section, we propose the definitions of the income measures, Lorenz curve and Gini index which reflect the inequalities in the data taking into account the information regarding the two different states of the income generating source. We also study the theoretical properties of the inequality measures. In the third section, characterizations using these measures are discussed.
5.2 Definitions and Relationships

Let \((X_1, X_2)\) be a vector of non-negative random variables admitting absolutely continuous distribution function \(F(x_1, x_2)\) and density function \(f(x_1, x_2)\). Let \(\bar{F}(x_1, x_2) = P[X_1 \geq x_1, X_2 \geq x_2]\). Also assume that \(E[X_i] < \infty, i = 1, 2\), then \((X_1, X_2)\) could represent income from two different crops or income in a household unit from two independent sources. When the income from a unit goes below a threshold level say \(x, x \in \mathbb{R}^+\), then the load of generating more income falls on the other source. Assume that the income is reported only if at least one of the sources has an income that exceeds \(x\). Then the possibilities are

(i) Income from both the sources exceed \(x\).

(ii) At least one of the sources have an income larger than \(x\), while the other has an income less than \(x\).

In the former situation if \(Z = \min(X_1, X_2)\) and \(F_Z, f_Z, \bar{F}_Z\) denotes the distribution function, density function and survival function respectively, associated with \(Z\), then the proportion of units when both the sources are generating income greater than \(x\), and up to \(t\) is given by

\[
P_x(t) = \frac{\int_0^t dF_Z(y)}{\bar{F}_Z(x)}.
\]

(5.3)

The cumulative income share of this population is given by

\[
L(P_x(t)) = \frac{\int_x^t ydF_Z(y)}{\int_x^\infty ydF_Z(y)}, x < t.
\]

(5.4)

Thus \((P_x(t), L(P_x(t)))\) represents the inequality in income of the population when both the sources generate an income of at least \(x\). However when the income from one source falls below the threshold value \(x\), then the above measure ceases to be an apt measure. Assume that the income from the second source is \(x_2, x_2 < x\) and income of the first source is greater than \(x\), then
denotes the proportion of population whose first resource has an income greater than \( x \) and up to \( t \) while the second source generates an income \( x_2 < x \). The cumulative income share of the population (5.5) is

\[
P_{\Delta x_2} (t) = \frac{\int_{x}^{t} f(y, x_2) dy}{\int_{x}^{\infty} f(y, x_2) dy}, \quad x_2 < x < t \tag{5.5}
\]

Similar interpretation follows for

\[
L \left( P_{\Delta x_1} (t) \right) = \frac{\int_{x}^{t} y f(y, x_2) dy}{\int_{x}^{\infty} y f(y, x_2) dy}, \quad x_2 < x \tag{5.6}
\]

The three measures namely

\[
\left( P_x (t), L \left( P_x (t) \right) \right), \left( P_{\Delta x_2} (t), L \left( P_{\Delta x_2} (t) \right) \right) \text{ and } \left( P_{\Delta x_1} (t), L \left( P_{\Delta x_1} (t) \right) \right) \tag{5.9}
\]

are consistent with the univariate definition of the Lorenz curve and reflect inequality in a bivariate income data under specific situations mentioned in (i) and (ii).

**Note:** Since the above definitions can be seen as univariate measures in their domains of definition it is not difficult to observe that
(i) \( P_x(t), P_{\alpha_{x_i}}(t), i = 1, 2 \) are continuous in \([0,1]\).

(ii) \( L\left(P_x(t)\right), L\left(P_{\alpha_{x_i}}(t)\right), i = 1, 2 \rightarrow 0(1) \) according as \( P_x(t), P_{\alpha_{x_i}}(t) \rightarrow 0(1) \).

(iii) \( L\left(P_x(t)\right), L\left(P_{\alpha_{x_i}}(t)\right), i = 1, 2 \) are increasing in \( x \).

(iv) \( L\left(P_x(t)\right), L\left(P_{\alpha_{x_i}}(t)\right), i = 1, 2 \) is convex in \( P_x(t), P_{\alpha_{x_i}}(t) \) respectively.

The Gini index is a popular inequality measure closely related to the Lorenz curve, though there are various definitions for the Gini index unrelated to Lorenz curve. But it should also be noted the various definitions agree with each other. One definition closely associated to the Lorenz function is given by

\[
G_x = 2 \int_0^\infty P_x(t) \frac{d}{dt} L(P_x(t)) - 1. \tag{5.10}
\]

Analogously using the inequality measures in (5.3) to (5.8), we have the definition for Gini index as \( G(x) = \left(G_x, G_{\alpha_{x_2}}, G_{\alpha_{x_i}}\right) \), where \( G_x \) is defined in (5.10) and

\[
G_{\alpha_{x_i}} = 2 \int_0^\infty P_{\alpha_{x_i}}(t) \frac{d}{dt} L(P_{\alpha_{x_i}}(t)) - 1, x_i < x, i = 1, 2. \tag{5.11}
\]

We can use the Lorenz curve to obtain a different interpretation of lifetime data. Let us illustrate the Lorenz function for a Freund bivariate exponential (refer Table 5.2) data (Kim and Kvam (2004)). The data is shown to exhibit a load sharing dependence (Deshpande et al. (2007)). The estimates of the parameters are obtained as \( \alpha_1 = 0.18, \alpha_2 = 0.35, \alpha_1' = 0.22 \) and \( \alpha_2' = 0.29 \). The values of the Lorenz functions are given in Table 5.1.
Table 5.1 Values of the Lorenz functions

| $P_x(t)$ | $L(P_x(t))$ | $P_{x|y}(x)$ | $L(P_{x|y}(x))$ | $P_{x|z}(x)$ | $L(P_{x|z}(x))$ |
|----------|--------------|--------------|-----------------|--------------|-----------------|
| 0.9275   | 0.7716       | 0.1814       | 0.0471          | 0.0314       | 0.0074          |
| 0.2354   | 0.0672       | 0.2354       | 0.0672          | 0.0618       | 0.0154          |
| 0.3240   | 0.1070       | 0.3240       | 0.1070          | 0.1043       | 0.0278          |
| 0.5646   | 0.2679       | 0.4231       | 0.1629          | 0.1450       | 0.0412          |
| 0.9833   | 0.9271       | 0.5329       | 0.2415          | 0.2275       | 0.0729          |
| 0.1814   | 0.0471       | 0.5646       | 0.2679          | 0.2430       | 0.0796          |
| 0.4231   | 0.1629       | 0.7962       | 0.5305          | 0.3298       | 0.1219          |
| 0.5329   | 0.2415       | 0.8705       | 0.6535          | 0.5213       | 0.2479          |
| 0.7962   | 0.5305       | 0.9275       | 0.7716          | 0.6480       | 0.3630          |
| 0.8705   | 0.6535       | 0.9833       | 0.9271          | 0.8900       | 0.7017          |

Figure 5.1 Lorenz curve corresponding to the failure time data

From the Figure 5.1, it can be inferred that there is less disparity among the samples when both the units have exceeded the truncation point (assumed as unity in this case) than the Lorenz curve when one of the components lie below unity.

As suggested in Chandra and Singpurwalla (1981) it is interesting to use certain ideas used in reliability theory to derive the theoretical properties of the
Lorenz curve and Gini index. This motivated us to study the relationship between the bivariate Lorenz curve and the bivariate mean residual life function, defined as

\[ m(x) = \left( m(x), m_{12}(x|x_2), m_{21}(x|x_1) \right) \]  

where

\[
m(x) = \frac{\int_{x}^{\infty} F_Z(y)dy}{F_Z(x)}
\]

\[
m_{ij}(x|x_j) = \frac{\int (y-x_j)f(y,x_j)dy}{\left[-\frac{\partial F(x,u)}{\partial u}\right]_{u=x_j}}, \quad i \neq j = 1, 2.
\]

The mean residual life function defined above is related to the Cox’s (1972) failure rate \( \dot{\lambda}(x) = (\dot{\lambda}(x), \lambda_{12}(x|x_2), \lambda_{21}(x|x_1)) \) by

\[
1 + \frac{d}{dx} \frac{m(x)}{m(x)} = \dot{\lambda}(x) = \frac{f_Z(x)}{F_Z(x)}
\]

\[
1 + \frac{\partial}{\partial x} \frac{m_{ij}(x|x_j)}{m_{ij}(x|x_j)} = \dot{\lambda}_{ij}(x|x_j) = \frac{\partial^2}{\partial x \partial x_j} F(x,x_j)
\]

\[
, \quad x_j < x, \quad i = 3 - j, \quad j = 1, 2.
\]

Theorem 5.1 Let \( (X_1, X_2) \) be a bivariate random variable admitting an absolutely continuous distribution function and having finite expectations. Further if \( L(P_x(t)), L(P_{x|x_2}(t)), L(P_{x|x_1}(t)) \) are differentiable, then

\[
-\frac{d}{dt} \log \left[ 1 - L(P_x(t)) \right] = \frac{t \left( 1 + \frac{d}{dt} m(t) \right)}{m(t) [t + m(t)]} \quad (5.14)
\]

\[
-\frac{\partial}{\partial t} \log \left[ 1 - L(P_{x|x_2}(t)) \right] = \frac{t \left( 1 + \frac{\partial}{\partial t} m_{12}(t|x_2) \right)}{m_{12}(t|x_2) [t + m_{12}(t|x_2)]} \quad (5.15)
\]
\[-\frac{\partial}{\partial t} \log \left[ 1 - L \left( P_{x \mid x} (t) \right) \right] = \frac{t \left( 1 + \frac{\partial}{\partial t} m_{21}(t \mid x) \right)}{m_{21}(t \mid x) [t + m_{21}(t \mid x) ]} \].  \hspace{1cm} (5.16)

Proof

From (5.4), we have

\[
\int_{x}^{t} y dF_z(y) = \frac{L(P_x(t))}{\int_{x}^{t} y dF_z(y)}.
\]

\[
1 - L(P_x(t)) = \frac{\int_{x}^{t} y dF_z(y)}{\int_{x}^{t} y dF_z(y)} = t F_z(t) + \int_{t}^{\infty} F_z(y) dy = \frac{\mu_z(x)}{\mu_z(x)}
\]

where \( \mu_z(x) = \int_{x}^{\infty} y dF_z(y) \).

\[
\mu_z(x) \left[ 1 - L(P_x(t)) \right] = t F_z(t) + \int_{t}^{\infty} F_z(y) dy
\]

From (5.12) the above relation becomes

\[
F_z(x) [x + m(x)] \left[ 1 - L(P_x(t)) \right] = F_z(t) [t + m(t)]. \hspace{1cm} (5.17)
\]

Note that, since (5.17) represent the cumulative income larger than \( t \), it is only natural that right hand side of (5.17) is independent of \( x \) since \( t > x \).

Since \( \frac{d}{dt} L(P_x(t)) = \frac{t f(t)}{F_z(x) [x + m(x)]} \), we have using (5.17),

\[
\left( \frac{1}{1 - L(P_x(t))} \right) \frac{d}{dt} L(P_x(t)) = \frac{t \lambda(t)}{t + m(t)}
\]

where \( \lambda(t) \) is as in (5.13). Substituting for \( \lambda(t) \), we have
- \frac{d}{dt} \log \left[ 1 - L(P_x(t)) \right] = \frac{t}{m(t)} \left( 1 + \frac{d}{dt} \frac{m(t)}{t + m(t)} \right)

Now consider $L(P_{dx_2}(t))$ as in (5.6)

\[ L(P_{dx_2}(t)) = \int_{x}^{\infty} y f(y, x_2) dy \]

or

\[ 1 - L(P_{dx_2}(t)) = \int_{x}^{\infty} y f(y, x_2) dy \]

\[ = t \left[ \frac{\partial}{\partial u} \tilde{F}(t, u) \right]_{u=x_2} + \int_{x}^{\infty} \frac{\partial}{\partial u} \tilde{F}(y, u) \right]_{u=x_2} dy \]

\[ \mu(x, x_2) \]

where $\mu(x, x_2) = \int_{x}^{\infty} y f(y, x_2) dy$, $x_2 < x$.

So that

\[ \mu(x, x_2) \left[ 1 - L(P_{dx_2}(t)) \right] = \left[ \frac{\partial}{\partial u} \tilde{F}(t, u) \right]_{u=x_2} \left[ t + m_{12} \left( t | x_2 \right) \right], x_2 < t. \]

Since \( \frac{\partial}{\partial u} L(P_{dx_2}(t)) = \frac{t f(t, x_2)}{\mu(x, x_2)} \), we have

\[ -\frac{\partial}{\partial t} \log \left[ 1 - L(P_{dx_2}(t)) \right] = \frac{tf(t, x_2)}{\left[ \frac{\partial}{\partial u} \tilde{F}(t, u) \right]_{u=x_2} \left[ t + m_{12} \left( t | x_2 \right) \right]} \]

\[ = \frac{t}{m_{12} \left( t | x_2 \right)} \left[ t + m_{12} \left( t | x_2 \right) \right]. \]

Proceeding in a similar manner with $L(P_{dx_1}(t))$ we can prove that
\[ -\frac{d}{dt} \log \left[ 1 - L\left( P_{\alpha_0}(t) \right) \right] = \frac{t \left[ 1 + \frac{d}{dt} m_{21}(t \mid x_1) \right]}{m_{21}(t \mid x_1) \left[ t + m_{21}(t \mid x_1) \right]} \cdot \]

**Remark 5.1** In equation (5.17), when \( x = 0 \), it reduces to the result of Chandra and Singpurwalla (1981).

It is natural to investigate if there exist a similar relationship between the bivariate Gini index and bivariate mean residual life function. The next theorem deals with this.

**Theorem 5.2** Under the usual assumptions and conditions specified in Theorem 5.1, the following relationships hold,

\[
\left[ 1 - G_x \right]\left[ x + m(x) \right] = x + \frac{1}{\left[ F_Z(x) \right]^2} \int_x^\infty \left[ F_Z(t) \right]^2 dt \tag{5.18}
\]

\[
\left[ 1 - G_{\alpha_0 x_2} \right]\left[ x + m_{22}(x \mid x_2) \right] = x + \frac{1}{\left[ \frac{\partial}{\partial t} F(x, u) \right]_{u=x_2}} \left[ \left[ \frac{\partial}{\partial u} F(x, u) \right]_{u=x_2} \right]^2 dt \tag{5.19}
\]

\[
\left[ 1 - G_{\alpha_1 x} \right]\left[ x + m_{21}(x \mid x_1) \right] = x + \frac{1}{\left[ \frac{\partial}{\partial t} F(u, x) \right]_{u=x_1}} \left[ \left[ \frac{\partial}{\partial u} F(u, x) \right]_{u=x_1} \right]^2 dt. \tag{5.20}
\]

**Proof**

From (5.10) it follows that

\[ G_x = 2 \int_x^\infty P_x(t) \frac{dL(P_x(t))}{dt} - 1. \]

Which from (5.3) and (5.4) can be rewritten as
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\[ G_x = \frac{2 \int \left[ \frac{F_Z(t) - F_Z(x)}{F_Z(x)} \right] t f_Z(t) dt}{\int_y f_Z(y) dy} - 1. \]

Which is same as

\[ G_x \int_x y f_Z(y) dy = 2 \int_x \left[ \frac{F_Z(t) - F_Z(x)}{F_Z(x)} \right] t f_Z(t) dt - \int_x y f_Z(y) dy \]

or

\[ G_x \int_x y f_Z(y) dy = 2 \int_x \frac{F_Z(t) - F_Z(x)}{F_Z(x)} \int_x \frac{F_Z(t) - F_Z(x)}{F_Z(x)} dt - \int_x y f_Z(y) dy \]

\[ = 2 \int_x \frac{F_Z(t) - F_Z(x)}{F_Z(x)} \int_x t f_Z(t) dt + \int_x t f_Z(t) dt - \int_x y f_Z(y) dy \]

So that

\[ G_x \int_x y f_Z(y) dy = 2 \int_x \frac{F_Z(t) - F_Z(x)}{F_Z(x)} \int_x t f_Z(t) dt - \int_x y f_Z(y) dy \]

\[ = 2 \int_x \frac{F_Z(t) - F_Z(x)}{F_Z(x)} \int_x t f_Z(t) dt + \int_x t f_Z(t) dt. \]

Observing that \( \frac{x}{F_Z(x)} = x + m(x) \), we have

\[ G_x [x + m(x)] = 2 \int_x \frac{F_Z(t) - F_Z(x)}{F_Z(x)} \int_x \left[ \frac{F_Z(t)}{F_Z(x)} \right]^2 dt - \int_x y f_Z(y) dy \]

or

\[ 1 - G_x [x + m(x)] = \frac{2}{\left[ F_Z(x) \right]^2} \int_x t f_Z(t) \overline{F}_Z(t) dt. \quad (5.21) \]

Consider \( I = \int_x t f_Z(t) \overline{F}_Z(t) dt \)

\[ = - \int_x \left( t \overline{F}_Z(t) \right) d\overline{F}_Z(t) \]

\[ = - \left( t \overline{F}_Z(t) \right) \int_x d\overline{F}_Z(t) + \int_x \left[ \left( t f(t) \right) \int_x d\overline{F}_Z(t) \right] dt - \int_x \left[ \overline{F}_Z(t) \int_x d\overline{F}_Z(t) \right] dt \]
Thus
\[ 2I = x \left[ F_Z(x) \right]^2 + \int_x^\infty \left[ F_Z(t) \right]^2 dt \]

Substituting for \( 2I \) in (5.21) we have the result.

To prove the second equality, from (5.11), (5.5) and (5.6), it follows that
\[
G_{d_{x_2}} = 2 \int_x^\infty P_{d_{x_2}}(t) \frac{dL(P_{x_{2}}(t))}{dt} - 1.
\]
\[
= \frac{1}{x} \left[ \frac{\partial F(t,u)}{\partial u} \right]_{u=x_2} \int_x^\infty \left[ \frac{\partial F(x,u)}{\partial u} \right]_{u=x_2} t f(t,x_2) dt
\]
\[
= \frac{1}{x} \left[ -\frac{\partial F(x,u)}{\partial u} \right]_{u=x_2} \int_x^\infty y f(y,x_2) dy
\]

That is
\[
G_{d_{x_2}} \int_x^\infty y f(y,x_2) dy = \frac{2}{x} \left[ -\frac{\partial F(t,u)}{\partial u} \right]_{u=x_2} t f(t,x_2) dt
\]
\[
+ \frac{1}{x} \left[ \frac{\partial F(x,u)}{\partial u} \right]_{u=x_2} \int_x^\infty y f(y,x_2) dy.
\]

Dividing through out by \( -\left[ \frac{\partial F(x,u)}{\partial u} \right]_{u=x_2} \), we get
\[
G_{d_{x_2}} \int_x^\infty y f(y,x_2) dy
\]
\[
= \frac{2}{x} \left[ -\frac{\partial F(t,u)}{\partial u} \right]_{u=x_2} t f(t,x_2) dt + \frac{1}{x} \left[ \frac{\partial F(x,u)}{\partial u} \right]_{u=x_2} \int_x^\infty y f(y,x_2) dy.
\]
From (5.12) we have

\[ G_{d|x_2} \left[ x + m_{12} \left( x \mid x_2 \right) \right] = \left\{ \frac{2}{\partial} \left[ \frac{\partial F(t,u)}{\partial u} \right] \right\} \left[ \int_{x_{12}}^{\infty} \frac{\partial F(t,u)}{\partial u} \right] \left[ \frac{\partial f(t,x_2)}{\partial t} \right] \left[ x + m_{12} \left( x \mid x_2 \right) \right] \]

Thus

\[ \left[ 1 - G_{d|x_2} \right] \left[ x + m_{12} \left( x \mid x_2 \right) \right] = \left\{ \frac{-2}{\partial} \left[ \frac{\partial F(t,u)}{\partial u} \right] \right\} \left[ \int_{x_{12}}^{\infty} \frac{\partial F(t,u)}{\partial u} \right] \left[ \frac{\partial f(t,x_2)}{\partial t} \right] \left[ x + m_{12} \left( x \mid x_2 \right) \right]. \]

(5.22)

Now consider the integral given by

\[ \int_{x_{12}}^{\infty} \frac{\partial F(t,u)}{\partial u} \left[ \frac{\partial f(t,x_2)}{\partial t} \right] dt = \int_{x_{12}}^{\infty} \frac{\partial F(t,u)}{\partial u} \left[ \frac{\partial F(t,u)}{\partial t} \right] dt + \int_{x_{12}}^{\infty} \frac{\partial^2 F(t,u)}{\partial t \partial u} \left[ \frac{\partial F(t,u)}{\partial u} \right] dt \]

\[ = -x \left\{ \frac{\partial F(x,u)}{\partial u} \right\} \left[ \frac{\partial F(t,u)}{\partial u} \right] \left[ \frac{\partial f(t,x_2)}{\partial t} \right] dt + \int_{x_{12}}^{\infty} \frac{\partial^2 F(t,u)}{\partial t \partial u} \left[ \frac{\partial F(t,u)}{\partial u} \right] dt \]

\[ = -x \left\{ \frac{\partial F(x,u)}{\partial u} \right\} \left[ \frac{\partial F(t,u)}{\partial u} \right] \left[ \frac{\partial f(t,x_2)}{\partial t} \right] dt + \int_{x_{12}}^{\infty} \frac{\partial^2 F(t,u)}{\partial t \partial u} \left[ \frac{\partial F(t,u)}{\partial u} \right] dt \]

i.e.
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\[ 2 \int_{x}^{\infty} \left[ \frac{\partial F(t,u)}{\partial u} \right]_{u=x_2} \frac{\partial}{\partial t} \left[ \frac{\partial F(t,u)}{\partial u} \right]_{u=x_2} = \]
\[ -x \left\{ \left[ \frac{\partial F(x,u)}{\partial u} \right]_{u=x_2} \right\}^2 - \int_{x}^{\infty} \left\{ \left[ \frac{\partial F(t,u)}{\partial u} \right]_{u=x_2} \right\}^2 dt \]

where

\[ f(t, x_2) = \frac{\partial}{\partial t} \left[ \frac{\partial F(t,u)}{\partial u} \right]_{u=x_2}. \]

Substituting the above expression in (5.22), we get (5.19).

Proceeding in similar arguments with (5.11), (5.7) and (5.8), the third equality can be claimed.

5.3 Characterizations

In the backdrop of Theorem 5.1 and Theorem 5.2 it is of interest to look into characterizations based on the Lorenz function and Gini index. One general result in this direction is given as follows.

**Theorem 5.3** Under the usual notations the relation

\[ G_x = \left( \frac{1-k}{x+m(x)} \right), x > 0 \] (5.23)

\[ G_{dx_{ij}} = \frac{(1-k_i)m_{ij}(x|x_j)}{x+m_{ij}(x|x_j)}, x > x_j > 0, i \neq j = 1,2 \] (5.24)

holds for a bivariate vector \((X_1, X_2)\) with

(i) \(k, k_i = \frac{1}{2}, i = 1,2\) if and only if \((X_1, X_2)\) is distributed as Freund bivariate exponential distribution.

(ii) \(k, k_i < \frac{1}{2}, i = 1,2\) if and only if \((X_1, X_2)\) is distributed as bivariate Pareto I distribution.

(iii) \(k, k_i > \frac{1}{2}, i = 1,2\) if and only if \((X_1, X_2)\) is distributed as bivariate finite range distribution.
Proof

The if part of the theorem can be verified from Table 5.2. To prove the converse, consider (5.23), the equation becomes

\[ G_x = \frac{m(x)}{x + m(x)} - \frac{k m(x)}{x + m(x)} \]

\[ = 1 - \frac{x}{x + m(x)} - \frac{k m(x)}{x + m(x)} \]

\[ 1 - G_x = \frac{x + k m(x)}{x + m(x)}. \]  

(5.25)

From equation (5.25), the equation (5.18) becomes

\[ x + \frac{1}{\left[ \frac{\bar{F}_Z(x)}{x} \right]^2} \int_{x}^{\infty} \left[ \bar{F}_Z(t) \right]^2 dt = x + km(x). \]

or

\[ \int_{x}^{\infty} \left[ \bar{F}_Z(t) \right]^2 dt = k m(x) \left[ \bar{F}_Z(x) \right]^2. \]  

(5.26)

Differentiating (5.26) with respect to \( t \), we get

\[ -\left[ \bar{F}_Z(x) \right]^2 = -2k m(x) \bar{F}_Z(x) f_Z(x) + k \left[ \bar{F}_Z(x) \right]^2 \frac{d}{dx} m(x) \]

or

\[-1 = \frac{-2k m(x) f_Z(x)}{\bar{F}_Z(x)} + k \frac{d}{dx} m(x). \]

Now using (5.13), this equation becomes

\[ -1 = -2k m(x) \lambda(x) + k \left[ \lambda(x)m(x) - 1 \right] \]

or

\[ 1 = k m(x) \lambda(x) + k \]

so that

\[ m(x) \lambda(x) = \frac{1-k}{k}. \]  

(5.27)

Once again using (5.13) we have

\[ \frac{d}{dx} m(x) = \frac{1-2k}{k} \]

which on integration with respect to \( x \) gives
\[ m(x) = \left( \frac{1-2k}{k} \right) x + c \]

or
\[ \lambda(x) = \frac{1-k}{(1-2k)x + c}. \quad (5.28) \]

Now consider the second equation (5.24), which is equivalent to
\[ 1 - G_{x|x_j} = \frac{x + k_i m_j(x|x_j)}{x + m_i(x|x_j)}. \quad (5.29) \]

Now equation (5.29) implies
\[
x + \left\{ \frac{1}{\left[ -\frac{\partial F(t,u)}{\partial u} \right]_{u=x_2}} \right\}^2 \int_x^\infty \left[ -\frac{\partial F(t,u)}{\partial u} \right]_{u=x_2}^2 dt = x + k_1 m_{i_2} \left( x | x_2 \right). \]

or
\[
\int_x^\infty \left[ -\frac{\partial F(t,u)}{\partial u} \right]_{u=x_2}^2 dt = k_1 m_{i_2} \left( x | x_2 \right) \left\{ -\frac{\partial F(t,u)}{\partial u} \right\}_{u=x_2}^2. \]

Proceeding in the same manner as for (5.26) we will arrive at
\[ m_{i_2} \left( x | x_2 \right) A_{i_2} \left( x | x_2 \right) = \frac{1-k_1}{k_1}. \]

Using (5.13) we have
\[ \lambda_{i_2} \left( x | x_2 \right) = \frac{1-k_1}{(1-2k_1)x + c_1}, \quad x_2 < x. \quad (5.30) \]

Similar arguments hold for
\[ G_{x|x_i} = \frac{(1-k_2) m_{i_2}(x|x_i)}{x + m_{i_2}(x|x_i)}, \quad x_i < x \]

to get
\[ \lambda_{i_2} \left( x | x_i \right) = \frac{1-k_2}{(1-2k_2)x + c_2}, \quad x_2 < x. \quad (5.31) \]

Thus from Cox (1972) uniqueness property given in (1.23), we have
Inequality Measures for Bivariate Distributions with Load Sharing Dependence

\begin{eqnarray*}
f(x_1, x_2) = \\
\begin{cases} 
\exp \left[ - \frac{x_1 (1-k)}{(1-2k)} - \frac{x_2 (1-k_2)}{(1-2k_2)} \right] \frac{p_1 (1-k)}{(1-2k_1)} \frac{(1-k_2)}{(1-2k_2)} - \frac{x_1 u + c_{x_2}}{(1-2k_2)} \int_{x_1}^{x_2} \frac{du}{x_1 + c_{x_2}} - x_2 + c_{x_2} \right] \frac{p_2 (1-k)}{(1-2k_1)} \frac{(1-k_2)}{(1-2k_2)} \ ; x_1 < x_2 \\
\exp \left[ - \frac{x_1 (1-k)}{(1-2k)} - \frac{x_2 (1-k_2)}{(1-2k_2)} \right] \frac{p_1 (1-k)}{(1-2k_1)} \frac{(1-k_2)}{(1-2k_2)} - \frac{x_1 u + c_{x_2}}{(1-2k_2)} \int_{x_1}^{x_2} \frac{du}{x_1 + c_{x_2}} - x_2 + c_{x_2} \right] \frac{p_2 (1-k)}{(1-2k_1)} \frac{(1-k_2)}{(1-2k_2)} \ ; x_2 < x_1
\end{cases}
\end{eqnarray*}

Which when \( k, k_i = \frac{1}{2}, i = 1, 2 \), becomes

\begin{equation}
f(x_1, x_2) = \\
\begin{cases} 
\alpha_i \alpha_i' \exp \left[ - \alpha_i' x_2 - (\alpha_i + \alpha_i') x_1 \right] ; 0 < x_1 < x_2 < \infty \\
\alpha_i \alpha_i' \exp \left[ - \alpha_i' x_1 - (\alpha_i + \alpha_i') x_2 \right] ; 0 < x_2 < x_1 < \infty
\end{cases}
\end{equation}

which is the Freund’s (1961) bivariate exponential distribution with \( \alpha_i + \alpha_i' = \frac{1}{2c} \), \( \alpha_i = \frac{p_i}{2c} \) and \( \alpha_i' = \frac{1}{2c_i}, i = 1, 2 \).

When \( k, k_i < \frac{1}{2}, i = 1, 2 \) and \( c, c_i = 0 \), it becomes

\begin{equation}
f(x_1, x_2) = \\
\begin{cases} 
\frac{\alpha_i \alpha_i'}{\sigma^{-(\alpha_i + \alpha_i')}} x_1^{-(\alpha_i + \alpha_i' + 1)} x_2^{-(\alpha_i' + 1)} ; \sigma < x_1 < x_2 < \infty \\
\frac{\alpha_i \alpha_i'}{\sigma^{-(\alpha_i + \alpha_i')}} x_2^{-(\alpha_i + \alpha_i' + 1)} x_1^{-(\alpha_i' + 1)} ; \sigma < x_2 < x_1 < \infty
\end{cases}
\end{equation}

which is the bivariate Pareto I \((BPI(\sigma, \alpha_i, \alpha_i', \alpha_i', \alpha_i'))\) distribution (Asha and Jagathnath (2008)) with parameters

\( \alpha_i + \alpha_i' = \frac{1-k}{1-2k}, \alpha_i = \frac{p_i (1-k)}{1-2k} \) and \( \alpha_i' = \frac{1-k_i}{1-2k_i}, i = 1, 2 \).

When \( k, k_i > \frac{1}{2} \), and \( \frac{c}{1-2k} = \frac{c_i}{1-2k_i}, i = 1, 2 \), it becomes

\begin{equation}
f(x_1, x_2) = \\
\begin{cases} 
\frac{\alpha_i \alpha_i'}{R^2} \left( 1 - \frac{x_1}{R} \right)^{(\alpha_i + \alpha_i' - 1)} \left( 1 - \frac{x_2}{R} \right)^{(\alpha_i' - 1)} ; 0 < x_1 < x_2 < R \\
\frac{\alpha_i \alpha_i'}{R^2} \left( 1 - \frac{x_2}{R} \right)^{(\alpha_i + \alpha_i' - 1)} \left( 1 - \frac{x_1}{R} \right)^{(\alpha_i' - 1)} ; 0 < x_2 < x_1 < R
\end{cases}
\end{equation}
which is the bivariate finite range distribution with parameters, \( \alpha_1 + \alpha_2 = \frac{k-1}{2k-1} \).

\[
\alpha_i = \frac{p_i(k-1)}{2k-1}, \quad \alpha_i' = \frac{k_i-1}{2k_i-1}, \quad i = 1, 2 \quad \text{and} \quad R = \frac{c}{1-2k}.
\]

**Remark 5.2** Under conditions of the Theorem 5.3, equation (5.23) and (5.24) are equivalent to stating

\[
1 - G_x = \frac{x}{x + m(x)} + \frac{k}{\lambda(x)}
\]

and

\[
1 - G_{ixj} = \frac{x}{x + m_j(x \mid x_j)} + \frac{k_i}{\lambda_{ij}(x \mid x_j)}.
\]

This result helps us to deduce many results which are analogous to popular results in the univariate case. The piecewise constancy of the bivariate Gini index can be deduced from Theorem 5.3, which in a way extends the truncation invariance property (Ord et al. (1983)) of the univariate Pareto I distribution.

**Remark 5.3** Under the conditions in Theorem 3.1, the bivariate Gini index is of the form

\[
G(x) = (g, g_1, g_2)
\]

where \( 0 < g, g_i < 1, i = 1, 2 \) if and only if \((X_1, X_2)\) has a bivariate Pareto I distribution (refer Table 5.2).

**Remark 5.4** The quantity \( \frac{m(x)}{x + m(x)} \) is referred to as income gap ratio (Belzunce et al. (1998)). It measures the proportion of the people, whose income from both the sources are greater than the threshold value \( x \). So \( \frac{m_{ij}(x \mid x_j)}{x + m_{ij}(x \mid x_j)} \) can be used to
measure the proportion of the people whose income from one source falls below the threshold value \( x \), when \( x > x_j \). Thus

\[
\beta(x) = \left( \frac{m(x)}{x + m(x)}, \frac{m_{12}(x \mid x_2)}{x + m_{12}(x \mid x_2)}, \frac{m_{21}(x \mid x_1)}{x + m_{21}(x \mid x_1)} \right)
\]

can be viewed as a bivariate income gap ratio for the rich.

**Corollary 5.1** Under the conditions of Theorem 5.3, the following statements are equivalent.

(i) The bivariate Gini index is of the form (5.33) if and only if the bivariate income gap ratio is of the form

\[
\left( \frac{m(x)}{x + m(x)}, \frac{m_{12}(x \mid x_2)}{x + m_{12}(x \mid x_2)}, \frac{m_{21}(x \mid x_1)}{x + m_{21}(x \mid x_1)} \right) = (\beta, \beta_1, \beta_2),
\]

where \( \beta, \beta_1 \) and \( \beta_2 \) are some constant such that \( 0 < \beta, \beta_i < 1, i = 1, 2 \).

(ii) \((X_1, X_2)\) has a bivariate Pareto I distribution.
### Table 5.2: Expression for Gini index and income gap ratio

<table>
<thead>
<tr>
<th>Bivariate density function</th>
<th>( \lambda(x) )</th>
<th>( m(x) )</th>
<th>( G(x) )</th>
<th>( \beta(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bivariate exponential by Freund (1961)</strong></td>
<td>( (\alpha_1 + \alpha_2, \alpha_1', \alpha_2') )</td>
<td>( \frac{1}{\alpha_1} )</td>
<td>( \frac{1}{2\left((\alpha_1 + \alpha_2) x + 1\right)} )</td>
<td>( \frac{1}{\alpha_1 x + 1} )</td>
</tr>
<tr>
<td>( f(x_1, x_2) = \begin{cases} \frac{\alpha_1 \alpha_2}{\sigma^{-(\alpha_1 + \alpha_2)}} x_1^{-(\alpha_1 + \alpha_2)} x_2^{-(\alpha_2 + 1)}; \sigma &lt; x_1 &lt; x_2 \ \frac{\alpha_2 \alpha_1'}{\sigma^{-(\alpha_1 + \alpha_2)}} x_2^{-(\alpha_1 + \alpha_2 + 1)} x_1^{-(\alpha_2 + 1)}; \sigma &lt; x_2 &lt; x_1 \end{cases} ) (Asha &amp; Jagathnath Krishna, 2008)</td>
<td>( \frac{x}{\alpha_1 + \alpha_2 - 1} )</td>
<td>( \frac{x}{2\alpha_1' - 1} )</td>
<td>( \frac{1}{\alpha_1 + \alpha_2} )</td>
<td>( \frac{1}{\alpha_1', \alpha_2'} )</td>
</tr>
<tr>
<td>( \alpha_i &gt; 1, \alpha_i' &gt; 1, \alpha_1 + \alpha_2 \neq \alpha_i', \sigma &gt; 0, i = 1, 2. )</td>
<td>( \frac{R - x}{\alpha_1 + \alpha_2 + 1} )</td>
<td>( \frac{2\alpha_1' + 1}{\alpha_1' + 1} \frac{R + \alpha_1'}{R + \alpha_2'} )</td>
<td>( \frac{(R - x)}{(R + \alpha_i' x)} )</td>
<td>( \frac{(R - x)}{(R + \alpha_i' x)} )</td>
</tr>
<tr>
<td><strong>Bivariate finite range distribution</strong></td>
<td>( \frac{\alpha_1 \alpha_2}{R^2} \left(1 - \frac{x_1}{R}\right)^{(\alpha_1 + \alpha_2 - \alpha_1')}; \frac{1 - x_2}{R} \right)^{(\alpha_2' - 1)}; 0 &lt; x_1 &lt; x_2 &lt; R )</td>
<td>( \frac{R - x}{\alpha_1 + \alpha_2} )</td>
<td>( \frac{(\alpha_1 + \alpha_2)(R - x)}{(2\alpha_1 + R + \alpha_1)} )</td>
<td>( \frac{(R - x)}{(R + \alpha_i' x)} )</td>
</tr>
<tr>
<td>( \frac{\alpha_2 \alpha_1'}{R^2} \left(1 - \frac{x_2}{R}\right)^{-(\alpha_1 + \alpha_2 - \alpha_1')} \left(1 - \frac{x_1}{R}\right)^{(\alpha_2 - 1)}; 0 &lt; x_2 &lt; x_1 &lt; R )</td>
<td>( \frac{R - x}{\alpha_2' + 1} \frac{R - x}{\alpha_1' + 1} )</td>
<td>( \frac{2\alpha_2' + 1}{\alpha_2'} \frac{R + \alpha_2'}{R + \alpha_1'} )</td>
<td>( \frac{(R - x)}{(R + \alpha_2' x)} )</td>
<td>( \frac{(R - x)}{(R + \alpha_2' x)} )</td>
</tr>
<tr>
<td>( \alpha_i &gt; 0, \alpha_i' &gt; 0, \alpha_1 + \alpha_2 \neq \alpha_i', R &gt; 0, i = 1, 2. )</td>
<td></td>
<td></td>
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