Chapter 7

Sequential Interval Estimation of the Limiting Interval Reliability

7.1 Introduction

In this chapter, we consider the sequential interval estimation of the limiting interval reliability of a repairable system when the sequences of failure and repair times are generated by a bivariate stationary dependent sequence. In Section 7.2, we discuss the estimation of the limiting interval reliability $R(x)$ for a stationary strong mixing bivariate sequence of failure and repair times. Section 7.3 considers the sequential interval estimation of $R(x)$. In Section 7.4, we consider the sequential interval estimation in the case of a bivariate exponential autoregressive (BEAR) model. A numerical study is also performed in Section 7.5 to assess the performance of the proposed sequential decision rule. Finally, Section 7.6 provides brief conclusions of the study.

7.2 Estimation of the Limiting Interval Reliability

Suppose that $\{(X_n, Y_n), n \geq 1\}$ is strictly stationary and strong mixing in the sense that as $h \to \infty$,

$$\alpha(h) = \sup \left\{ \left| P(A \cap B) - P(A)P(B) \right| : A \in \mathcal{F}^i(X, Y) \text{ and } B \in \mathcal{F}^{\infty}_{k+h}(X, Y) \right\} \to 0,$$

where

$$\mathcal{F}^i(X, Y) = \sigma(X_i, Y_i; 1 \leq i \leq k) \text{ and } \mathcal{F}^{\infty}_{k+h}(X, Y) = \sigma(X_i, Y_i; i \geq k+h).$$

The results in this chapter have been accepted for publication as entitled ‘Sequential Interval Estimation of the Limiting Interval Availability for a Bivariate Stationary Dependent Sequence’ in the journal *Statistics* (See Balakrishna and Mathew, 2010).
When the observations on the failure and repair times of ‘n’ complete cycles, \((X_1,Y_1),(X_2,Y_2),..., (X_n,Y_n)\), are available, a natural estimator for the limiting interval reliability \(R(x)\) is

\[
\hat{R}_n(x) = \frac{\bar{U}_n}{\bar{X}_n + \bar{Y}_n},
\]

where \(\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}, \bar{Y}_n = \frac{\sum_{i=1}^{n} Y_i}{n}\) and \(\bar{U}_n = \frac{\sum_{i=1}^{n} U_i}{n}\) with \(U_i = (X_i - x)I_{(X_i > x)}\).

Since \(\{(X_n,Y_n), n \geq 1\}\) is strictly stationary it follows that \(\bar{X}_n \to \mu_X\), \(\bar{Y}_n \to \mu_Y\) and \(\bar{U}_n \to \nu(x)\) almost surely as \(n \to \infty\) and hence we conclude that \(\hat{R}_n(x) \to R(x)\) almost surely as \(n \to \infty\).

In order to establish the asymptotic normality of the estimator \(\hat{R}_n(x)\), we assume that for some \(\delta > 0\), \(E(X_i^{2+\delta}) < \infty\), \(E(Y_i^{2+\delta}) < \infty\) and \(\sum_{h=1}^{\infty} \alpha^{d(2+\delta)}(h) < \infty\).

Since \(\{(X_n,Y_n), n \geq 1\}\) is strictly stationary and strong mixing with mixing coefficient \(\alpha(h)\), under the above assumptions, by the central limit theorem for such sequences (Lemma 1.7) we have as \(n \to \infty\),

\[
\sqrt{n}(\bar{X}_n - \mu_X, \bar{Y}_n - \mu_Y) \xrightarrow{L} N_2(0, \Sigma_2),
\]

where \(N_2(0, \Sigma_2)\) is a bivariate normal vector with mean \(0 = (0,0)\) and dispersion matrix

\[
\Sigma_2 = \begin{pmatrix}
\sigma_{XX} & \sigma_{XY} \\
\sigma_{XY} & \sigma_{YY}
\end{pmatrix},
\]

with \(\sigma_{XX} = \text{var}(X_1) + 2 \sum_{h=2}^{\infty} \text{cov}(X_1, X_h), \sigma_{YY} = \text{var}(Y_1) + 2 \sum_{h=2}^{\infty} \text{cov}(Y_1, Y_h)\)

and \(\sigma_{XY} = \text{cov}(X_1, Y_1) + \sum_{h=2}^{\infty} \text{cov}(X_1, Y_h) + \sum_{h=2}^{\infty} \text{cov}(X_h, Y_1)\).
If we define, \(Z_n = \frac{1}{n} \sum_{i=1}^{n} Z_i\), with \(Z_i = X_i + Y_i\), then it follows that as \(n \to \infty\)

\[
\sqrt{n} \left( Z_n - \mu_Z \right) \xrightarrow{L} N(0, \sigma_{ZZ}),
\]

where \(\mu_Z = \mu_X + \mu_Y\) and \(\sigma_{ZZ} = \sigma_{XX} + 2\sigma_{XY} + \sigma_{YY}\).

It is to be noted that \(\{(U_n, Z_n), n \geq 1\}\) is also strictly stationary and by the Cramer-Wold device (Billingsley, 1968, pp.49), it may be verified that as \(n \to \infty\)

\[
\sqrt{n} \left( \bar{U}_n - v(x), \bar{Z}_n - \mu_Z \right) \xrightarrow{L} N_2(0, \Sigma^*_2), \tag{7.2}
\]

where \(\Sigma^*_2 = \begin{pmatrix} \sigma_{UU} & \sigma_{UZ} \\ \sigma_{UZ} & \sigma_{ZZ} \end{pmatrix}\),

with \(\sigma_{UU} = \text{var}(U_i) + 2\sum_{h=2}^{\infty} \text{cov}(U_i, U_h)\) and

\[
\sigma_{UZ} = \text{cov}(U_i, Z_i) + \sum_{h=2}^{\infty} \text{cov}(U_i, Z_h) + \sum_{h=2}^{\infty} \text{cov}(U_h, Z_i).
\]

If we define \(g(x, y) = x / y\), then \(g(\bar{U}_n, \bar{Z}_n) = \hat{R}_n(x)\).

Now, the partial derivatives of \(g(.)\) are

\[
\frac{\partial g}{\partial x} \bigg|_{(x, y, \mu_z)} = \frac{1}{\mu_z}, \text{ and } \frac{\partial g}{\partial y} \bigg|_{(x, y, \mu_z)} = -\frac{v(x)}{\mu_z^2}.
\]

Hence, by using Lemma 1.4, we can show that as \(n \to \infty\),

\[
\sqrt{n} \left( \hat{R}_n(x) - R(x) \right) \xrightarrow{L} N(0, \tau^2(x)),
\]

where

\[
\tau^2(x) = \frac{\sigma_{UU}}{\mu_z} - 2\frac{\mu(w)\sigma_{UZ}}{\mu_z^2} + \frac{\mu^2(w)\sigma_{ZZ}}{\mu_z^2}. \tag{7.3}
\]

Thus, we proved the following theorem.
Theorem 7.1

If \( \{(X_n, Y_n), \ n \geq 1\} \) is a strictly stationary and strong mixing sequence of bivariate random vectors on \( R^2 = \{(x, y) : 0 \leq x < \infty, 0 \leq y < \infty\} \) such that for some \( \delta > 0 \),
\[ E(X_i^{2+\delta}) < \infty, \ E(Y_i^{2+\delta}) < \infty \] and \( \sum_{h=1}^{\infty} \alpha^{\delta/(2+\delta)}(h) < \infty \), then \( \hat{R}_n(x) \) is a consistent and asymptotically normal (CAN) estimator for \( R(x) \).

Thus if \( \tau^2(x) \) is known, for a given significance level \( \alpha \in (0,1) \), a 100(1-\( \alpha \))% confidence interval for \( R(x) \) is
\[ \hat{R}_n(x) - z_{\alpha/2} \frac{\tau(x)}{\sqrt{n}} \leq R(x) \leq \hat{R}_n(x) + z_{\alpha/2} \frac{\tau(x)}{\sqrt{n}}. \]

Remark 7.1 When \( x = 0 \), the estimator \( \hat{R}_n(x) \) reduces to \( \hat{R}_n(0) = \frac{\bar{X}_n}{\bar{X}_n + \bar{Y}_n} = \hat{A}_n \), which is a CAN estimator for the limiting availability \( A = \mu_x / (\mu_x + \mu_y) \). Also it is straightforward to see that as \( n \to \infty \)
\[ \sqrt{n} \left( \hat{A}_n - A \right) \xrightarrow{d} N(0, \gamma^2), \]
where \( \gamma^2 = (\mu_y^2 \sigma_{xx} + \mu_x^2 \sigma_{yy} - 2 \mu_x \mu_y \sigma_{xy}) / (\mu_x + \mu_y)^4 \).

In the next section, we discuss the sequential confidence interval estimation for the limiting interval reliability.

7.3 Sequential Interval Estimation

In sequential Interval estimation our prime objective is to locate an interval, say \( I_n \), based on the observations, \( (X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n) \), such that

i) \( P[R(x) \in I_n] \geq 1 - \alpha \) and

ii) width of \( I_n \leq 2d \),

where \( \alpha \) (0 < \( \alpha < 1 \)) and \( d \) (\( d > 0 \)) are preassigned numbers.
Theorem 7.1 ensures that for large $n$,

$$P\left[ \sqrt{n}\left| \hat{R}_n(x) - R(x)\right| \leq z_{\alpha/2}\tau(x) \right] \geq 1 - \alpha,$$

(7.4)

where $z_{\alpha/2}$ denotes the upper $\alpha/2$ quantile of the standard normal distribution.

Define, $I_n = \left[ \hat{R}_n(x) - d, \hat{R}_n(x) + d \right]$.

If $\tau^2(x)$ is known, we could take $n_d = \min\{n : z_{\alpha/2}^2 \tau^2(x) d^{-2} \leq n\}$ as the number of observations. Then $I_{n_d}$ is the fixed accuracy confidence interval for the limiting interval reliability $R(x)$ of fixed width $2d$ with coverage probability

$$P[R(x) \in I_{n_d}] = P\left[ \sqrt{n_d}\left| \hat{R}_{n_d}(x) - R(x)\right| \leq d\sqrt{n_d} \right],$$

which converges to $1 - \alpha$ as $d \to 0$ due to (7.4) and the fact that

$$\lim_{d \to 0} \frac{d^2 n_d}{z_{\alpha/2}^2 \tau^2(x)} = 1.$$

However, $\tau^2(x)$ is unknown in practice, so we should replace it by a consistent estimator. A consistent estimator $\hat{\tau}^2_n(x)$ of $\tau^2(x)$ can be obtained by replacing $\nu(x), \mu_Z, \sigma_{UU}, \sigma_{ZZ}$ and $\sigma_{UZ}$ with their corresponding consistent estimators in (7.3). Obviously $\bar{U}_n$ and $\bar{Z}_n$ are the consistent estimators for $\nu(x)$ and $\mu_z$ respectively. In order to construct consistent estimators for $\sigma_{UU}, \sigma_{ZZ}$ and $\sigma_{UZ}$, we use the moving-block jackknife method for variance estimation with dependent data (Kunch, 1989). The moving-block jackknife estimators for $\sigma_{UU}, \sigma_{ZZ}$ and $\sigma_{UZ}$, respectively, are

$$\hat{\sigma}_{UU}^2 = \frac{l}{n-l+1} \sum_{i=1}^{n-l+1} \left( \bar{U}^{(i)}_i - (n+1-l)^{-1} \sum_{j=1}^{n-l+1} \bar{U}^{(j)}_i \right)^2,$$

$$\hat{\sigma}_{ZZ}^2 = \frac{l}{n-l+1} \sum_{i=1}^{n-l+1} \left( \bar{Z}^{(i)}_i - (n+1-l)^{-1} \sum_{j=1}^{n-l+1} \bar{Z}^{(j)}_i \right)^2$$

and

$$\hat{\sigma}_{UZ}^2 = \frac{l}{n-l+1} \sum_{i=1}^{n-l+1} \left( \bar{U}^{(i)}_i - (n+1-l)^{-1} \sum_{j=1}^{n-l+1} \bar{U}^{(j)}_i \right) \left( \bar{Z}^{(i)}_i - (n+1-l)^{-1} \sum_{j=1}^{n-l+1} \bar{Z}^{(j)}_i \right).$$
where \( \overline{U}^{(i)} = \sum_{j=i}^{i+l-1} U_j \), \( \overline{Z}^{(i)} = \sum_{j=i}^{i+l-1} Z_j \) and \( l = l(n) \) is the block size.

To establish the optimal properties of the sequential procedure we make the following assumption:

\[ A_i: \text{For some } \delta > 0 , \ E\left[\left|X_1\right|^{6+\delta}\right] < \infty, \ E\left[\left|Y_1\right|^{6+\delta}\right] < \infty \text{ and } \sum k^2 \alpha(k)^{6/(6+\delta)} < \infty. \]

Under the above assumption the estimators \( \hat{\sigma}_{UU}^2, \hat{\sigma}_{ZZ}^2 \) and \( \hat{\sigma}_{UZ} \) converge almost surely to \( \sigma_{UU}, \sigma_{ZZ} \) and \( \sigma_{UZ} \) respectively if \( l = o(n) \) and \( l \to \infty \) (Kunch, 1989).

Then, it is easy to see that

\[ \hat{\tau}_n^2(x) \to \tau^2(x) \text{ almost surely as } n \to \infty. \] (7.5)

Now, consider the stopping rule

\[ N_d = \inf \{ n \geq m : nd^2 \geq z_{\alpha/2}^2 \hat{\tau}_n^2(x) \}, \] (7.6)

where \( m \) is the initial sample size.

The bounded length confidence interval is then

\[ I_{N_d} = \left[ \hat{R}_{N_d}(x) - d, \hat{R}_{N_d}(x) + d \right]. \]

The various steps involved in the construction of sequential confidence interval for the limiting interval reliability are summarized below:

1) Take a preliminary sample of appropriate size \( m \), \( (X_i, Y_i), i = 1, 2, \ldots, m \) and transform the data into \( (U_i, Z_i), i = 1, 2, \ldots, m \), where \( U_i = (X_i - x)I_{(X_i > x)} \) and \( Z_i = X_i + Y_i \).

2) Estimate the unknown parameter \( \tau^2(x) \) by

\[ \hat{\tau}_n^2(x) = \frac{\hat{\sigma}_{UU}^2}{Z_n^2} - 2 \frac{\overline{U}_n \hat{\sigma}_{UZ} \overline{Z}_n}{Z_n^2} + \frac{\hat{\sigma}_{ZZ}^2}{Z_n^2}. \]

3) For preassigned \( d, (0 < d \leq 0.5) \), calculate the stopping number \( N_d \) defined by (7.6).
4) Take $N_d - m$ additional samples $(X_i, Y_i)$, $i = m + 1, m + 2, ..., N_d$. Then with
the total sample of size $N_d$ construct the confidence interval

$$I_{N_d} = \left[ \hat{R}_{N_d}(x) - d, \, \hat{R}_{N_d}(x) + d \right].$$

The desirable asymptotic properties of the stopping rule defined by (7.6) are given
in the following theorem.

**Theorem 7.2**

*Under the assumption $A_i$, as $d \to 0$,

(i) $\frac{N_d}{n_d} \to 1$ almost surely

(ii) $P[R(x) \in I_{N_d}] \to 1 - \alpha$ (asymptotic consistency)

(iii) $E\left( \frac{N_d}{n_d} \right) \to 1$ (asymptotic efficiency).

*Proof*

In order to prove (i) note that

$$d^{-2} z_{\alpha/2}^2 \left( \hat{z}_{N_d}^2(x) + N_d^{-h} \right) \leq N_d \leq (m - 1)I_{(N_d=m)} + d^{-2} z_{\alpha/2}^2 \left( \hat{z}_{N_d-1}^2(x) + (N_d - 1)^{-h} \right) + 1$$

$$\leq m + d^{-2} z_{\alpha/2}^2 \left( \hat{z}_{N_d-1}^2(x) + (N_d - 1)^{-h} \right).$$

Hence,

$$\frac{d^{-2} z_{\alpha/2}^2 \left( \hat{z}_{N_d}^2(x) + N_d^{-h} \right)}{n_d} \leq \frac{N_d}{n_d} \leq \frac{m}{n_d} + \frac{d^{-2} z_{\alpha/2}^2 \left( \hat{z}_{N_d-1}^2(x) + (N_d - 1)^{-h} \right)}{n_d}.$$ 

Now using the fact that $\lim_{d \to 0} \frac{d^2 n_d}{z_{\alpha/2}^2} = \tau^2(x)$ and from (7.5) it follows that as $d \to 0$,

$$\frac{N_d}{n_d} \to 1 \text{ almost surely.} \quad (7.7)$$

If we define,

$$\xi_j = \mu_x U_j - \nu(x) Z_j, \quad j = 1, 2, ..., $$
then $\xi_j$'s are also strictly stationary and it follows from (7.2) that as $n \to \infty$

$$
\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \xi_j \xrightarrow{d} N\left(0, \gamma^2(x)\right),
$$

where

$$
\gamma^2(x) = \mu_x^2 \sigma_{Ux} + \nu^2(x) \sigma_{Zx} - 2 \nu(x) \mu_x \sigma_{UX}.
$$

(7.8)

To establish the asymptotic consistency property we use Anscombe’s theorem (Lemma 1.8), which requires \(\left\{\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \xi_j\right\}\) to be uniformly continuous in probability (u.c.i.p). (See Definition 1.5).

Letting $Q_n = \sum_{j=1}^{n} \xi_j$ and following Woodroofe (1982, pp.11), we can write

$$
\left| \frac{Q_{n+k}}{\sqrt{n+k}} - \frac{Q_n}{\sqrt{n}} \right| \leq \sqrt{n} \left| Q_{n+k} - Q_n \right| + \left[ 1 - \frac{n}{n+k} \right] \frac{Q_n}{\sqrt{n}}
$$

for $k, n \geq 1$.

If $\epsilon, \delta > 0$ and $k \geq n \delta$, then the second term on the right is bounded by $C(\delta) Q_n / \sqrt{n}$, where $C(\delta) = 1 - (1 + \delta)^{-1/2}$ and

$$
P\left[ \operatorname{Max}_{0 \leq k \leq n \delta} \left| \frac{Q_{n+k}}{\sqrt{n+k}} - \frac{Q_n}{\sqrt{n}} \right| > \frac{\epsilon}{2} \right] \leq P\left[ \frac{Q_n}{\sqrt{n}} > \frac{\epsilon}{2C(\delta)} \right].
$$

which tends to zero as $\delta \to 0$ uniformly in $n \geq 1$, since $\{Q_n / \sqrt{n}, n \geq 1\}$ are stochastically bounded.

Since $\{\xi_j\}$ is a strong mixing sequence of random variables, by the maximal inequality for such random variables (Rio, 1995), we have,

$$
P\left[ \operatorname{Max}_{0 \leq k \leq n \delta} \left| Q_{n+k} - Q_n \right| > \frac{\epsilon \sqrt{n}}{2} \right] \leq \frac{64}{\epsilon^2} \operatorname{Var}\left( \sum_{j=n+1}^{n+n \delta} \xi_j \right)
$$

$$
\leq \frac{64}{\epsilon^2} \delta \gamma^2(w),
$$

which is independent of $n \geq 1$ and tends to zero as $\delta \to 0$.
Thus, \( \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \xi_j \), \( n \geq 1 \) is u.c.i.p.

Now, by Anscombe’s theorem (Lemma 1.8), we have as \( d \to 0 \)
\[
\frac{1}{\sqrt{N_d}} \sum_{j=1}^{N_d} \xi_j = \frac{1}{\sqrt{N_d}} \sum_{j=1}^{N_d} [\mu_j \xi_j - \nu(x) Z_j] \xrightarrow{L} N \left( 0, \gamma^2(x) \right).
\]

Note that
\[
\sqrt{N_d} \left( \hat{R}_{N_d}(x) - R(x) \right) = \sqrt{N_d} \left( \frac{\bar{U}_{N_d}}{Z_{N_d}} - \frac{\nu(x)}{\mu_z} \right)
\]
\[
= \frac{1}{\sqrt{N_d}} \sum_{j=1}^{N_d} [\mu_j \xi_j - \nu(x) Z_j]
\]
\[
= \frac{\mu_j Z_{N_d}}{\mu_z Z_{N_d}}.
\]

Since \( \bar{Z}_{N_d} \xrightarrow{p} \mu_z \) as \( d \to 0 \) it follows from Slutsky’s theorem (Lemma 1.5) that
\[
\sqrt{N_d} \left( \hat{R}_{N_d}(x) - R(x) \right) \xrightarrow{L} N \left( 0, \tau^2(x) \right).
\]

Now,
\[
P[R(x) \in I_{N_d}] = P \left[ \left| \hat{R}_{N_d}(x) - R(x) \right| \leq d \right]
\]
\[
= P \left[ \sqrt{N_d} \left| \hat{R}_{N_d}(x) - R(x) \right| \leq d \sqrt{n_d} \frac{N_d}{\tau(x) \sqrt{n_d}} \right],
\]
which converges to \( 1 - \alpha \) as \( d \to 0 \) due to (7.7) and the fact that
\[
\lim_{d \to 0} \frac{d^2 n_d}{\tau^2_{\alpha/2}(x)} = 1.
\]

Let \( 0 < \varepsilon < 1 \) be given, and define \( a = (1 - \varepsilon)n_d \) and \( b = (1 + \varepsilon)n_d \).

Note that,
\[
E \left( \sum_{n=\infty}^{N_d} nP[N_d = n] \right) \geq aP[N_d \geq a]
\]
and hence
\[
E \left( \frac{N_d}{n_d} \right) \geq (1 - \varepsilon)P[N_d \geq a].
\]
Now, using (7.7)

\[
\liminf_{d \to 0} E \left( \frac{N_d}{n_d} \right) \geq (1 - \varepsilon) \quad (7.9)
\]

Also note that,

\[
E(N_d) = \sum_{n=m}^{\infty} nP[N_d = n] \leq bP[N_d \leq b] + \sum_{n=b+1}^{\infty} nP[N_d = n] = b + T(b),
\]

where \( T(b) = \sum_{n=b}^{\infty} P[N_d > n] \).

Now,

\[
E \left( \frac{N_d}{n_d} \right) \leq (1 + \varepsilon) + \frac{T(b)}{n_d}. \quad (7.10)
\]

Consider,

\[
T(b) = \sum_{n=b}^{\infty} P[N_d > n] \leq \sum_{n=b}^{\infty} P\left[ n < c \left( \hat{\tau}_n^2(x) + n^{-h} \right) \right]
\]

where \( c = d^{-2}z_{\alpha/2}^2 \)

\[
\leq \sum_{n=b}^{\infty} P[\hat{\tau}_n^2(x) > c^{-1}n - n^{-h}]
\]

\[
\leq \sum_{n=b}^{\infty} P[\hat{\tau}_n^2(x) > c^{-1}b - b^{-h}]
\]

\[
\leq \sum_{n=b}^{\infty} P[\hat{\tau}_n^2(x) - \tau^2(x) > c^{-1}(b - n_d) - b^{-h}]
\]

\[
\leq \sum_{n=b}^{\infty} P[\hat{\tau}_n^2(x) - \tau^2(x) > \varepsilon\tau^2(x) - \left\{ d^2/[z_{\alpha/2}^2(1 + \varepsilon)\tau^2(x)] \right\}^b]
\]

If we choose \( d \) small enough so that

\[
\varepsilon\tau^2(x) - \left\{ d^2/[z_{\alpha/2}^2(1 + \varepsilon)\tau^2(x)] \right\}^b > \frac{1}{2}\varepsilon\tau^2(x),
\]

then

\[
T(b) \leq \sum_{n=b}^{\infty} P \left[ \left| \hat{\tau}_n^2(x) - \tau^2(x) \right| > \frac{1}{2}\varepsilon\tau^2(x) \right] < \infty.
\]
It is clear that for sufficiently small $d$, since $T(b) < \infty$, (7.5) together with (7.10) imply that,

$$\limsup_{d \to 0} E \left( \frac{N_d}{n_d} \right) \leq (1 + \varepsilon).$$

Combining this with (7.9) we get (iii). This completes the proof.

In the next section we discuss the sequential estimation for a specific bivariate sequence.

### 7.4 Sequential Interval Estimation for a BEAR(1) Process

In this section we discuss the application of the results obtained in Section 7.2 and 7.3 for a BEAR(1) model.

Let $(N_1, N_2)$ be a bivariate geometric random vector with support $S = \{(i, j) : i, j \geq 1\}$ defined by Block et. al. (1988) with probability mass function

$$P[N_1 = n_1, N_2 = n_2] = \begin{cases} p_{00} p_{01}^{n_0-1} (p_{00} + p_{01})^{n_0-n_1} (1 - (p_{00} + p_{01})) & n_1 < n_2 \\ p_{10} p_{11}^{n_1-1} (p_{10} + p_{11})^{n_1-n_2} (1 - (p_{10} + p_{11})) & n_1 > n_2 \\ p_{01}^{n_1} p_{00} & n_1 = n_2 \end{cases}, \quad (7.11)$$

where $0 \leq p_{ij} \leq 1$, $i, j = 0, 1$ such that $p_{00} + p_{10} + p_{01} + p_{11} = 1$, $0 < p_{01} + p_{11} < 1$ and $0 < p_{10} + p_{11} < 1$.

Let $\{(I_1(n), I_2(n))\}$ be a sequence i.i.d. bivariate Bernoulli random vectors with $P[I_1(n) = i, I_2(n) = j] = p_{ij}$, $i, j = 0, 1$, where $p_{ij}$’s are as in (7.11).

Suppose $\{(E_{1n}, E_{2n}), n = 0, \pm 1, \pm 2, \ldots\}$ is a sequence of i.i.d. bivariate exponential random vector denoted by $BVE(\lambda_1, \lambda_2, \rho)$ with mean $(\lambda_1^{-1}, \lambda_2^{-1})$ and correlation coefficient $\rho$ such that the sequences $(E_1, E_2)$, $(I_1, I_2)$ and $(N_1, N_2)$ are mutually independent.
Define,

\[
X(n) = \begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} I_1(n)X_{n-1} + \pi_1E_{in} \\ I_2(n)Y_{n-1} + \pi_2E_{2n} \end{pmatrix}, \quad n = 2, 3, \ldots \quad (7.12)
\]

where

\[
X(1) = (X_1, Y_1)' = \left( \pi_1 \sum_{j=1}^{N_1} E_{1,-j}, \pi_2 \sum_{j=1}^{N_2} E_{2,-j} \right)'.
\]

with \( \pi_1 = p_{01} + p_{00} \) and \( \pi_2 = p_{10} + p_{00} \).

The sequence \( \{X(n), n \geq 1\} \) defined by (7.12) is referred to as a Bivariate exponential autoregressive process of order 1 (BEAR(1)) process. For each \( n \geq 1 \), \( X(n) \) has \( BVE(\lambda_1, \lambda_2, \rho) \) distribution. It is shown in Abraham and Balakrishna (2000) that the BEAR(1) sequence \( \{X(n)\} \) is stationary, ergodic and strong mixing with mixing parameter \( \alpha(h) = (p_{01} + p_{11})^{h-1} + (p_{01} + p_{11})^{h-1}, \quad h = 1, 2, \ldots \).

In particular, if \( (E_1, E_2) \) has a Marshall-Olkin bivariate exponential distribution with survival function (Marshall and Olkin, 1967)

\[
\bar{F}(x, y) = \exp[-b_1x-b_2y-b_{12}\max(x, y)], \quad x, y \geq 0,
\]

where \( b_1, b_2 \) and \( b_{12} \) are non-negative real numbers such that \( \lambda_1 = b_1 + b_{12}, \lambda_2 = b_2 + b_{12} \) and \( \rho = b_{12}/(b_1 + b_2 + b_{12}) \) and if we choose \( 0 < \theta < (\lambda_1 + \lambda_2)^{-1} \), \( \pi_1 = \lambda_1\theta \), \( \pi_2 = \lambda_2\theta \), \( p_{01} = \theta b_{12}, \), \( p_{01} = \theta b_1 \), \( p_{10} = \theta b_2 \) and \( p_{11} = 1-\theta(b_1 + b_2 + b_{12}) \), then the resulting BEAR(1) sequence \( \{X(n), n \geq 1\} \) is stationary and strong mixing with mixing parameter \( \alpha(h) = (1-\pi_1)^{h-1} + (1-\pi_2)^{h-1} \) and each \( X(n) \) has a Marshall-Olkin bivariate exponential distribution for \( n \geq 1 \) (See Block et. al., 1988).

If we define \( V(n) = (X_n, Y_n, U_n)' \), the autocovariance matrix \( \Gamma_V(k) \) of \( \{V(n)\} \) becomes
\[ \Gamma_v(k) = \text{Cov}(V(n), V(n+k)) \]
\[
\begin{pmatrix}
(1 - \pi_1)^k \lambda_1^{-2} & (1 - \pi_2)^k \rho \lambda_1^{-1} \lambda_2^{-1} & (1 - \pi_1)^k \lambda_1^{-2} (1 + \lambda_1 x) e^{-\lambda_1 x} \\
(1 - \pi_1)^k \rho \lambda_1^{-1} \lambda_2^{-1} & (1 - \pi_2)^k \lambda_2^{-2} & (1 - \pi_1)^k \rho(x) \lambda_1^{-1} \lambda_2^{-1} e^{-\lambda_1 x} \\
(1 - \pi_1)^k \lambda_1^{-2} (1 + \lambda_1 x) e^{-\lambda_1 x} & (1 - \pi_2)^k \rho(x) \lambda_1^{-1} \lambda_2^{-1} e^{-\lambda_1 x} & (1 - \pi_1)^k \lambda_1^{-2} e^{-\lambda_1 x} (2 - e^{-\lambda_1 x})
\end{pmatrix}
\]
where \( \rho(x) \) is the correlation coefficient between \( U_n \) and \( Y_n \).

For the \( BEAR(1) \) sequence all the moments of \( X_n \) and \( Y_n \) are finite and hence those of \( U_n \) and \( Z_n \). In this case \( \nu(x) = \lambda_1^{-1} e^{-\lambda_1 x} \) and \( \mu_Z = (\lambda_1 + \lambda_2) \lambda_1^{-1} \lambda_2^{-1} \).

Also, it can be verified that \( \sum_{h=1}^{\infty} \alpha^{\delta(h+2)}(h) < \infty \). Thus it follows that
\[
\sqrt{n} \left( \bar{U}_n - \mu(w), \bar{Z}_n - \mu_Z \right) \xrightarrow{L} N(0, \Sigma^*_2),
\]
where \( \Sigma^*_2 = \begin{pmatrix} \sigma_{UU} & \sigma_{UZ} \\ \sigma_{UZ} & \sigma_{ZZ} \end{pmatrix} \),
\[
\sigma_{UU} = \frac{2 - \pi_1}{\pi_1 \lambda_1^2} e^{-\lambda_1 x} (2 - e^{-\lambda_1 x}),
\]
\[
\sigma_{UU} = \frac{2 - \pi_1}{\pi_1 \lambda_1^2} e^{-\lambda_1 x} (1 + \lambda_1 x) + \frac{\rho(x)e^{-\lambda_1 x}}{\lambda_1 \lambda_2} \left( \frac{\pi_1 + \pi_2 - \pi_1 \pi_2}{\pi_1 \pi_2} \right)
\]
and
\[
\sigma_{ZZ} = \frac{2 - \pi_1}{\pi_1 \lambda_1^2} + \frac{2 \rho}{\pi_1 \lambda_1^2} \left( \frac{\pi_1 + \pi_2 - \pi_1 \pi_2}{\pi_1 \pi_2} \right) + \frac{2 - \pi_2}{\pi_2 \lambda_2^2}.
\]

Hence by applying the results of Theorem 7.1, we get
\[
\sqrt{n} \left( \hat{R}_n(x) - R(x) \right) \xrightarrow{L} N(0, \tau^2(x)), \tag{7.13}
\]
where \( \tau^2(x) = \frac{\lambda_2^2 e^{-2\lambda_2 x}}{(\lambda_1 + \lambda_2)^2} \left[ \frac{2 - \pi_1}{\pi_1} \left( \frac{\lambda_1^2}{(\lambda_1 + \lambda_2)^2} - \frac{2 \lambda_2 (1 + \lambda_1 x)}{\lambda_1 + \lambda_2} + 2 e^{\lambda_1 x} - 1 \right) + \frac{2 \lambda_1 \pi_1}{\pi_1 \pi_2 (\lambda_1 + \lambda_2)} \left( \frac{\rho \lambda_2}{\lambda_1 + \lambda_2} - \rho(x) \right) + \frac{2 - \pi_2}{\pi_2} \left( \frac{\lambda_2^2}{(\lambda_1 + \lambda_2)^2} \right) \right].
\]

This in turn implies that, for a \( BEAR(1) \) sequence the estimator \( \hat{R}_n(x) \) is CAN for the limiting interval availability \( R(x) = \lambda_2 e^{-\lambda_2 x} / (\lambda_1 + \lambda_2) \).
In order to establish the importance of \( \text{BEAR}(1) \) model compared to the \( \text{i.i.d. } BVE(\lambda_1, \lambda_2, \rho) \) sequence, without loss of generality we consider the case of \( x = 0 \). In this case, \( R(x) \) reduces to the limiting availability \( A = \lambda_2 / (\lambda_1 + \lambda_2) \) and hence (7.13) becomes

\[
\sqrt{n} \left( \hat{R}_n(0) - A \right) \xrightarrow{L} N \left( 0, \tau^2 \right),
\]

where

\[
\tau^2 = \frac{2(1 - \rho)(\lambda_1 \lambda_2)^2(\pi_1 + \pi_2 - \pi_1 \pi_2)}{\pi_1 \pi_2 (\lambda_1 + \lambda_2)^2}.
\]

Note that the \( \text{BEAR}(1) \) sequence reduces to the \( \text{i.i.d. } BVE(\lambda_1, \lambda_2, \rho) \) sequence when \( \pi_1 = \pi_2 = 1 \). Let \( \tau_*^2 \) be the asymptotic variance of \( \hat{R}_n(0) \) in the case of \( \text{i.i.d. } BVE(\lambda_1, \lambda_2, \rho) \) case.

Then,

\[
\tau_*^2 = \frac{2(1 - \rho)(\lambda_1 \lambda_2)^2}{(\lambda_1 + \lambda_2)^2}.
\]

Let \( n_d \) and \( n_d^* \) denote the number of observations required to construct sequential confidence interval for the limiting availability \( A \), of fixed width \( 2d \) and coverage probability \( 1 - \alpha \) in the case of \( \text{BEAR}(1) \) sequence and \( \text{i.i.d. } BVE(\lambda_1, \lambda_2, \rho) \) sequence of failure and repair times. Then assuming the asymptotic variance \( \tau^2 \) and \( \tau_*^2 \) are known,

\[
n_d = \min\{n : n \geq d^2 z_{\alpha/2}^2 \tau^2 \} \quad \text{and} \quad n_d^* = \min\{n : n \geq d^2 z_{\alpha/2}^2 \tau_*^2 \}.
\]

Consider the ratio,

\[
\frac{n_d}{n_d^*} = \frac{\tau^2}{\tau_*^2} = \frac{\pi_1 + \pi_2 - \pi_1 \pi_2}{\pi_1 \pi_2} \cdot \frac{1}{1 - \rho_1} + \frac{1}{1 - \rho_2} - 1.
\]
where \( \rho_1 = 1 - \pi_1 \) and \( \rho_2 = 1 - \pi_2 \) are the marginal lag 1 autocorrelations of the sequences \( \{X_n\} \) and \( \{Y_n\} \) respectively. The following table gives the values of the ratio \( n_d / n_d^\ast \) for a few values of \( \rho_1 \) and \( \rho_2 \).

<table>
<thead>
<tr>
<th>( \rho_1 )</th>
<th>0.2</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>1.50</td>
<td>2.25</td>
<td>3.58</td>
<td>10.25</td>
</tr>
<tr>
<td>0.5</td>
<td>2.25</td>
<td>3.00</td>
<td>4.33</td>
<td>11.00</td>
</tr>
<tr>
<td>0.7</td>
<td>3.58</td>
<td>4.33</td>
<td>5.67</td>
<td>12.33</td>
</tr>
<tr>
<td>0.9</td>
<td>10.25</td>
<td>11.00</td>
<td>12.33</td>
<td>19.00</td>
</tr>
</tbody>
</table>

Note that the ratio \( n_d / n_d^\ast \) is always greater than unity and increases as the marginal autocorrelations \( \rho_1 \) and \( \rho_2 \) increase. This indicates that under the assumption of independence the sample size is significantly underestimated if the true process is BEAR(1). For example, even when the autocorrelation is small \( (\rho_1 = 0.2, \rho_2 = 0.2) \) the ratio \( n_d / n_d^\ast \) is approximately equal to 1.50, indicating underestimation of 50%. Thus, when the successive sequences of failure and repair times are dependent, the assumption of independence make erroneous conclusions.

### 7.5. Numerical Study

In order to compare the performance of the sequential decision rule defined by (7.6) in the case of bivariate stationary dependent sequence with that of i.i.d. sequence, a simulation study is performed in this section. A sequence of failure and repair times are generated by a BEAR(1) model having bivariate Marshall-Olkin distribution with parameters \( \lambda_1 = 0.06, \lambda_2 = 0.36 \) and \( \lambda_{12} = 0.14 \). So the bivariate random vector has \( BVE(0.2, 0.5, 0.25) \) distribution with mean \( (5, 2) \) and correlation coefficient \( \rho = 0.25 \). Here we assume that \( p_{00} = 0.14, p_{01} = 0.06, p_{10} = 0.36 \) and \( p_{11} = 0.44 \) so that the marginal autocorrelations of the failure and repair times are \( \rho_1 = 0.8 \) and \( \rho_2 = 0.5 \) respectively.
The 95% sequential confidence interval for \( R(x) \) for several values of ‘\( x \)’ and ‘\( d \)’ are constructed for the \( BEAR(1) \) process. We also construct such confidence intervals for \( R(x) \) by treating the above data are as generated by an i.i.d. \( BVE(0.2, 0.5, 0.25) \) distribution. We repeat this experiment 5000 times and then compute the empirical coverage probabilities in both cases. The results of the simulation study are summarized in Table 7.1, where \( n_d \) represents the actual sample size required to construct a sequential confidence interval for \( R(x) \) of width \( 2d \). The notations \( \bar{N}_d, \hat{R}_{\bar{N}_d}(x), CP \) and \( \bar{N}_d^*, \hat{R}_{\bar{N}_d^*}(x), CP^* \) denote the average sample size required, average value of the estimated \( R(x) \), empirical coverage probability for the sequential confidence interval in the case of \( BEAR(1) \) process and i.i.d. bivariate exponential model respectively. The initial sample size \( m \) is taken as 10 in the simulation study.

Table 7.1 Simulated coverage probabilities for limiting interval reliability

<table>
<thead>
<tr>
<th>( x )</th>
<th>( R(x) )</th>
<th>( d )</th>
<th>( n_d )</th>
<th>( BEAR(1) ) Case</th>
<th>( Bivariate ) i.i.d. Case</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \bar{N}<em>d ) ( \widehat{R}</em>{\bar{N}_d}(x) ) ( CP )</td>
<td>( \bar{N}<em>d^* ) ( \widehat{R}</em>{\bar{N}_d^<em>}(x) ) ( CP^</em> )</td>
<td></td>
</tr>
<tr>
<td>0.050</td>
<td>0.576</td>
<td>562.71 0.71333 0.9182</td>
<td>97.87 0.70895 0.6020</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.075</td>
<td>0.256</td>
<td>247.75 0.71241 0.9298</td>
<td>49.79 0.70132 0.6152</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.71429</td>
<td>0.100 0.144</td>
<td>137.92 0.70965 0.9384</td>
<td>30.70 0.68527 0.6274</td>
<td></td>
</tr>
<tr>
<td>0.125</td>
<td>0.93</td>
<td>89.44 0.71029 0.9402</td>
<td>20.30 0.68758 0.6706</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.150</td>
<td>0.64</td>
<td>63.59 0.70934 0.9496</td>
<td>15.12 0.67991 0.6827</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.050</td>
<td>0.866</td>
<td>843.05 0.64495 0.9187</td>
<td>139.83 0.63916 0.6118</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.075</td>
<td>0.385</td>
<td>379.28 0.64791 0.9221</td>
<td>61.15 0.63544 0.6191</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.64631</td>
<td>0.100 0.217</td>
<td>212.04 0.64349 0.9274</td>
<td>34.60 0.62721 0.6242</td>
<td></td>
</tr>
<tr>
<td>0.125</td>
<td>0.139</td>
<td>126.94 0.63794 0.9343</td>
<td>23.15 0.61183 0.6605</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.150</td>
<td>0.097</td>
<td>89.53 0.63317 0.9424</td>
<td>16.73 0.58345 0.6785</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.050</td>
<td>1.142</td>
<td>1095.61 0.58212 0.9103</td>
<td>155.08 0.57982 0.6194</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.075</td>
<td>0.508</td>
<td>485.37 0.58964 0.9186</td>
<td>69.81 0.56392 0.6256</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.58481</td>
<td>0.100 0.286</td>
<td>252.81 0.57565 0.9287</td>
<td>40.07 0.55152 0.6350</td>
<td></td>
</tr>
<tr>
<td>0.125</td>
<td>0.183</td>
<td>175.25 0.57921 0.9298</td>
<td>25.87 0.53395 0.6455</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.150</td>
<td>0.127</td>
<td>113.85 0.57539 0.9389</td>
<td>18.15 0.52437 0.6625</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 7.1 reveals that the coverage probabilities of $R(x)$ under the assumption of i.i.d. model are significantly smaller than those under the $BEAR(1)$ case. This also indicates that the ignorance of autocorrelations present in the sequence will significantly under estimates the sample size.

7.6 Conclusion

In this chapter we have discussed the sequential confidence interval estimation of the limiting interval availability when the failure and repair times of a system form a stationary strong mixing sequence of bivariate random vectors. It is shown that the confidence interval is asymptotically consistent and the proposed stopping rule is asymptotically efficient as the width of the interval approaches zero. The general theory is applied to a stationary $BEAR(1)$ sequence and the resulting stopping rule is compared with the stopping rule under the i.i.d. set-up. It is observed that when the true model is $BEAR(1)$, the assumption of an i.i.d. sequence underestimates the sample size and leads to poor coverage probability. A simulation study also confirmed the same result.

7.7 Plan for Future Work

In Chapter 2 and 3, we consider the nonparametric estimation of the average availability and the interval reliability under three different sampling schemes. The estimation was carried out by assuming that the sequences of failure and repair times are two independent sequences of i.i.d. random variables. However, this assumption need not hold good in many situations. The repair times may depend on the previous failure time due to the influence of the operating environment on the system. When the failure and repair times form a bivariate i.i.d. sequence, the estimation of the availability measures; point availability, average availability and interval reliability, is an interesting research problem which is to be addressed.

The availability behavior and the estimation of the limiting interval reliability when the sequences of failure and repair times are generated by
stationary dependent sequences of random variables were discussed in Chapter 5 and 6. When the system is working in a random environment, it is natural to observe dependence among successive sequences of failure times. The inference procedures for estimating various quantities in the survival analysis are discussed by several authors in this set-up. See, for example, Ying and Wei (1994), Cai and Roussas (1998), Cai (2001). However, the estimation of the availability measures; point availability, average availability and interval reliability, is not discussed in the literature when the sequences of failure and repair times are generated by some stationary mixing sequences of dependent random variables, except the case of limiting measures.

Throughout this thesis, we use the empirical distribution function and the Kaplan-Meier product limit estimator as a nonparametric estimator of the cumulative distribution function in the case of complete and censored observations respectively. These estimators can only give a step function as the estimates. There are several works available in the literature dealing with the estimation of smooth distribution functions using kernel type estimators. See, Reiss (1981) and Ghorai and Susarla (1990). The nonparametric estimators of the availability measures using smoothly estimated distribution functions may reduce the mean square errors of the estimators significantly. This can be considered as a future work in this direction.