CHAPTER 3

TOOTH GEOMETRY

3.1 GEOMETRIC PARAMETERS

Figure 3.1 shows a portion of the involute curve bounded by the outside where root diameters have been used as tooth profile. In a properly designed gear mesh the involute curve merges with the root fillet at a point below the final contact of the mating gear. This intersection of involute and root fillet, is called the form diameter. Figure 3.1 shows gear tooth parameters which are self explanatory.

Figure 3.1 Gear Tooth Parameters
3.1.1 Module of Gear (m)

It is defined as the ratio of the pitch diameter to the number of teeth of a gear. The value of the module is expressed in millimeters. The module is one of the major terms for determining geometric parameters of gear (Maitra 1996). Also it is equal to the ratio of the outside diameter to the number of teeth plus 2, (Colbourne 1987).

\[
m = \frac{2r_p}{Z} = \frac{2r_a}{Z + 2} \quad (3.1)
\]

in (Inch):

\[
m = 25.4 \times D_p \text{ (inch)}^{-1} \quad (3.2)
\]

where:

\[
D_p = \frac{Z}{2r_p} \text{ (inch)}^{-1} = \frac{m}{25.4}
\]

For example, if a gear has a pitch diameter of 2 inches and 40 teeth then it is said to be 20Dp, it is not usual to encounter odd Dp number after 10Dp. Gears below 10Dp will be in the main larger than the amateur would be to cut. This arrangement makes the setting out of gear train easy (Law 1990). Older text books quote it as being reciprocal of the Dp, but more recently it has become the ‘metric’ way of quoting the size of the teeth. Preferred module values for gears, measured in mms, are as follows: 1,1.25,1.5,2,2.5,3,4,5,7,8,10,12,17,20,25,32,40,50,etc; (Colbourne 1987, Norton 1998).

3.1.2 Circular Tooth Thickness (CTT)

The tooth thickness at any radius is defined as the arc length between opposite faces of a tooth, measured around the circumference of the circle at this radius (Maitra 1996). Knowing the tooth thickness at one radius, it can be calculated at any other. Thus, it is only necessary to specify the tooth
thickness at one particular radius, and for this purpose the standard pitch circle is generally chosen.

Referring to Equation (2.21) the pressure angle can be determined at any point on the involute curve in relation to the parameters of another known point. The tooth thickness at the pitch circle is given by:

$$\text{CTT} = \frac{\pi m}{2}$$

The top land thickness ($\text{CTT}_a$) at pressure angle $20^\circ$ is given by:

$$\text{CTT}_a = 2r_p \left[ \frac{\text{CTT}}{2r_a} + \text{inv}(20) - \text{inv}(\phi_a) \right]$$

$$\text{CTT}_a = 2r_a \left[ \frac{\pi m/2}{d_p} + \text{inv}(20) - \text{inv}(\phi_a) \right]$$

So, $\phi_a$ can be obtained from Equation (2.21), the tooth thickness at pitch circle of a positively corrected gear is not equal to ($\pi m/2$), but somewhat greater, depending upon the correction factor chosen (Chironis 1967).

$$\text{CTT} = \frac{\pi m}{2} + 2m C_r \tan \phi$$

Sometimes, for positively corrected gears, it may be necessary to calculate the diameter or radius at which the tip becomes pointed or “peaked”. Obviously, at this point ($\text{CTT}_a = 0$), substituting this value in
Equations (3.3) and (3.4), \((r_a)\) for peaked tooth or the pitch diameter \((d_p)\) can be calculated (Maitra 1996).

### 3.1.3 Base and Root Circles

If the module and the pressure angle remain the same, the base circle and the root circle will depend upon the number of teeth for any particular basic rack. It is wrong to presume that the root circles are the smallest circle in a gear. As an example for that taking a standard basic rack where \(\phi = 20^\circ\), basic circle diameter and root circle diameter are:

\[
d_b = d_p \cos \phi = mZ \cos \phi \quad (3.6)
\]

\[
d_r = d_p - 2(1.25m) = mZ - 2.5m = m(Z - 2.5) \quad (3.7)
\]

If \(d_b = d_r\), then:

\[
m Z \cos \phi = m (Z - 2.5)
\]

Or

\[
Z \cos \phi = Z - 2.5
\]

\[
Z (1 - \cos \phi) = 2.5
\]

\[
\therefore Z = \frac{2.5}{1 - \cos \phi} = \frac{2.5}{1 - 0.93969} \approx 41
\]

This being the borderline case, if the number of teeth exceeds 41; the root circle becomes greater than the base circle.
For example, if \( Z = 52, m = 5, \phi = 20^\circ \), then:

\[
d_b = m \, Z \cos \phi = 5 \, (52 \cos 20^\circ) = 244.32 \text{ mm}
\]

\[
d_f = m \, (Z - 2.5) = 5 \, (52 - 2.5) = 247.5 \text{ mm}
\]

Hence, in this case, root circle is greater than base circle. Therefore theoretically the involute has already started before the dedendum circle or root circle. In actual practice, fillets with suitable radii are provided at the roots of the teeth to nullify the detrimental affects of stress concentration and notch effect, irrespective of whether the base circle or the root circle is the bigger of the two.

This work will take all cases until those cases when the pressure angle does not remain the same. Like in case of \( \phi = 25^\circ \) and \( Z=25 \) teeth) where \( Z \) is constant and with different numbers of correction factor values \( (C_f) \), the \( d_f \) and \( d_b \) values are:

<table>
<thead>
<tr>
<th>Profile shift factor ( (C_f) )</th>
<th>Base Diameter ( (d_b) ) mm</th>
<th>Root Diameter ( (d_f) ) mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>112.5</td>
<td>113.289</td>
</tr>
<tr>
<td>0.25</td>
<td>115</td>
<td>113.289</td>
</tr>
<tr>
<td>0.5</td>
<td>117.5</td>
<td>113.289</td>
</tr>
</tbody>
</table>
This example shows that when Z and m are constant, the increase in pressure angle and correction factor (profile shift) yields a root circle larger than a base circle. Chapter 7 shows the tables and curves which explain this case (root circle bigger than base circle) by influence of pressure angle and profile shift factor.

### 3.1.4 Interference and Undercutting

If contact takes place the tips of the teeth in one gear will dig out material from the fillets of the other and smooth running of the gear pair is impossible. This phenomenon is known as interference and gear pair must always be designed so that it will not occur (Colbourne 1987). Any meshing outside of the involute portion will result in non-conjugate action and that portion of teeth profile which lies between the base circle and the root circle comprises the (trochoid) curve (Maitra 1996).

Undercutting results when the cutting occurs in the portion of tooth below the base circle, which weakens the tooth by removing material at its root. The maximum moment and maximum shear from the tooth loaded as a cantilever beam both occur in this region. Severe undercutting will cause early tooth failure. Interference and its attendant undercutting can be prevented simply by avoiding gears with too few teeth. The minimum number of full-depth teeth required to avoid interference on pinion running against a standard rack can be calculated from (Norton 1998):

\[
Z_{\text{min}} = \frac{2}{\sin^2 \phi} \quad (3.8)
\]
The other condition for no interference as in (Colbourne 1987):

\[ r_f \leq r_L - 0.025 \text{ m} \]  \hspace{1cm} (3.9)

\[ r_L = \sqrt{r_b^2 + (r_b \tan \phi - \frac{Addc}{\sin \phi})^2} \]

where \( r_L \) - the radius of limit circle.
\( r_f \) - the radius of root circle.

And for no undercutting the condition is:

\[ \frac{h - C_f}{\sin \phi} \leq r_b \tan \phi \]  \hspace{1cm} (3.10)

where, \( h \) - Addc \(-r_c\)
\( C_f \) - correction factor (profile shift) = offset of cutter
\( r_c \) - radius of cutter fillet (tip of cutter).

The interference with the matching tooth will create a dynamic load which will increase tooth stresses and system noise and vibration levels (Lynwander 1983).

3.1.5 **Pressure Angle** (\( \phi \))

The pressure angle \( \phi \) of a gear is defined as the gear profile angle at the standard pitch circle. The name (pressure angle) is used for several types of angles (Maitra 1996).
The operating pressure angle ($\phi_{op}$) of a gear pair only exists when the two gears are meshed and it is the angle between the line of the action and the common tangent to the two pitch circles, but for a pinion meshed with its basic rack, all pressure angles are equal in value (Colbourne 1987).

In corrected gears, the operating (working) pressure angle is quite different from the standard pressure angle, depending upon the correction factor involved and the mounting dimensions. Since the involute is generated from the base circle, the pressure angle at the starting point is zero (Maitra 1996). The value of pressure angle for the standard tooth as the basic rack is $20^\circ$, but the pressure angle can also have other values: $14^\circ\frac{1}{2}$, $15^\circ$, $20^\circ$, $22^\circ\frac{1}{4}$, $25^\circ$, $30^\circ$, etc (Norton 1998).

The standard values for the pressure angle $\phi$ of the basic rack are $14.5^\circ$, $20^\circ$, $25^\circ$ and $30^\circ$, with $20^\circ$ being by far the most commonly used. The corresponding systems of gears therefore have the same four standard values for the pressure angle $\phi$.

To illustrate the effect of the value of $\phi$ on the tooth shape, four gear profiles are shown in Chapter 7, each with 20 teeth, and the pressure angles are equal to the four standard values. The pressure angle of $14.5^\circ$ is no longer recommended for new designs, because the teeth are relatively weak, and the gears are subject to a problem known as Undercutting. The profiles which represent $30^\circ$ pressure angle showing wide base in the root portion of the tooth have the space between the tooth at root and base circles reduced, which will generate a stress concentration region.
3.1.6 Profile Shift Factor

Any gear whose tooth thickness (CTT) is not equal to \((0.5 \, P_c)\) is said to be cut with profile shift, Where \((P_c)\) is the standard circular pitch which is equal to \((2\pi r_p / Z)\). The correction factor given by:

\[
f_t = \frac{1}{2 \tan \phi} \left( \frac{CTT - \pi m}{m} \right) \tag{3.11}
\]

If the profile shift (correction factor) is too large, the teeth may become pointed. It is normal practice to design gear with a minimum tooth thickness at the tip circle of 0.25m (where \(m\) is the module) and this condition places an upper limit on the value of the profile shift (Colbourne 1987).

By use of Equation (3.8) a theoretical expression for the profile shift which is the minimum value a gear with a number of teeth \((Z)\) must have to avoid undercutting is given as:

\[
f_t = \frac{Z_{\text{min}} - Z}{Z_{\text{min}}} \tag{3.12}
\]

For \(\phi=20^\circ\), the theoretical value of \(Z_{\text{min}}\) is 17. Therefore, the following relation is most commonly used for the correction factor (Maitra 1996).

\[
f_t = \frac{14 - Z}{17} \tag{3.13}
\]

The AGMA defines addendum modification coefficients (the correction factors) \(C_{t1}, C_{t2}\) which always add to zero, being equal in magnitude and opposite in sign. The positive coefficient \((C_{t1})\) is applied to
increase the pinion addendum and the negative ($C_{12}$) decreases the gear addendum by the same mount. But the total tooth depth remains the same. The net effect is to shift the pitch circles away from the pinion’s base circles and eliminate that non involute (trochoid) portion of pinion tooth below the base circle. The standard coefficients are ($\pm 0.25$) and ($\pm 0.5$), with add/subtract (25%) or (50%) of the standard addendum respectively. The limit of this approach occurs when the pinion tooth becomes pointed (Norton 1998).

When the sum of profile (correction) factors of the two matching gears is not equal to zero, there is some relation between them as in (Nieman 1960). There are two types of corrected gears. All relations and information about them have been expressed at length in (Maitra 1996, Nieman 1960), and this work has used and recommended $S_o$-gears type.

### 3.1.7 Number of Teeth

The size of tooth depends on the number of teeth, and the module is the index of tooth size in SI. It is imperative to know the number of the teeth to know the module because the module is the ratio of the pitch diameter to the number of teeth. Therefore the number of teeth is the most required parameter in all equations and calculations of gear design. The number of teeth of gears is determined after plotting the speed charts and sketching the gearing diagram of any speed box (Mehta 2004). It is necessary to select the minimum number of teeth on the smallest gear (the pinion). The number of teeth on the smallest gear should be as small as possible because an increase in the size of this gear leads to an increase in the overall dimensions of gear and gear box. If the number of teeth of a gear is less than a certain value, then the teeth are undercut and weak.
To estimate the bending stress at the root of a loaded gear tooth, form factor and the stress concentration factor should be introduced, which are both defined as a function of the tooth thickness at the critical section and the distance between this section and the intersection of the line of action and the tooth centerline, when load acts at the highest point of single tooth contact. Both factors depend on many parameters of both mating gears and generating tool, but for any tool geometry, the critical section depends only on two: the number of teeth and the rack shift factor on the considered gear (Pedrero et al. 1999).

3.1.8 Rack Cutter Tip Radius

From the definition of the trochoid curve, the envelope of the path of a series of circles equal in size to the rounding of the rack cutter tip corner, and with their centers on the trochoidal path during the cutting operation the trochoid curve will obtained. Therefore the cutter tip radius is so important in forming of gear tooth profile and then on all other related parameters. It is well known that the most failure cases are happen in the root portion of the tooth, which exactly tracks the trochoid curve as a fillet of root tooth in this portion. So, the study of rack cutter tip radius about bending stress and stress concentration gives a good estimation in the root of gear tooth.

In order that the same typical basic rack can be used to define the tooth profiles for gears of any size, the dimensions of the basic rack might be expressed in terms of the module. The rack pitch is then equal to \(\pi m\), and the reference line is the line along which the tooth thickness and the space width that are each equal to \(\pi m/2\).

The essential difference between this typical basic rack and the one shown in Figure 2.3 is that, in this basic rack, the tooth profiles are rounded
near the tips of the teeth. For a gear, which is conjugate to the basic rack the shape of the involute part, each gear tooth is defined by the straight part of the basic rack tooth, while the shape of the gear tooth near its root is defined by the curved section at the tip of the basic rack tooth. This section of the gear tooth is known as the fillet, and it is shaped in a manner that blends smoothly into the root circle, in order to strengthen the tooth near its base.

3.2 STRESS CALCULATIONS

When a gear pair is designed, it is important to calculate the maximum tooth stresses, to ensure that the teeth will not be damaged during the operation of the gear pair. However, the shape of a gear tooth makes it impossible to calculate the stresses exactly, using the theory of elasticity. There are a number of theoretical methods such as the finite element method, by which the stresses can be found, but these methods require large computers and take considerable computing time.

There is a need for a simple theory, even if the results are only approximate. Approximate methods for calculating tooth stresses have, of course, existed for a long time, and descriptions of some of these methods have been made available in a large number of books and papers. Most of these approximate methods are based on beam theory (Lewis formula) for bending stress calculations and on Hertz theory for contact stress calculations. The stresses in a gear tooth depend primarily on the load and the tooth shape, but there are several other phenomena which must be taken into account. In this chapter we will describe only the calculation of the static stresses, since the subject of this chapter is tooth geometry, and it is the static stresses rather than the actual stresses, which are determined by the geometry.
There are two distinct types of stresses which are calculated in the design of a gear pair, because such types can cause damage to the teeth, and their eventual failure. First, there is the contact stress which occurs at the points where the meshing teeth are in contact. If the contact stress is too high, the tooth surface becomes pitted with small holes. This pitting may not be harmful, so long as the pits remain small, but if they become larger, the tooth surface is eventually destroyed.

The second type of stress which is often responsible for tooth damage is the tensile stress in the fillet, caused by a tooth load on the face of the tooth. If the tensile stress is too large, fatigue cracks will be formed in the fillet, and the tooth will eventually fracture. It is clear that both the contact stress and the fillet stress must always be calculated, and compared with values which the gear material can sustain without damage. This work describing two approximate methods for this purpose are as follows:

3.2.1 Single Tooth Contact Method

In the absence of friction, the tooth force $W$ is directed along the normal to the tooth profile, which is tangent to the base circle, as shown in Figure 3.2 if the torque applied to gear is $M_t$, the corresponding tooth force $W$ is found by taking moment about the gear axis.

$$M = W \times r_b$$ (3.14)

The load intensity ($W_{int}$) is defined as the tooth force per unit length of the contact line, the maximum load intensity occurs when there is only one tooth pair in contact, and is then equal to the tooth force divided by the gear face-width ($F$),
\[ W_{int} = \frac{W}{F} \]  

**3.2.1.1 Contact Stress (\( \sigma_c \))**

It occurs at the point where the meshing teeth are in contact. The contact stress at the points where the teeth touch each other is found by means of the Hertz contact stress theory, which is described in most books of elasticity. Since the contact takes place along a line, the teeth are represented in the vicinity of the contact line by two circular cylinders. The radii of these cylinders are equal to the radii of curvature \( L_1 \) and \( L_2 \) of the tooth profiles at the contact point. The maximum contact stress (\( \sigma_c \)) is given by (Colbourne 1987):

\[ \sigma_c = K_c - Z_E \sqrt{\frac{W_{int}}{m}} \]  

where \( Z_E \) is an elastic coefficient, defined as follows,

\[ \frac{1}{Z_E} = \sqrt{\frac{\pi(1 - \nu_1)^2}{E_1} + \frac{\pi(1 - \nu_2)^2}{E_2}} \]  

where \( E_1, \nu_1 \) and \( E_2, \nu_2 \) are Young’s modulus and Poisson’s ratio for the material of each gear, also \( Z_E \) can found from Table A1.1 in the Appendix 1 for different materials of pinion and gear.
$K_c$ is the geometry factor which given by:

$$K_c = \sqrt{\frac{C m \sin (\phi_{op})}{L_1 L_2}} \quad (3.18)$$

where $L_1$, $L_2$ are the radii of curvature of the tooth profiles at the contact point, and found from:

$$L_1 = \sqrt{(r_{a1})^2 - (r_{b1})^2} - P_b \quad (3.19)$$

$$L_2 = (r_{b1} + r_{b2}) \tan (\phi_{op}) - L_1 \quad (3.20)$$

where $L_1 + L_2 = C \sin (\phi_{op})$

$$P_b = \frac{2r_b \pi}{Z} = \pi m \cos \phi \quad (3.21)$$

and $\phi_{op} = \cos^{-1} \frac{r_{b1} + r_{b2}}{C} \quad (3.22)$

### 3.2.1.2 Bending Stress ($\sigma_b$)

If the tensile stress caused by a tooth load on the face in the fillet is too large, fatigue cracks will be formed in the fillet that will eventually lead to tooth fracture. A loaded gear tooth is shown in Figure 3.2, with the contact force applied at typical point $A_W$. The actual tooth force is $W$, but for the stress analysis it is convenient to consider a tooth of uniform-width, so applying force equal to the load intensity $W_{int}$, given by Equation (3.15).
The force $W$ at $A_w$ is considered to act at $D$, and it is then resolved into two components. The first component causes stresses in the tooth like the bending stresses in a cantilevered beam, in particular, there are tensile and compressive stresses at point $A$ and $A'$, the second component causes a radial compressive stress throughout the tooth, like the axial stress in a beam, and smaller in magnitude that the bending stresses at $A$ and $A'$ (Colbourne 1987).

The maximum fillet stress occurs when the load is applied at a radius ($r_w$) which is equal to radius of (HPSTC) and given by:

$$r_{wl}^2 = r_{b1}^2 + [(r_{b1} - r_{b2}) \tan \phi - \sqrt{r_{a2}^2 - r_{b2}^2 + P_b}]^2$$  \hspace{1cm} (3.23)
where \( P_b \) is the base pitch and \( C_R \) is the contact ratio which will be dealt with at length in chapter 5, for standard gear teeth expressed by (Lynwander 1983):

\[
C_R = \sqrt{\frac{\sin^2(\phi) + \frac{4}{Z_i} + \frac{4}{Z_i^2} + \sqrt{\frac{4i}{Z_i} + \frac{4}{Z_i^2}}}{-(1+i)\sin(\phi)}} \frac{2\pi \cos(\phi)}{}
\]

(3.24)

where \( i = \frac{Z_2}{Z_1} \)

Other parameters shown in Figure 3.15 may be obtained from:

\[
\cos(\phi_w) = \frac{r_{b1}}{r_{w1}}
\]

(3.25)

\[
\theta_w = \frac{CTT}{2r_p} + \sin(\phi - \sin(\phi_w)}
\]

(3.26)

\[
X_w = r_w \sin(\theta_w)
\]

(3.27)

So, x and y coordinates of HPSTC is given by:

\[
Y_w = r_w \cos(\theta_w)
\]

(3.28)

\[
\gamma_w = \phi_w - \theta_w
\]

(3.29)

The y-coordinate of point D, where the normal at \( A_w \) intersects the tooth center-line, can be found from:

\[
Y_D = Y_w - Y_w \tan(\gamma_w)
\]

(3.30)
The stress concentration factor $K_f$ is based on a formula deduced from photoelastic investigation of stress concentration in gear teeth over 50 years ago. For the fillet stress $K_f$ is given by the following expression (Colbourne 1987):

$$K_f = K_1 + \left( \frac{2(xx)}{r_i} \right)^{K_2} \cdot \left( \frac{2(xx)}{Y_D - (yy)} \right)^{K_3}$$  \hspace{1cm} (3.31)

where $K_1$, $K_2$ and $K_3$ coefficients can be calculated from the following three equations as a functions of the pressure angle $\phi$:

$$K_1 = 0.3054 - 0.00489\phi^o - 0.000069(\phi^o)^2$$  \hspace{1cm} (3.32)

$$K_2 = 0.362 - 0.01268\phi^o - 0.000104(\phi^o)^2$$  \hspace{1cm} (3.33)

$$K_3 = 0.2934 - 0.00609\phi^o - 0.000087(\phi^o)^2$$  \hspace{1cm} (3.34)

$r_i$ is the fillet radius of curvature which is given by the following equation:

$$r_i = r_c + \frac{(Add_c - mC_f - r_c)^2}{r_p + (Add_c - mC_f - r_c)}$$  \hspace{1cm} (3.35)

where, $r_c$ and $Add_c$ can be easily found from Equations (2.13) and (2.14) or taken as (0.3m) and (1.25m) respectively, $m$ is the module.

Now, the maximum fillet stress value can be expressed as:

$$\sigma_b = \frac{w_{inl}}{m}\cos(\gamma_w) \left[ K_f \left( \frac{1.5m(Y_D - yy)}{(xx)^2} - \frac{0.5m\tan \gamma_w}{(xx)} \right) \right]$$  \hspace{1cm} (3.36)

where $xx$ and $yy$ is the x and y coordinates of the intersect point of the critical section and the fillet curve (trochoid curve) which represented by point $A^\prime$ or
A at Figure 3.2. Programme 3 is developed based on the formulation of single tooth contact method to calculate bending and contact stress theoretically, Figure 3.3 shows the flow chart of this programme.

The simplest method for finding the critical section is to calculate \(\sigma_b\) at a number of points (xx, yy) along the fillet, and choose the largest value (Colbourne 1987).

![Flowchart of Gear Tooth Stresses Calculating Programme](Programme 3)
3.2.2 AGMA Method

The American Gear Manufacturers Association (AGMA) has for many years been the authority responsible for the dissemination of knowledge pertaining to the design and analysis of gearing. The methods this organization present are in general use in the United States where strength and wear are primary considerations. In view of this fact it is important that the AGMA approach to the subject be presented here. The general AGMA approach requires a great many charts and graphs (too many for a single chapter in this work).

3.2.2.1 AGMA Stress Equations

Two fundamental stress equations are used in AGMA methodology, one for bending stress and another for pitting resistance (contact stress). AGMA equation for contact stress is:

$$\sigma_c = Z_E \sqrt{W_t K_o K_v K_s K_H Z_R} / (2r_p FZ_t)$$  \hspace{1cm} (3.37)

and AGMA equation for bending stress is:

$$\sigma_b = W_t K_o K_v K_s K_R K_B / (m F Y_j)$$ \hspace{1cm} (3.38)

where

Transverse Velocity ($V$) = Angular velocity ($\omega$) $\times$ Pith radius ($r_p$)

$$V = \frac{r_p \pi n_1}{30}$$ \hspace{1cm} (3.39)
and

\[ W_t = \frac{30P}{r_p \pi n_1} \] (3.40)

- **K_0** is the **Overload factor**. It is intended to make an allowance for all externally applied loads in excess of the nominal tangential load \( W_t \) in a particular application. Examples include variations in torque from the mean value due to firing of cylinders in an internal combustion engine or reaction to torque variations in a position pump drive. Others call a similar factor an application factor or a service factor. These are established after considerable field experience in a particular application (it mostly taken as 1).

- **K_v** is the **Dynamic factor**. AGMA has defined a set of quality-control numbers. These numbers define the tolerance for gears of various size manufactured to a specified quality class. Classes 3 to 7 will include most commercial quality gears, classes 8 to 12 are of precision quality. The AGMA transmission accuracy level number \( Q_v \) can be taken as the same as quality number which can be obtained from Table A1.2 in the Appendix 1.

The following equations for the dynamic factor are based on these \( Q_v \) numbers:

\[ K_v = \left( \frac{A + \sqrt{200V}}{A} \right)^B \] (3.41)
where, \( V \) is the velocity in m/s.

And:

\[
A = 50 + 56(1-B)^{2/3} \\
B = 0.25 (12 - Q_v)^{2/3}
\]

An alternative way is to find \( K_v \) from Figure A1.1 in the Appendix 1 which is a graph of \( K_v \), the dynamic factor, as a function of pitch-line speed for graphical estimate of \( K_v \).

- \( K_s \) are the **Size factor.** It is reflect non-uniformity of material properties due to size. \( K_s \) can be found by:

\[
K_s = 0.904 \left(mF\sqrt{Y}\right)^{0.0535}
\]  
(3.42)

where \( Y \) is Lewis form factor which can be obtained from Table A1.3 in the Appendix 1 (PSG 2005).

- \( K_H \) is the **Load-Distribution factor.** It is modified the stress equations to reflect non-uniform distribution of load across the line of contact the ideal is to locate the gear “mid span” between two bearings at the zero slope place when the load is applied. However this is not always possible.

The Load-Distribution factor \( K_H \) is given by:

\[
K_H = 1 + C_{mc} \left(C_pf C_{pm} + C_{ma} C_e \right)
\]  
(3.43)

where, \( C_{mc} \) is 1 for uncrowned teeth and 0.8 for the crowned teeth.
\[
C_{pf} = \frac{F}{20r_p} - 0.025 \quad \text{F} \leq 25 \text{ mm}
\]
\[
C_{pf} = \frac{F}{20r_p} - 0.0375 + 4.92(10^{-4})F \quad 25 < F \leq 425 \text{ mm}
\]
\[
C_{pf} = \frac{F}{20r_p} - 0.1109 + 8.15(10^{-4})F - 3.35(10^{-7})F^2 \quad 425 < F \leq 1000 \text{ mm}
\]

\(C_{pm}\) is taken as 1 for straddle-mounted pinion with \(S_1/S\) is <0.175 and 1.1 for straddle-mounted pinion with \(S_1/S\) is \(\geq 0.175\). In the Appendix 1 Figure A1.2 used to define of \(S\) and \(S_1\).

\[
C_{ma} = A + BF + CF^2
\]

where, \(A, B\) and \(C\) are obtained from Table A1.4 in the Appendix 1.

\(C_e\) is taken as 0.8 for gearing adjusted at assembly, or compatibility is improved by lapping, or both, and 1 for all other conditions.

- \(Z_R\) is the Surface Condition factor. It is used only in the pitting resistance equation. Standard surface conditions for gear teeth have not yet been established, so AGMA has suggested a value of \(Z_R\); it is 1 or greater than unity.

- \(Z_E\) is the Elastic coefficient. This coefficient has been discussed previously in this chapter; however it can be taken from Table A1.1 in the Appendix 1.

- \(F\) is the Face width of the narrowest member.

- \(K_B\) is the Rim-thickness factor. When the rim thickness is not sufficient to provide full support for the tooth root, the
location of the bending fatigue failure may be through the gear rim rather than at the tooth fillet. In such cases, the use of stress-modifying factor $K_B$ is recommended.

This factor, adjusts the estimated bending stress for the thin-rimmed gear. It is a function of the backup ratio $m_B$, which is expressed at length in Chapter 4. The backup ratio $m_B$ and the rim-thickness factor $K_B$ can be obtained from Figures A1.3 and A1.4 in the Appendix 1.

- $Z_I$ is the **Surface-strength geometry factor**. It is also called the pitting-resistance geometry factor by AGMA, which is given for spur gear by the following equations:

$$Z_I = \frac{\cos \phi \sin \phi i}{2 i \pm 1}$$

where; the sign ($\pm$) is denoted to the external gear and internal gear respectively.

- $Y_J$ is the **Bending-strength geometry factor**. The AGMA factor $Y_J$ employs modified value of the Lewis form factor; also it is denoted by J in many other references. $Y_J$ factor can be obtained from Figure A1.5 in the Appendix 1.

The ability of a particular gear to resist bending stress is called allowable bending strength and “is a function of the hardness and residual stress near the surface of the root fillet and at the core. To determine failure, allowable bending stress is derated by factors such as dynamic loading, over loading, and reliability. This number is then compared to the bending stress; allowable compressive strength measures the tooth surface’s resistance to pitting. To increase compressive strength, aerospace gears are usually strengthened through carburized, case hardening. To determine failure,
allowable compressive strength is derated by factors such as surface condition, hardness, and dynamic factors. This value is then compared to the contact stress (Bellocchio 2005). Programme 4 is developed based on the formulation of AGMA method to calculate bending and contact stress theoretically. Figure 3.4 shows the flow chart of this programme.

![Flowchart of AGMA Gear Stresses Calculation Programme (Programme 4)](image-url)
For checking the resulting values of bending stress and contact stress from above method, it is preferable to compare these results with the allowable bending stress and allowable contact stress in Tables A1.5 and A1.6 respectively in Appendix 1.

Table A1.7 in this Appendix 1 gives the specified chemical composition and related mechanical properties of the carbon steel materials and including the yield stress and tensile strength for each carbon steel which will give a good idea whether the obtained stress results are outside or inside the reasonable stress ranges.

3.3 **FINITE ELEMENT THEORY AND APPLICATION ON SPUR GEAR TOOTH**

Finite Element Method (FEM) is presented in this work for stress analysis of a three teeth sector spur gear at high speed. Many FEM softwares have been developed over the years. It is verified that the developed FEM software can calculate correct stresses of the spur gear model. This work uses ANAYS software to find the stresses in spur gear models, because it is one of the best and most reliable finite element softwares which can be used for this purpose. The theoretical finite element formulation which is applied on gear models in this work is expressed as follows:

The strain or displacement relationship for most elastic problems may be written in the form.

\[
\{ \varepsilon \} = [L]\{ \delta \}  \tag{3.44}
\]
where \( \{ \varepsilon \} \) - strain vector
\( \{ \delta \} \) - Displacement vector.
\( \{ L \} \) - Matrix of displacement differential operators.
\[
\{ \delta \} = [N] \{ \delta^e \} \tag{3.45}
\]
where \( \{ \delta^e \} \) - Displacement vector for element.
\( \{ N \} \) - \([N_1 \ldots N_8]\) Matrix of shape function.

Substituting Equation (3.45) into Equation (3.44) yields:
\[
\{ \varepsilon \} = [L] \{ N \} \{ \delta^e \} = [B] \{ \delta^e \} \tag{3.46}
\]
where, \([B] = Element strain matrix\)

For the plane elasticity problems the 8-node quadratic quadrilateral element is chosen for the finite element analysis which is shown in Figure 3.5, this element belongs to the serendipity family of elements, so the shape functions \( N_i \) (i=1, 2… 8) can be expressed in the natural coordinates \((\xi, \eta)\) as (Chandrupatla and Belegundu 2002):

\[
\begin{align*}
N_1 (\xi, \eta) &= -(1-\xi) (1-\eta) (1+\xi+\eta)/4 \\
N_2 (\xi, \eta) &= -(1+\xi) (1-\eta) (1-\xi+\eta)/4 \\
N_3 (\xi, \eta) &= -(1+\xi) (1+\eta) (1-\xi-\eta)/4 \\
N_4 (\xi, \eta) &= -(1-\xi) (1+\eta) (1-\xi-\eta)/4 \\
N_5 (\xi, \eta) &= (1-\xi^2) (1-\eta)/2 \\
N_6 (\xi, \eta) &= (1+\xi) (1-\eta^2)/2 \\
N_7 (\xi, \eta) &= (1-\xi^2) (1+\eta)/2 \\
N_8 (\xi, \eta) &= (1- \xi) (1-\eta^2)/2
\end{align*} \tag{3.47}
\]
Figure 3.5 Eight-Node Quadrilateral Element (a) in \( x, y \) Space and (b) in \( \xi, \eta \) Space

For the plane stress / strain situation the strain/ displacement relationship may be written as:

\[
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix} \begin{bmatrix}
u \\
v
\end{bmatrix}
\]

(3.48)

But the strain matrix \([B]\) is:

\[
[B_i] = \begin{bmatrix}
\frac{\partial N_i}{\partial x} & 0 \\
0 & \frac{\partial N_i}{\partial y} \\
\frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x}
\end{bmatrix}, \quad i = 1, 2, \ldots, 8
\]

(3.49)
So,

\[
\{\varepsilon\} = \sum_{i=1}^{8} \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \sum_{i=1}^{8} [B_i] [\delta i] \tag{3.50}
\]

The stress/strain relationship for an elastic material may be written in the form:

\[
\{\sigma\} = [D] \{\varepsilon\} \tag{3.51}
\]

where, \([D]\) is the matrix of elastic constant. For plane stress situations and assuming isotropic materials (Zahavi 1991):

\[
[D] = \frac{E}{1-\nu} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{3.52}
\]

For plane strain situation (isotropic case) \([D]\) is:

\[
[D] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \tag{3.53}
\]

where, \(E\) and \(\nu\) are the elastic modulus and Poisson’s ratio respectively.
Since the strain/displacement relationship in the finite element approximation may be written as:

\[ \{ \varepsilon \} = [B_1, \ldots, B_8] \{ \delta^e \} \]

\[ \{ \sigma \} = [D] [B] [\delta^e] \]

Thus for a plane stress situation:

\[
\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \sum_{i=1}^{8} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ 0 \\ \frac{\partial N_i}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ 0 \\ \frac{\partial N_i}{\partial y} \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix}
\]

(3.54)

Therefore the element stiffness matrix is:

\[
\left[ K^e \right] = \iint [B]^T [D][B] dv
\]

In fact, a typical sub matrix of \([k^e]\) linking nodes i and j may be evaluated from the expression:

\[
K_{ij}^e = \iint [B]^T [D][B] |J| d\xi d\eta
\]

where, \((t)\) is the element thickness.

By using Jacobian matrix and its inverse the element stiffness matrix \([k^e]\) can be written as (Seshu 2003):
\[
\begin{bmatrix} K^e \end{bmatrix} = \int \int [B]^T \left[D\right][B] |J| d\xi \ d\eta
\]  
\quad (3.55)

where, \( [J] \) is the Jacobian matrix and \( \det J \) is the determinant of \( [J] \). The Jacobian matrix \( [J] \) can be written as:

\[
[J] = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
\end{bmatrix} = \sum_{i=1}^{s} \begin{bmatrix}
\frac{\partial Ni_x}{\partial \xi} & \frac{\partial Ni_y}{\partial \xi} \\
\frac{\partial Ni_x}{\partial \eta} & \frac{\partial Ni_y}{\partial \eta}
\end{bmatrix}
\quad (3.56)
\]

The determinate of \( [J] \) is:

\[
\det J = \frac{\partial x \partial y}{\partial \xi \partial \eta} - \frac{\partial x \partial y}{\partial \eta \partial \xi}
\]

The inverse matrix of \( [J] \) is:

\[
[J]^{-1} = \begin{bmatrix}
\frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\
\frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y}
\end{bmatrix} = \frac{1}{\det J} \begin{bmatrix}
\frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\
-\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi}
\end{bmatrix}
\quad (3.57)
\]

### 3.4 THE EFFECTS

To understand the effects of the geometric parameters on the bending stress in spur gear, four most important parameters are selected. For each selected geometric parameter, cases with suitable boundary conditions are proposed. For each case a three teeth sector of spur gear has been developed by using both the technique and programme developed in Chapter 2. It is necessary to know the theoretical stresses, which have been calculated
by using both the theoretical methods which are expressed previously in this chapter. A finite element model has been built for each case and stress analysis has been done by using of ANSYS software. The theoretical methods stress results and ANSYS stress results for each case have been shown in Chapter 7.

3.4.1 Effect of Pressure Angle on Bending Stress in Spur Gear

This analysis was executed for 5 mm module, 20 teeth and zero profile shift factor and varying the pressure angle values (14.5°, 20°, 25° and 30°). The programme 1 outputs for the spur gear tooth profiles for each pressure angle are shown in Chapter 7. Other important geometrical parameters such as root radius and base radius are also shown in these outputs. It is clear to notice from the outputs that the values of root radius decrease when the pressure angle values increase, which is shown by the numerical values and curve in Chapter 7.

Maitra (1996) has considered that the mutual position of the base circle and the root circle will depend upon the number of teeth for any particular basic rack and it is wrong to presume that the root circle is the smallest circle in a gear, and he proved that when the number of teeth exceeds 41, the root circle becomes greater than the basic circle (at 20° pressure angle). The increasing of pressure angle value will cause root circle bigger than base circle too, which is also an another fact found from this work.

This work tries to simulate the actual situation when a pair of spur gears (pinion and gear) is engaging. Gear ratio is 2 and pinion rotate with 1800rpm speed and the transmitted power is 10 kW. The torque applied to pinion causing the contact between the gears teeth, is 53.053 N.m. The normal load which is applied on the HPSTC point is 1061N. Back up ratio of 2.5 and
pinion face width of 25 mm are used. The material is assumed to be isotropic and homogeneous; the modules of Elasticity and Poisson’s ratio are 215000 N/mm² and 0.3 respectively.

For each selected pressure angle a model consisting of three tooth section of pinion was developed for stress analysis using finite element software ANSYS.

Figure 3.6 shows the finite element grid for the three tooth gear segment, a fine mesh was used in the root and fillet areas of the central tooth in each case as it can be seen. In order to obtain a curved edge element and higher order transformation, an eight node iso-parametric plane-stress quadrilateral quadratic element was used to build the finite element model inside the frame work which was obtained by using the spur gear tooth profile programme of Chapter 2 (Programme 1).

![Finite element model at \( \phi =20^\circ \)](image)
Full load was applied at the highest point of single tooth contact (HPSTC) on the central tooth in the direction of the line of action. To apply the load at a node, the grid had to have a node point at or near this loading point. The calculated values and curve relation between the contact ratio ($C_R$) and pressure angle ($\phi$) and the calculated values and curve relation of $\gamma_w$ and $r_w$ with pressure angles ($\phi$) at same conditions are shown in Chapter 7.

The outputs of Programme 3 and Programme 4 which calculate the theoretical contact and bending stresses for the case of pressure angle 20°, depending on the single tooth contact method and AGMA method are shown in Chapter 7, these thoratical results will used to compare with finite element stress analysis results which is also presented by contours, tables and curves in Chapter 7.

3.4.2 Effect of Profile Shift Factor on Bending Stress in Spur Gear:

This analysis was executed for 1mm module, 20 teeth, 20° pressure angle and zero rack cutter tip radius and varying the profile shift factor values (-0.2, 0, 0.2, 0.4 and 0.7). The outputs of Programme 1 for the spur gear tooth profiles for profile shift factor values (-0.2, 0.2, 0.4 and 0.6) are shown in Chapter 7.

The other important geometric parameters such as root radius and base radius are also shown in these outputs. It is clear to notice from the outputs that the values of root radius decrease when the profile shift factor values increase, which is shown by the table and curve in Chapter 7. The increasing of profile shift factor value will cause root circle to be bigger than base circle too, which is another fact found from this work.
This work tries to simulate the actual situation when a pair of spur gears (pinion and gear) is engaging. Gear ratio is 3 and pinion rotates with 1440 rpm speed and the transmitted power is 150 W. A clockwise torque is applied to the driving gear, causing the contact between the gear teeth, the torque applied to pinion is 0.995 N-m. Back up ratio of 2.5 and pinion face width of 7mm are used.

The material is assumed to be isotropic and homogeneous; the modules of Elasticity are 215000 N/mm² and Poisson’s ratio is 0.3.

For each selected profile shift factor a model consisting of three tooth section of pinion was developed for stress analysis purpose by using finite element software ANSYS. Figure 3.7 shows the finite element grid for the three tooth gear segment, a fine mesh was used in the root and fillet areas of the central tooth in each case as it can be seen.

![Finite Element Model at C_r = 0.7](image)

**Figure 3.7 Finite Element Model at C_r = 0.7**
In order to obtain a curved edge element and higher order transformation, an eight node iso-parametric plane-stress quadrilateral quadratic element was used to build the finite element model inside the frame works which was obtained from spur gear tooth profile programme previously in this work. Full load was applied at the highest point of single tooth contact (HPSTC) on the central tooth in the direction of the line of action. To apply the load at a node, the grid had to have a node point at or near this loading point.

The stress analysis has been performed for the mechanism of external gear. The results of $r_w$, $\gamma_w$ and maximum bending stress for each profile shift factor, the relations between the profile shift factor and $C_R$, CTT, $r_w$ and $\gamma_w$, the contours of maximum bending von Mises stress of the spur teeth for cases of profile shift values (-0.2 and 0.6) and the curve relation of profile shift factor values with maximum von Mises bending stress are all shown in Chapter 7.

3.4.3 Effect of Number of teeth on bending stress in spur gear

This analysis was executed for 1mm module, $20^\circ$ pressure angle zero profile shift factor and zero rack cutter tip radius and varying the number of teeth values (17, 20, 30, 50, 100 and 300). It is clear to notice from the calculations that the values of root radius decrease when the profile shift factor values increase, which is shown by both numerical values and curve in Chapter 7. The increasing of number of teeth value will cause a bigger root circle than base circle, which also an anther fact found from this work.

This work tries to simulate the actual situation when a pair of spur gears (pinion and gear) is engaging. Gear ratio is 3 and pinion rotates with 1440rpm speed and the transmitted power is 150 W. A clockwise torque is applied to the driving gear, causing the contact between the gear teeth, the
torque applied to pinion is 0.995 N-m. Back up ratio of 2.5 and pinion face width of 7 mm are used. The material is assumed to be isotropic and homogeneous; the modules of Elasticity is 215000 N/mm$^2$ and Poisson’s ratio is 0.3.

For each selected number of teeth a model consisting of three tooth section of pinion was developed for stress analysis purpose by using finite element software ANSYS.

Figure 3.8 shows the finite element grid for the three tooth gear segment, a fine mesh being used in the root and fillet areas of the central tooth in each case as it can be seen. In order to obtain a curved edge element and a higher order transformation, an eight node iso-parametric plane-stress quadrilateral quadratic element was used to build the finite element model inside the frame work which was obtained from spur gear tooth profile programme previously in this work.

![Finite Element Model at Z= 25 Teeth](image)

**Figure 3.8  Finite Element Model at Z= 25 Teeth**
Full load was applied at the highest point of single tooth contact (HPSTC) on the central tooth in the direction of the line of action. To apply the load at a node, the grid had to have a node point at or near this loading point. The stress analysis has been performed for the mechanism of external gear.

The calculated values of $r_w$ and $\gamma_w$ and maximum von Mises bending stress is obtained by finite element analysis software (ANSYS) for selected cases of number of teeth (17, 20, 25, 30, 50, 100 and 300) at 1mm module, 20° pressure angle and zero profile shift factor are shown in Chapter 7.

The curves which represent the relation between $r_w$ and $\gamma_w$ with the number of teeth, the von Mises bending stress contours of cases of 17 teeth and 300 teeth and the curve of the maximum von Mises bending stress results with the selected values of number of teeth are also presented in Chapter 7.

### 3.4.4 Effect of Rack Cutter Tip Radius on Bending Stress in Spur Gear

This analysis was executed for 1mm module, 20 teeth, 20° pressure angle and zero profile shift factor and varying the rack cutter tip radius values (0, 0.1, 0.2, 0.3 and 0.4) mm.

This work tries to simulate the actual situation when a pair of spur gears (pinion and gear) is engaging. Gear ratio is 3 and the pinion rotates with 1440rpm speed and the transmitted power is 150W. A clockwise torque is applied to the driving gear, causing the contact between the gear teeth, the torque applied to pinion is 0.995N-m. Back up ratio of 2.5 and pinion face width of 7mm are used. The material is assumed to be isotropic and
homogeneous; the modules of Elasticity are 215000 N/mm² and Poisson’s ratio is 0.3.

For each selected rack cutter tip radius a model consisting of three tooth section of pinion was developed for stress analysis purpose by using finite element software ANSYS.

Figure 3.9 shows the finite element grid for the three tooth gear segment, a fine mesh was used in the root and fillet areas of the central tooth in each case as it can be seen.

![Finite Element Grid](image)

**Figure 3.9** Finite Element Model at Cutter Tip Radius of 0.4mm

In order to obtain curved edge element and higher order transformation, an eight node iso-parametric plane-stress quadrilateral quadratic element was used to build the finite element model inside the frame.
work which was obtained from spur gear tooth profile programme previously in this work. Full load was applied at the highest point of single tooth contact (HPSTC) on the central tooth in the direction of the line of action. To apply the load at a node, the grid had to have a node point at or near this loading point. The stress analysis has been performed for the mechanism of external gear.

The maximum von Mises bending stress results which was obtained from finite element analysis by ANSYS software for the selected cases of rack cutter tip radius values (0, 0.1, 0.2, 0.3 and 0.4) at 1mm module, 20 teeth, 20°pressure angle and zero profile shift factor are presented in Chapter 7.

The curve of relation between these stress results and the selected crack cutter tip radius and the von Mises stress contours for two of selected cases of rack cutter tip radius values (0 mm and 0.4mm) are all shown in Chapter 7, too.