CHAPTER 6

DYNAMIC ANALYSIS

Rotating members like shafts, pulleys, gears, etc are subjected to dynamic loads. The dynamic load creates bending stresses at the tooth root which can lead to fatigue failure. One of the major concerns in the design of power transmission gears is the reduction of dynamic load. Research on gear noise and vibration has revealed that the basic mechanism of noise generated from gears is due to vibration excited by the dynamic load. The life and reliability of a gear transmission is reduced by high dynamic load. Minimizing gear dynamic load will decrease gear noise, increase efficiency, improve pitting fatigue life, and help prevent gear tooth fracture. One major cause of gear failure is fracture at the base of the gear tooth due to bending stress. Design models for this mode of failure use a parabolic beam with stress concentration correction (ANSI/AGMA 1988).

The classical approach to the problem of spur gear tooth stress distribution makes the use of cantilever beam theory, with the addition of empirical stress concentration factors related to radius of curvature of tooth fillet (Ramamurti and Reddy 2001). The bending strength is influenced by the gear size, described by the diametral pitch; the shape of the tooth, described by the number of teeth on the gear; the highest location of the full load, described by the number of teeth on the mating gear; and the fillet geometry of the gear tooth. The present AGMA design model treats these factors directly and by extrapolating limited experimental data for the stress concentration correction.
Here, three tooth segments of a 20 tooth and 30 tooth pinions in mesh with a 40 tooth and 60 tooth gears respectively are studied. A rack tip generated trochoid fillet is at the base of the involute to accurately describe the structural geometry of the tooth.

6.1 GEAR TOOTH GEOMETRY

The selected models pinions analyzed have pitch diameters of 240mm and 750mm, number of teeth 20 and 30 respectively and a nominal pressure angle of 20 degree. The pinions and mating of 40 and 60 tooth gears have standard full depth teeth with addendum of 1.0 m and dedendum of 1.25 m. The rack form cutter tip has a sharp corner and the face widths of the gears are 60mm and 125 respectively. A 6425N load acts between the two gears along the line of action for the first model and a 28000N load acts between the two gears along the line of action for the second model. Development of the finite element models begins with data describing the outline of a single tooth and its fillets from the center of the tooth space on one side to the center of the tooth space on the other side. Several different curves make up the tooth outline: concentric circular arc at the tooth tip defining the addendum circle, involutes on the two sides of the tooth, and trochoides between the involutes and the bottom lands at the base of the tooth. The tooth side involutes, fillet trochoides, and bottom lands are shaped to model a gear cut with a rack form cutter. Coordinates for the surface profile of the tooth come from a kinematic analysis of the cutting process (Hefeng et al. 1985). Both the rack form cutter and the resulting gear surface are tangent to each other at the cutting points, which generate the gear shape from the rack shape. The involute is generated by points on the side of the rack form, the gear tooth fillet is generated by the tip of the rack form, and the bottom land is generated by the top surface of the rack form tooth. With the appropriate rotations, this slope and radius locates the direction and point of application of
the gear mesh force on the central teeth in the three tooth segments models. In this work the pressure angle (φ) is 20 degree, gear ratio (i) is 2, modulus of elasticity (E) is $2.15 \times 10^5$ N/mm$^2$, poisson’s ratio (ν) is 0.3, steel mass density (ρ) is $8.75 \times 10^{-9}$ N.s$^2$/mm$^4$ and the rotating speed (n) is 1440 rpm the in both of the two models. Table 6.1 shows other important parameters which are not equal in these two models of the present work in this chapter.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>First model</th>
<th>Second model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth of pinion</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>Module (m)</td>
<td>12 mm</td>
<td>25 mm</td>
</tr>
<tr>
<td>Face width (F)</td>
<td>60 mm</td>
<td>125 mm</td>
</tr>
<tr>
<td>Applied Load (W)</td>
<td>6425 N</td>
<td>28000 N</td>
</tr>
</tbody>
</table>

6.2 BEAM FORMULATION

The beam strength of the gear tooth is determined from Lewis equation and the load carrying capacity of the gear tooth as determined by this equation gives satisfactory results. In the investigation, Lewis assumed that as the load is being transmitted from one gear to another, it is all given and taken by one tooth, because it is not always safe to assume the load is distributed among several teeth. When contact begins, the load is assumed to be at the end of driven teeth and as contact ceases, it is at the end of the driving teeth.

This may not be true when the number of teeth in a pair of mating gears is large, because the load may be distributed among several teeth. But it is almost certain that at some time during the contact of teeth, the proper
distribution of load does not exist and that one tooth must transmit the full load. The gear which has the fewer teeth (pinion) will be the weaker, because the tendency toward undercutting of the teeth becomes more pronounced in gears as the number of teeth becomes smaller (Maitra 1996).

### 6.2.1 Natural Frequency Calculation

Figure 6.1 shows the gear tooth as a cantilever beam. It is easy to calculate the natural frequency of this beam by the following equation (Ramamurti 2002):

\[
\omega_n = \sqrt{\frac{k}{\text{mass}}} \tag{6.1}
\]

k is the stiffness of the beam and it can be calculated from the following equations:

\[
k = \frac{3EI}{l^3} \tag{6.2}
\]

\[
I = \frac{F \cdot t^3}{12} \tag{6.3}
\]
The length of the beam $l$ will take the same value of the height of the tooth while the width of the beam $t$ will take the same value of tooth thickness at pitch circle CTT which equal to $(\pi m/2)$.

\[
\text{mass} = \rho (1 \times t \times F)
\]  

(6.4)

where, $\rho$ is the mass density of steel and $F$ is the tooth face width. By using the above formula, the natural frequencies of a single tooth for the first and second models are 89065rad/sec and 44600rad/sec respectively.

Rayliegh’s method can be applied to find the natural frequency of continuous systems. This method is much simpler for systems with varying distribution of mass and stiffness. Although the method is applicable to all continuous systems, it is applied to beam in this section; the natural frequency by this theory is (Rao 2004):

\[
\omega_n^2 = \frac{\int_0^l EI \left(\frac{d^2 u(x)}{dx^2}\right)^2 dx}{\int_0^l \rho A(u(x))^2 dx}
\]  

(6.5)

The shape function $u(x)$ of cantilever beam can be taken as (Ramamurti 2002):

\[
u(x) = C_1 \left(\frac{x^5}{l^5} - \frac{5x}{l} + 4\right)
\]  

(6.6)

By substituting Equation (6.6) in Equation (6.5) and applying the required gear details of the selected model from Table 6.1 the resulting values of the natural frequency for the first and second models are (100423rad/sec) and (50968 rad/sec) respectively.
It is to be noted that the assumed shape \( u(x) \) unintentionally introduces a constraint on the system (which amounts to adding additional stiffness to the system), and so the frequency given by Equation (6.5) is higher than the exact value. Therefore the natural frequency values obtained from Equation (6.1) are more reasonable and recommended.

6.2.2 Stress Calculation

The load acting during the entire period of engagement is not uniform. In the beginning, at the start of engagement, two pairs of teeth will be in contact and each pair will carry only half of the load, and the contact point will be in the tip of one of these teeth. Hence, a load equal to 6425N and 28000N (the total loads for the first and second models, respectively) are applied to the end of the cantilever beam at an angle equal to the pressure angle of the two mating gears, namely 20 degree. The height of the beam (\( l \)) is assumed to be the distance between the pitch radius and root radius of the gear.

The axial and radial components of these loads are taken as the \( x \)-direction and \( y \)-direction components of the load at a point corresponding to the pitch radius of the cantilever beam where the loads are full value, which are 2197.4N and 6037.5N for the first model and 26311.4N and 9576.5N for the second model respectively. The stresses in \( x \) and \( y \) directions can be calculated by the following equations:

\[
\sigma_x = \frac{W \sin \phi}{F \cdot t} \tag{6.7}
\]

\[
\sigma_y = \frac{W \cos \phi \times l}{F \cdot t^2 \cdot 6} \tag{6.8}
\]
where, \( t \) is the thickness of the tooth at pitch circle of the gear, \( F \) is the face width of the tooth, \( W \) is the total load and \( \sigma_x \) and \( \sigma_y \) are the maximum stresses of the beam in \( x \) and \( y \) axes respectively. From this formulation, the obtained values of stresses in \( x \) direction for the first and second proposed models are 1.94MPa and 1.95MPa respectively and the stresses in \( y \) direction are 25.488MPa and 39.006MPa, respectively.

6.3 FINITE ELEMENT MODEL

For two models consisting of a three tooth section of a 20 of 12mm module and 30 tooth of 25mm module, pinions were developed with the general purpose finite element software ANSYS.

Figures 6.2 and 6.3 show the finite element grid for the three tooth gear segments for the first and second models. Successive reflections of the coordinates for the initial tooth generated segments of three equally spaced, identical teeth. Both the tooth surface and the inside rim surface are unconstrained. The total angles subtended by the segments are 52 degree and 35 degree, respectively. The radial lines defining the ends of the three tooth segments are at \( \pm 26 \) degree and \( \pm 17.5 \) degree respectively from the centre line for each model. An eight nodded iso-parametric plane stress quadratic quadrilateral element was used to build the finite element models inside the frameworks described above. This element has a quadratic displacement function and is well-suited for analyzing irregular shapes.
Figure 6.2 Finite Element Mesh of First Model

Figure 6.3 Finite Element Mesh of Second Model
Each node in the element has two degrees of freedom translations in the x and y directions. The plane stress option with unit thickness was used and scaled to the actual model thickness of 60 mm and 125 mm respectively.

To specify the boundary conditions, all the nodes on the two radial lines defining the ends of the segments and the bottom rim were selected and given zero displacement in both directions. To apply the load at a node, the grid had to have a node point at or near this loading point. Figures 6.4 and 6.5 show the right and left sides of the central tooth (tension and the compression sides), with the nodes numbers in position and stresses at these nodes will be evaluated for comparison purposes later. The complete model has 320 elements, 1071 nodes, and 2142 degrees of freedom in the first model, and 326 elements, 1097 nodes, and 2194 degrees of freedom in the second model. The backup ratio in both of these two models is 2.5 and the rim thicknesses are 67.5 mm and 140.625 mm respectively.

Figure 6.4  Magnified View of Central Tooth in First Model
6.4 MODAL ANALYSIS

If a system, after an initial disturbance, is left to vibrate on its own, the frequency with which it oscillates without external forces is known as its natural frequency.

A vibratory system having n degrees of freedom will have, in general, n distinct natural frequencies of vibration (Rao 2004).

Having obtained the finite element model, a modal analysis was conducted on the same. The global stiffness matrix $[K]$ and global mass matrix $[M]$ was obtained by assembling the element stiffness and mass matrices, respectively. The natural frequencies of the gear tooth were obtained by solving the eigenvalue problem given by the following equation:
\[ [K] \{U\} = \omega_n^2 [M] \{U\} \]  

where \( \omega_n \) is the natural frequency of the system and \( \{U\} \) is the corresponding normalized eigen vector (mode shape). The eigen values and eigen vectors were obtained using Block Lanczos method (Seshu 2004). The first five natural frequencies and corresponding normalized eigen vectors were calculated using this technique by ANSYS software. These first five natural frequencies of the gear tooth obtained for the two proposed models are given in Chapter 7.

### 6.5 LOAD DISTRIBUTION ON GEAR TOOTH

In order to conduct a static stress analysis the loads have to be evaluated. The load on the central tooth of the finite element model, which produces the largest bending stress, is the full load acting at the highest point of single tooth contact (HPSTC). The magnitude of load at any point of contact on profile of gear tooth as the load moves from root to tip of tooth depends on the contact ratio.

The contact ratio is defined as the ratio of length of path of contact to base circle pitch, which is explained at length in Chapters 3 and 5. The contact ratios of the spur gear, for the present models (20 and 30 teeth pinions), with mating gears having 40 and 60 teeth are 1.63 and 1.7 respectively. Figure 6.6, shows the magnitude of loads at various points along the path of contact, where three important distinguishable regions are observed. The first region that is, two pairs of teeth will be in contact and both pairs are assumed to share normal load equally. In the second region, a single pair of gear tooth will be in contact and full load will act on the gear pair under study. In the third region again, two pairs of gear teeth will be in contact and share load equally.
The maximum normal load $W$ acting on the gear tooth sectors of the two proposed models are assumed to be 6425N and 28000N respectively based on the strength of gear tooth material. This normal load is considered in terms of its components in radial and tangential directions with account taken of the pressure angle (Load angle $\gamma_w$) at any point under consideration, and then the tangential load and radial load can be expressed as:

$$W_t = W \cos \gamma_w$$

$$W_r = W \sin \gamma_w$$

Figures 6.7 and 6.8 show the right side (Tension side) of the profile of the central tooth of the presented models, shown in Figures 6.4 and 6.5, stretched out into straight lines with the nodes indicated for the purpose of application of loads. The loads that have been applied for the various runs of static analysis are indicated on the respective nodes starting from the extreme left end (node 48) to the final point of contact (node 54) in the
first model and from the extreme left end (node 52) to the final point of contact (node 58) in the second model.

Figure 6.7 Loads on Various Nodes of First Model

Figure 6.8 Loads on Various Nodes of Second Model

6.6 STATIC STRESS ANALYSIS

A quasi-static analysis was carried out to determine the maximum static stress in the gear tooth for the moving load. The governing equation is expressed as:
The moving load during the time of contact is considered in 9 intervals. At each time interval, the position and magnitude of the load is different as shown in Figures 6.5 and 6.8 but, without considering of the time because the analysis which done here in this section is static analysis. The nodes on which loads have been applied and the static bending stresses in x and y directions at the critical nodes (Node 92 at the first model and Node 102 at the second model) on the compression side of the central teeth are shown in Chapter 7. These static stresses on all the elements for each load are obtained by using ANSYS Finite Element software.

6.7 TRANSIENT ANALYSIS

After computing the natural frequencies and the mode shapes the forced response is obtained using the modal superposition technique (Ramamurti 2002, Seshu 2004). The method is computationally efficient particularly for a large sized problem. The governing differential equation of gear tooth considering the damping can be written as:

\[ [K] \{ \delta \} = [W] \]  

(6.10)

\[ [M]\{\delta^{\infty}\} + [C]\{\delta^\circ\} + [k]\{\delta\} = \{W(t)\} \]

(6.11)

where, [M], [C] and [K] are global mass, damping and stiffness matrices of size \((n \times n)\), respectively. The \([W(t)]\) is the external time varying load of size \((n \times 1)\). The symbols \(\{\delta\}, \{\delta^\circ\}, \{\delta^{\infty}\}\) are the displacement, velocity, acceleration vectors of size \(n \times 1\). The Equation (6.11) has to be recast such that \(m\) uncoupled equations in a single degree are obtained. The recast equation can be obtained by substituting for \(\{\delta\}\), as \(\{\delta\} = [U]\{p\}\), where \([U]\)
is a matrix size \((n \times m)\) of the first \(m\) eigen vectors \((m << n)\), and \(\{p\}\) is a generalized displacement vector of size \((m \times 1)\). Pre multiplying Equation (6.11) by \([U]^T\), gives:

\[
\]

(6.12)

Here \([U]^T[M][U]\) is a unity diagonal matrix of size \((m \times m)\) and \([U]^T[C][U]\) \([U]\) is a diagonal matrix of size \((m \times m)\). The diagonal elements of these matrices are \(2\zeta_i \omega_i\) where \(\zeta_i\) are the damping ratio and \(\omega_i\) are the natural frequencies for \(i = 1, 2, 3, m\). The \([U]^T[K][U]\) \([U]\) is also a diagonal matrix with the square of the natural frequencies \((\omega_1^2, \omega_2^2, \omega_3^2, \ldots, \omega_m^2)\) as the diagonal terms. The resulting decoupled set of equations are solved as standard single degree of freedom system and the resulting displacement vector is transformed back to represent the gear tooth displacements.

\[
[U]^T[C][U] = 2 \begin{bmatrix}
\zeta_1 \omega_1 & 0 & 0 & 0 \\
0 & \zeta_2 \omega_2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & \zeta_m \omega_m
\end{bmatrix}
\]

(6.13)

In order to obtain the diagonal elements of the \([U]^T[C][U]\) given in Equation (6.13), the damping matrix is expressed as linear combination of \([M]\) and \([K]\) (Rayleigh damping), that is:

\[
[C] = \alpha [M] + \beta [K]
\]

(6.14)

where \(\alpha\) and \(\beta\) are constants to be determined. Then,

\[
2 \zeta_1 \omega_1 = \alpha + \beta \omega_1^2, \quad 2 \zeta_2 \omega_2 = \alpha + \beta \omega_2^2, \ldots, 2 \zeta_m \omega_m = \alpha + \beta \omega_m^2
\]

(6.15)
From the finding of Mohammad et al. (Mohammad et al. 1995) the values for mild steel material, $\zeta \omega_1=2.0$ and $\zeta \omega_3=3.2$, where $\omega_1$ and $\omega_3$ are the first and third natural frequencies in rad/sec. Using these two expressions, the damping ratio for other modes are calculated by substituting in Equation (6.15). Since for most industrial problems ($\omega_3 \gg \omega_1$) including the first three natural frequencies computing the dynamic transient response is fairly accurate. The first five natural frequencies obtained of the first model are (92199, 97917, 98432, 140542 and 167886) rad/second, and the first five natural frequencies obtained of the second model are (46135, 48050, 49932, 65954 and 79557) rad/second. After substituting the values of $\omega_1$ and $\omega_3$ of the first model in Equation (6.15), the values of $\alpha$ and $\beta$ are obtained as 13.2 and $8 \times 10^{-8}$ respectively, and by substituting the values of $\omega_1$ and $\omega_3$ of the second model in the same equation the values of $\alpha$ and $\beta$ are obtained as 9.98 and $2.5 \times 10^{-7}$ respectively.

6.7.1 Time Steps and Load Steps

The time of contact $T$ of gear tooth depends on the rotational speed of the gear. If the gear is assumed to run at a speed of $n$-rpm, the time taken for one revolution of the gear will be $(60/n)$ sec. In one revolution, $Z$ number of teeth will get engaged and disengaged, where $Z$ is the number of teeth in the gear. Then the time taken for one pair of teeth in engagement will be $(60/ n.Z)$ sec (Ramamurti and Reddy 2001), which will be $(2.09 \times 10^{-3}$ sec) in the first model and $(1.388 \times 10^{-3}$ sec) in the second model.

Time corresponding to angle of contact (the time elapsed from the time the contact is at the tip to the root of any gear tooth) is whenever in the figure represents the dynamic response and this time duration is more than the time taken for a pair of teeth to be in contact. However this work has
recommended the above method suggested by Ramamurti and Reddy 2001 as it is sufficient for this analysis.

The time taken for one pair of teeth in engagement can be divided into required number of steps (NTS - number of time steps). One time step $\Delta T$ can be calculated by considering the number of modes, which are expected to contribute to the dynamic response. So:

$$
\Delta T = \frac{1}{10\omega_m}
$$

(6.16)

For the third frequency of the first model is (15666 cycle/sec) which is taken from the finite element modal analysis results shown in Chapter 7; the time interval obtained is $(6.38 \times 10^{-6}$ sec) as shown in Figure 6.7, and the third frequency for the second model is (7947 cycle/sec) which is taken from the same results table shown in Chapter 7; the time interval obtained is $(1.258 \times 10^{-5}$ sec) as shown in Figure 6.8.

If the natural period of the $m^{th}$ mode is $T_m$, a choice of $\Delta T$ equal to $T_m/10$ should give a reasonable dynamic response up to $m^{th}$ mode (Ramamurti 1998). Therefore,

$$
T_{NTS} = \frac{60}{\Delta T.n.Z}
$$

(6.17)

where, $T_{NTS}$ is the total number of time steps or intervals. So, the total number of intervals (time steps) for the first model is 326 intervals that mean there are about 40 intervals between each two nodes shown in Figure 6.7, and the total number of intervals for the second model is 110 intervals that mean there are about 13 intervals between each two nodes shown in Figure 6.8.
At any time, two nodes are considered to calculate the load vector at that time interval. The actual load at a point is distributed during the motion of the load between each two nodes under consideration as shown in Figures 6.4, 6.7, 6.5 and 6.8.

The initial conditions for displacements and velocities for the gear tooth in the proposed model are taken as zero for all degrees of freedom. The third mode is selected for the mode superposition technique.

At each time interval, the load acting is calculated and fed into the mode superposition part in ANSYS software and the corresponding deformation and stress are thus obtained.

Node 92 in the first model (shown in Figure 6.4) and node 102 in the second model (shown in Figure 6.5) are considered as the critical points in the central tooth root portion of the selected gears models.

For these nodes corresponding to 1440rpm the dynamic displacements in the x and y directions, the dynamic bending stresses in x and y directions, the dynamic shear stress and dynamic von Misses bending stress are shown in Chapter 7.

The dynamic analysis is carried out also for the following four speeds of gear namely, 360 rpm, 720 rpm, 1440 rpm and 1800 rpm. The maximum dynamic stresses (\(\sigma_x\), \(\sigma_y\) and von Mises) in the gear tooth, which are at (node 92 in central tooth root portion of first model and at node 102 in the second model), are plotted and shown in Chapter 7.