Chapter 3

Walsh Fourier Coefficients

3.1 Order of magnitude of Walsh Fourier coefficients of functions of generalized bounded variation

It appears that while the study of the order of magnitude of the trigonometric Fourier coefficients for the functions of various classes of functions of generalized bounded variation such as $\text{BV}^{(p)} (p \geq 1)$ [77], $\phi\text{BV}$ [78], $\Lambda\text{BV}$ [76], $\Lambda\text{BV}^{(p)} (p \geq 1)$ [58], $\phi\Lambda\text{BV}$ [56] etc. (refer also pages 2 to 6 for definitions of these classes) has been carried out, such a study for the Walsh Fourier coefficients has not yet been done. The only results available are due to N. J. Fine [11], who proved Theorems 1.1.35 and 1.1.36, where, in proving Theorem 1.1.36 he used second mean value theorem. In this section we carry out this study. Interestingly, here no use of the second mean value theorem is made. For the classes $\text{BV}^{(p)}$ and $\phi\text{BV}$, Taibleson-like technique [63] for Walsh Fourier coefficients is developed. However, for the classes $\Lambda\text{BV}^{(p)}$ and $\phi\Lambda\text{BV}$ this technique seems to be not working and hence classical technique [57] is applied. In the case of $\Lambda\text{BV}$, it is also shown that the result is best possible in a certain sense. Results of this section are published in the form of a paper in [13] (see also MR2417326).

Theorem 3.1.1. For $p \geq 1$, if $f \in \text{BV}^{(p)}[0,1]$ then $\hat{f}(n) = O(1/(n^{\frac{1}{p}}))$.

Proof. Let $n \in \mathbb{N}$. Let $k \in \mathbb{N} \cup \{0\}$ be such that $2^k \leq n < 2^{k+1}$ and put $a_i = (i/2^k)$ for $i = 0, 1, 2, 3, ..., 2^k$. Since $\varphi_n$ takes the value 1 on one half of each of the intervals
\((a_{i-1}, a_i)\) and the value \(-1\) on the other half, we have
\[
\int_{a_{i-1}}^{a_i} \varphi_n(x)dx = 0, \text{ for all } i = 1, 2, 3, \ldots, 2^k.
\]
Define a step function \(g\) by \(g(x) = f(a_{i-1})\) on \([a_{i-1}, a_i)\), \(i = 1, 2, 3, \ldots, 2^k\). Then
\[
\int_0^1 g(x)\varphi_n(x)dx = \sum_{i=1}^{2^k} f(a_{i-1}) \int_{a_{i-1}}^{a_i} \varphi_n(x)dx = 0.
\]
Therefore,
\[
|\hat{f}(n)| = \left| \int_0^1 [f(x) - g(x)]\varphi_n(x)dx \right| 
\leq \int_0^1 |f(x) - g(x)|dx 
\leq ||f - g||_p|1||_q 
= \left( \sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} |f(x) - f(a_{i-1})|^pdx \right)^{\frac{1}{p}},
\]
by Hölder’s inequality as \(f, g \in \text{BV}^{(p)}[0, 1]\) and \(\text{BV}^{(p)}[0, 1] \subset L^p[0, 1]\). Hence,
\[
|\hat{f}(n)|^p \leq \sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} |f(x) - f(a_{i-1})|^pdx 
\leq \sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} (V_p(f; [a_{i-1}, a_i]))^pdx 
= \sum_{i=1}^{2^k} (V_p(f; [a_{i-1}, a_i]))^p \left( \frac{1}{2^k} \right) 
\leq \left( \frac{1}{2^k} \right) (V_p(f; [0, 1]))^p 
\leq \left( \frac{2}{n} \right) (V_p(f; [0, 1]))^p,
\]
which completes the proof. \(\square\)

**Remark 3.1.2.** Theorem 3.1.1 with \(p = 1\) gives Theorem 1.1.36 of Fine [11, Theorem VI].
Theorem 3.1.3. If \( f \in \phi \text{BV}[0, 1] \) then \( \hat{f}(n) = O(\phi^{-1}(1/n)) \).

Proof. Let \( c > 0 \). Using Jensen’s inequality and proceeding as in Theorem 3.1.1, we get

\[
\phi \left( c \int_0^1 |f(x) - g(x)| \, dx \right) \leq \int_0^1 \phi(c|f(x) - g(x)|) \, dx \\
= \sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} \phi(c|f(x) - f(a_{i-1})|) \, dx \\
\leq \sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} V_\phi(cf; [a_{i-1}, a_i]) \, dx \\
= \sum_{i=1}^{2^k} V_\phi(cf; [a_{i-1}, a_i]) \left( \frac{1}{2^k} \right) \\
\leq \left( \frac{2}{n} \right) V_\phi(cf; [0, 1]).
\]

Since \( \phi \) is convex and \( \phi(0) = 0 \), for sufficiently small \( c \in (0, 1) \), \( V_\phi(cf; [0, 1]) < 1/2 \). This completes the proof in view of (3.1).

Remark 3.1.4. If \( \phi(x) = x^p, \ p \geq 1 \), then the class \( \phi \text{BV} \) coincides with the class \( \text{BV}^{(p)} \) and Theorem 3.1.3 with Theorem 3.1.1.

Remark 3.1.5. Note that in the proof of Theorems 3.1.1 and 3.1.3, we have used the fact that if \( a = a_0 < a_1 < \ldots < a_n = b \), then

\[
\sum_{i=1}^{n} (V_p(f; [a_{i-1}, a_i]))^p \leq (V_p(f; [a, b]))^p
\]

and

\[
\sum_{i=1}^{n} V_\phi(f; [a_{i-1}, a_i]) \leq V_\phi(f; [a, b]),
\]

for any \( n \geq 2 \) (see [35, 1.17, p. 15]). Such inequalities for functions of the class \( \Lambda \text{BV}^{(p)} (p \geq 1) \) (resp., \( \phi \Lambda \text{BV} \)), which contains \( \text{BV}^{(p)} \) (resp., \( \phi \text{BV} \)) properly, do not hold true.

In fact, the following proposition shows that the validity of such inequality for the class \( \Lambda \text{BV}^{(p)} \) (resp., \( \phi \Lambda \text{BV} \)) virtually reduces the class to \( \text{BV}^{(p)} \) (resp., \( \phi \text{BV} \)) and hence we prove Theorem 3.1.9 and Theorem 3.1.10 applying a technique different from Taibleson-like technique [63] which we have applied in proving Theorem 3.1.1 and Theorem 3.1.3.
Proposition 3.1.6. Let $f \in \phi \Lambda BV[a, b]$. If there is a constant $C$ such that

$$\sum_{i=1}^{n} V_{\phi \Lambda}(f; [a_{i-1}, a_{i}]) \leq CV_{\phi \Lambda}(f; [a, b]),$$

for any sequence of points $\{a_{i}\}_{i=0}^{n}$ with $a = a_{0} < a_{1} < \ldots < a_{n} = b$, then $f \in \phi BV[a, b]$.

Proof. For any partition $a = x_{0} < x_{1} < \ldots < x_{n} = b$ of $[a, b]$, we have

$$\sum_{i=1}^{n} \phi(|f(x_{i}) - f(x_{i-1})|) = \lambda_{1} \sum_{i=1}^{n} \frac{\phi(|f(x_{i}) - f(x_{i-1})|)}{\lambda_{1}} \leq \lambda_{1} \sum_{i=1}^{n} V_{\phi \Lambda}(f; [x_{i-1}, x_{i}]) \leq \lambda_{1} CV_{\phi \Lambda}(f; [a, b]),$$

which shows that $f \in \phi BV[a, b]$. \qed

Remark 3.1.7. $\phi(x) = x^{p}$ $(p \geq 1)$ in this proposition will give analogous result for $\Lambda BV^{(p)}$.

To prove Theorem 3.1.9 and Theorem 3.1.10, we need the following lemma.

Lemma 3.1.8. For any $n \in \mathbb{N}$, $|\hat{f}(n)| \leq \omega_{p}(1/n; f)$, where $\omega_{p}(\delta; f)$ $(\delta > 0, p \geq 1)$ denotes the integral modulus of continuity of order $p$ of $f$ given by

$$\omega_{p}(\delta; f) = \sup_{|h| \leq \delta} \left( \int_{0}^{1} |f(x + h) - f(x)|^{p} dx \right)^{1/p}.$$

Proof. The inequality [11, Theorem IV, p. 382] $|\hat{f}(n)| \leq \omega_{1}(1/n; f)$ and the fact that $\omega_{1}(1/n; f) \leq \omega_{p}(1/n; f)$ for $p \geq 1$ immediately proves the lemma. \qed

Theorem 3.1.9. If $1$–periodic $f \in \Lambda BV^{(p)}[0, 1]$ $(p \geq 1)$ then

$$\hat{f}(n) = O\left(1/\left(\sum_{j=1}^{n} \frac{1}{\lambda_{j}}\right)^{\frac{p}{2}}\right).$$
Proof. For any \( n \in \mathbb{N} \), put \( \theta_n = \sum_{j=1}^{n} 1/\lambda_j \). Let \( f \in \Lambda_{BV}^{(p)}[0, 1] \). For \( 0 < h \leq 1/n \), put \( k = [1/h] \). Then for a given \( x \in \mathbb{R} \) all the points \( x + jh, \ j = 0, 1, ..., k \) lies in the interval \([x, x + 1]\) of length 1 and

\[
\int_{0}^{1} |f(x) - f(x + h)|^p dx = \int_{0}^{1} |f_j(x)|^p dx, \ j = 1, 2, ..., k,
\]

where \( f_j(x) = f(x + (j-1)h) - f(x + jh) \), for all \( j = 1, 2, ..., k \). Since the left hand side of this equation is independent of \( j \), multiplying both sides by \( 1/(\lambda_j \theta_k) \) and summing over \( j = 1, 2, ..., k \), we get

\[
\int_{0}^{1} |f(x) - f(x + h)|^p dx \leq \left( \frac{1}{\theta_k} \right) \int_{0}^{1} \sum_{j=1}^{k} \left( \frac{|f_j(x)|^p}{\lambda_j} \right) dx
\]

\[
\leq \frac{(V_{p\mathcal{A}}(f; [0, 1]))^p}{\theta_k}
\]

\[
\leq \frac{(V_{p\mathcal{A}}(f; [0, 1]))^p}{\theta_n},
\]

because \( \{\lambda_j\} \) is non-decreasing and \( 0 < h \leq 1/n \). The case \(-1/n \leq h < 0\) is similar and using Lemma 3.1.8 we get

\[
|\hat{f}(n)|^p \leq (\omega_p(1/n; f))^p \leq \frac{(V_{p\mathcal{A}}(f; [0, 1]))^p}{\theta_n}.
\]

This completes the proof.

\[\square\]

**Theorem 3.1.10.** If 1-periodic \( f \in \phi \Lambda_{BV}[0, 1] \) then

\[\hat{f}(n) = O\left( \phi^{-1} \left( 1 \left( \sum_{j=1}^{n} \frac{1}{\lambda_j} \right) \right) \right).\]

Proof. Let \( f \in \phi \Lambda_{BV}[0, 1] \). Then for \( h, k \) and \( f_j(x) \) as in the proof of Theorem 3.1.9 and for \( c > 0 \) by Jensen’s inequality,

\[
\phi \left( \int_{0}^{1} |f(x) - f(x + h)|^p dx \right) \leq \int_{0}^{1} \phi(c|f(x) - f(x + h)|) dx
\]

\[
= \int_{0}^{1} \phi(c|f_j(x)|) dx, \ j = 1, 2, ..., k.
\]
Multiplying both sides by $1/(\lambda_j \theta_k)$ and summing over $j = 1, 2, \ldots, k$, we get

$$\phi \left( c \int_0^1 |f(x) - f(x + h)| \, dx \right) \leq \left( \frac{1}{\theta_k} \right) \int_0^1 \sum_{j=1}^{k} \left( \frac{\phi(c|f_j(x)|)}{\lambda_j} \right) \, dx$$

$$\leq \frac{V_{\phi_{\Lambda}}(cf; [0, 1])}{\theta_k} \leq \frac{V_{\phi_{\Lambda}}(cf; [0, 1])}{\theta_n}.$$  

Since $\phi$ is convex and $\phi(0) = 0$, $\phi(\alpha x) \leq \alpha \phi(x)$, for $0 < \alpha < 1$. So we may choose $c$ sufficiently small so that $V_{\phi_{\Lambda}}(cf; [0, 1]) \leq 1$. But then we have

$$\int_0^1 |f(x) - f(x + h)| \, dx \leq \frac{1}{c} \phi^{-1} \left( \frac{1}{\theta_n} \right).$$

Thus it follows in view of Lemma 3.1.8 that

$$|\hat{f}(n)| \leq \omega_1(1/n; f) \leq \frac{1}{c} \phi^{-1} \left( \frac{1}{\theta_n} \right),$$

which completes the proof. \qed

Following theorem shows that Theorem 3.1.9 with $p = 1$ is best possible in a certain sense.

**Theorem 3.1.11.** If $\Gamma BV[0, 1] \supseteq \Lambda BV[0, 1]$ properly, then there exists $f \in \Gamma BV[0, 1]$ such that

$$\hat{f}(n) \neq O \left( \frac{1}{\left( \sum_{j=1}^{n} \frac{1}{\lambda_j} \right)} \right).$$

**Proof.** It is known [52] that if $\Gamma BV$ contains $\Lambda BV$ properly with $\Gamma = \{\gamma_n\}$ then $\theta_n \neq O(\rho_n)$, where in $\rho_n = \sum_{j=1}^{n} \frac{1}{\gamma_j}$ for each $n$. Also, if $c_0 = 0$, $c_{n+1} = 1$ and $c_1 < c_2 < \ldots < c_n$ denote all the $n$ points of $(0, 1)$ where the function $\varphi_n$ changes its sign in $(0, 1)$, $n_0 \in \mathbb{N}$ is such that $\rho_n \geq \frac{1}{2}$ for all $n \geq n_0$ and $E = \{n \in \mathbb{N} : n \geq n_0 \text{ is even}\}$, then for each $n \in E$, for the function

$$f_n = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{4\rho_n} \chi_{[c_{k-1},c_k)}$$

extended 1-periodically on $\mathbb{R}$,

$$V_{\Gamma}(f_n, [0, 1]) = \sum_{k=1}^{n+1} \frac{|f_n(c_k) - f_n(c_{k-1})|}{\gamma_k} = \sum_{k=1}^{n} \frac{1}{\gamma_k} \cdot \frac{1}{2\rho_n} = \frac{1}{2}.$$
because 
\[ f_n(c_{n+1}) = f_n(1) = f_n(0) = \frac{1}{4\rho_n} = f_n(c_n) \]
as \( \varphi_n \equiv 1 \) on \([c_0, c_1]\). Hence \( \|f_n\| = \frac{1}{4\rho_n} + \frac{1}{2} \leq 1 \) for each \( n \in E \) in the Banach space \( \Gamma BV[0, 1] \) with \( \|f\| = |f(0)| + V_T(f, [0, 1]) \). Observe that for \( f \in \Gamma BV[0, 1] \)
\[ \|f\|_1 \leq \int_0^1 \left( \frac{|f(x) - f(0)|}{\gamma_1} + |f(0)| \right) dx \leq C\|f\|, \quad C = \max\{1, \gamma_1\}, \]
and hence, for each \( n \in \mathbb{N} \) the linear map \( T_n : \Gamma BV[0, 1] \to \mathbb{R} \) defined by \( T_n(f) = \theta_n \hat{f}(n) \) is bounded as
\[ |T_n(f)| = \theta_n |\hat{f}(n)| \leq \theta_n \|f\|_1 \leq \theta_n C\|f\|, \quad \forall f \in \Gamma BV[0, 1]. \]
Next, for each \( n \in E \) since \( f_n \cdot \varphi_n = \frac{1}{4\rho_n} \) on \([0, 1]\), we see that
\[ T_n(f_n) = \theta_n \hat{f}_n(n) = \theta_n \int_0^1 f_n(x) \varphi_n(x) dx = \frac{1}{4} \left( \theta_n \rho_n \right) \neq O(1) \]
and hence
\[ \sup\{\|T_n\| : n \in \mathbb{N}\} \geq \sup\{\|T_n\| : n \in E\} \geq \sup\{|T_n(f_n)| : n \in E\} = \infty. \]
Therefore, an application of the Banach-Steinhaus theorem gives an \( f \in \Gamma BV[0, 1] \) such that \( \sup\{|T_n(f)| : n \in \mathbb{N}\} = \infty \). It follows that \( \theta_n \hat{f}(n) = T_n(f) \neq O(1) \) and hence the theorem is proved. \( \Box \)

3.2 Order of magnitude of Walsh Fourier coefficients of series with small gaps for functions of generalized bounded variation

In Section 3.1 we have studied the order of magnitude of Walsh Fourier coefficients of the functions of various classes of generalized bounded variation. Here we continue this study further and obtain the analogous results for the lacunary Walsh Fourier series with small gaps. Interestingly, here also we use the technique which we have developed in Section 3.1 and prove the corresponding results. We also use the results for non-lacunary Walsh Fourier series to prove the results for lacunary Walsh Fourier series in an elegant way. The results of this section are published in [15].
Definition 3.2.1. Let \( \{n_k\}_{k=1}^{\infty} \) be an increasing sequence of positive integers. A function \( f \in L^1[0,1] \) is said to have a lacunary Walsh Fourier series with small gaps if its Walsh Fourier coefficients \( \hat{f}(n) \) vanish for \( n \neq n_k, k \in \mathbb{N} \), where \( \{n_k\} \) satisfies the small gap condition (1.11) or, in particular, more stringent small gap condition (1.5).

Theorem 3.2.2. Let a 1-periodic \( f \in L^1[0,1] \) possess a lacunary Walsh Fourier series
\[
\sum_{k=1}^{\infty} \hat{f}(n_k) \varphi_{n_k}(x)
\]
with small gaps (1.11) and \( I = [0, 2^{-N}) \) be an interval of length \( |I| = 2^{-N} \geq 1/q \). Then \( f \in BV^{(p)}(I) \) \((p \geq 1)\) implies \( \hat{f}(n_k) = O(1/(n_k)^{1/p}) \).

Proof. Consider the polynomial \( P_N(x) \) (this is essentially the same polynomial as considered by Patadia (see [43, p. 20]) defined as follows: If \( N = 0 \), put \( P_N \equiv 1 \) and if \( N \in \mathbb{N} \) then put \( P_N(x) = \prod_{k=0}^{N-1} (1 + r_k(x)) \). Then
\[
x \in I = [0, 2^{-N}) \Rightarrow 1 + r_k(x) = 1 + \varphi_{2k}(x) = 1 + 1 = 2, \ \forall k = 0, 1, ..., N - 1
\Rightarrow P_N(x) = 2^N.
\]
On the other hand, if \( x \in [0,1) \setminus I \) then exactly one of the following holds:
\[
x \in [1/2, 1), \ x \in [1/2^2, 1/2), ..., x \in [1/2^N, 1/2^{N-1}) .
\]
Thus at least one of the following holds:
\[
r_0(x) = -1, \ r_1(x) = -1, ..., r_{N-1}(x) = -1.
\]
This shows that at least one of \( 1 + r_k(x) \) is 0 and hence \( P_N(x) = 0 \). We claim that if \( k \in \mathbb{N} \) is such that \( \hat{f}(n_k) \neq 0 \) then
\[
(f P_N)'(n_k) = \hat{f}(n_k) \ (k = 1, 2, 3, ...).
\]
(3.3)
Let \( k \in \mathbb{N} \) be such that \( \hat{f}(n_k) \neq 0 \). Then
\[
(f P_N)'(n_k) = \int_{0}^{1} f(x)P_N(x)\varphi_{n_k}(x)dx
= \hat{f}(n_k) + \sum_{i=0}^{N-1} \hat{f}(r_i \varphi_{n_k}) + \sum_{i,j=0}^{N-1} \hat{f}(r_i r_j \varphi_{n_k}) + ... + \hat{f}(r_0 r_1 ... r_{N-1} \varphi_{n_k}).
\]
(3.4)
By our assumption the first term in the right hand side of (3.4) is nonzero. The characters appearing in the other terms in right hand side of (3.4) are of the form \( \varphi \varphi_{n_k} \) wherein \( \varphi \) is such that \( \deg \varphi \) is positive and \( \leq N \). In view of the Payley ordering of Walsh characters, for each \( j \in \mathbb{N} \) there are totally \( 2^{j-1} \) characters of degree \( j \), namely \( \varphi_{2^{j-1}} \equiv r_{j-1}, \varphi_{2^{j-1}+1} \equiv r_{j-1}\varphi_1, \varphi_{2^{j-1}+2} \equiv r_{j-1}\varphi_2, \ldots, \varphi_{2^j-1} \equiv r_{j-1}\varphi_{2^j-1} \equiv r_{j-1}r_{j-2} \ldots r_1r_0 \). Consequently, total number of characters of positive degree \( \leq N \) is given by \( 2^0 + 2^1 + 2^2 + \ldots + 2^{N-1} = 2^N - 1 \); they are from \( \varphi_1 \) to \( \varphi_{2^N-1} \). It follows that when \( \varphi_{n_k} \) is multiplied by any character of positive degree \( \leq N \) the resulting character \( \varphi_m \) is such that

\[
n_k < m \leq n_k + 2^N - 1 < n_k + 2^N \leq n_k + q \leq n_{k+1},
\]

since the lacunary Walsh Fourier series (3.2) of \( f \) has gaps (1.11) with \( q \geq 2^N \). Since \( \hat{f}(n_k) \neq 0 \), all the terms of the right hand side of (3.4) vanish except the first. This means that (3.3) holds.

Let \( k \) be large enough and \( m \in \mathbb{N} \cup \{0\} \) be such that \( \hat{f}(n_k) \neq 0 \), \( 2^m \leq n_k < 2^{m+1} \) and \( m > N \). Then

\[
\hat{f}(n_k) = (f_P_N)^*(n_k) = 2^N \int_0^{1/2^N} f(x) \varphi_{n_k}(x) dx,
\]

since \( P_N = 2^N \) on \( I = [0, 2^{-N}] \) and \( P_N = 0 \) on \( [0, 1) \setminus I \). Put \( a_i = (i/2^m) \) for \( i = 0, 1, 2, 3, \ldots, 2^m \). Then, since \( 2^m \leq n_k < 2^{m+1} \), \( \varphi_{n_k} \) takes the value 1 on one half of each of the intervals \( (a_{i-1}, a_i) \) and the value -1 on the other half. Therefore we have

\[
\int_{a_{i-1}}^{a_i} \varphi_{n_k}(x) dx = 0, \quad \text{for all } i = 1, 2, 3, \ldots, 2^m.
\]

Define a step function \( g \) by \( g(x) = f(a_{i-1}) \) on \( [a_{i-1}, a_i) \), \( i = 1, 2, 3, \ldots, 2^m-N \). Then in view of (3.6) we have

\[
\int_0^{1/2^N} g(x) \varphi_{n_k}(x) dx = \sum_{i=1}^{2^m-N} f(a_{i-1}) \int_{a_{i-1}}^{a_i} \varphi_{n_k}(x) dx = 0.
\]

Thus in view of (3.5) we have

\[
|\hat{f}(n_k)| = 2^N \left| \int_0^{1/2^N} [f(x) - g(x)] \varphi_{n_k}(x) dx \right| \leq 2^N \int_0^{1/2^N} |f(x) - g(x)| dx.
\]

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Therefore
\[
|\hat{f}(n_k)| \leq 2^N \|f - g\|_{p,I}1_{q,I} = 2^{N/p} \left( \sum_{i=1}^{2^{m-N}} \int_{a_{i-1}}^{a_i} |f(x) - f(a_{i-1})|^p dx \right)^{\frac{1}{p}},
\]
by Hölder’s inequality as \( f, g \in BV^{(p)}(I) \) and \( BV^{(p)}(I) \subset L^p(I) \). Hence,
\[
|\hat{f}(n_k)|^p \leq 2^N \sum_{i=1}^{2^{m-N}} \int_{a_{i-1}}^{a_i} |f(x) - f(a_{i-1})|^p dx
\]
\[
\leq 2^N \sum_{i=1}^{2^{m-N}} \int_{a_{i-1}}^{a_i} (V_p(f; [a_{i-1}, a_i])|^p dx
\]
\[
= 2^N \sum_{i=1}^{2^{m-N}} (V_p(f; [a_{i-1}, a_i])|^p \left( \frac{1}{2^m} \right)
\]
\[
\leq \left( \frac{2^N}{2^m} \right) (V_p(f; I))^p
\]
\[
\leq \left( \frac{2 \cdot 2^N}{n_k} \right) (V_p(f; I))^p,
\]
which completes the proof. 

\[\square\]

**Theorem 3.2.3.** Let \( f \) and \( I \) be as in Theorem 3.2.2. Then \( f \in \phi BV(I) \) implies \( \hat{f}(n_k) = O(\phi^{-1}(1/n_k)) \).

**Proof.** Proceeding as in the proof of Theorem 3.2.2 we get (3.7). For \( c > 0 \) using Jensen’s inequality, we have
\[
\phi \left( 2^N c \int_0^{1/2^N} |f(x) - g(x)| dx \right) \leq 2^N \int_0^{1/2^N} \phi(c|f(x) - g(x)|) dx
\]
\[
= 2^N \sum_{i=1}^{2^{m-N}} \int_{a_{i-1}}^{a_i} \phi(c|f(x) - f(a_{i-1})|) dx
\]
\[
\leq 2^N \sum_{i=1}^{2^{m-N}} V_\phi(cf; [a_{i-1}, a_i]) dx
\]
\[
= 2^N \sum_{i=1}^{2^{m-N}} V_\phi(cf; [a_{i-1}, a_i]) \left( \frac{1}{2^m} \right)
\]
\[
\leq 2^N \left( \frac{2}{n_k} \right) V_\phi(cf; I).
\]
Since \( \phi \) is convex and \( \phi(0) = 0 \), for sufficiently small \( c \in (0, 1) \), \( V_\phi(cf; I) \leq 1/2^{N+1} \).
This completes the proof in view of (3.7).

**Remark 3.2.4.** If \( \phi(x) = x^p, p \geq 1 \), then the class \( \phiBV(I) \) coincides with the class \( BV^{(p)}(I) \) and Theorem 3.2.3 with Theorem 3.2.2.

**Theorem 3.2.5.** Let \( f \) and \( I \) be as in Theorem 3.2.2. Then \( f \in \LambdaBV^{(p)}(I) \) \( (p \geq 1) \) implies

\[
\hat{f}(n_k) = O\left(1/\left(\sum_{j=1}^{n_k} \frac{1}{\lambda_j}\right)^{\frac{1}{2}}\right).
\]

**Proof.** Proceeding as in the proof of Theorem 3.2.2 we get (3.6). Define \( h \) on \( [0, 1) \) by \( h = f \) on \( I = [0, 1/2^N) \), \( h(x) = f(1/2^N), \forall x \in [1/2^N, 1) \); and extend \( h \), \( 1 \)-periodically on \( \mathbb{R} \). We claim that \( h \in \LambdaBV^{(p)}[0, 1] \). Let \( \{I_n\} \) be a sequence of non-overlapping intervals in \( [0, 1] \) and consider the sum \( S = \sum_n |h(I_n)|^p/\lambda_n \) where \( h(I_n) = h(b_n) - h(a_n) \) if \( I_n = [a_n, b_n] \). If \( I_n \subset [1/2^N, 1) \) then by definition of \( h, h(I_n) = 0 \). Thus, \( S = \sum_k |h(I_{nk})|^p/\lambda_{nk}, \) where no \( I_{nk} \) is contained in \( [1/2^N, 1) \).

Since the sequence \( \{I_n\} \) is non-overlapping, there can be at most one interval, say, \( I_{n_j} \) which intersects \( (1/2^N, 1) \). If \( I_{n_j} = [a, b] \), let \( I_{n_j}' = I_{n_j} \cap [0, 1/2^N], I_{n_j}'' = I_{n_j} \cap [1/2^N, b] \). Then again by the definition of \( h, h(I_{n_j}'') = 0 \) and hence \( h(I_{n_j}) = h(I_{n_j}') + h(I_{n_j}'') = h(I_{n_j}). \) Hence

\[
S = \sum_{k, k \neq j} \frac{|h(I_{nk})|^p}{\lambda_{nk}} + \frac{|h(I'_{nj})|^p}{\lambda_{nj}}.
\]

Also, in \( \{I_n\} \), there can be at most one interval, say, \( I_{nt} \) of the form \([c, 1]\) where \( c \in (1/2^N, 1) \). But then

\[
S = \sum_{k, k \neq j, k \neq t} \frac{|h(I_{nk})|^p}{\lambda_{nk}} + \frac{|h(I'_{nj})|^p}{\lambda_{nj}} + \frac{|h(1) - h(c)|^p}{\lambda_{nt}}
\]
\[
= \sum_{k, k \neq j, k \neq t} \frac{|f(I_{nk})|^p}{\lambda_{nk}} + \frac{|f(I'_{nj})|^p}{\lambda_{nj}} + \frac{|f(0) - f(1/2^N)|^p}{\lambda_{nt}}
\]
\[
\leq V_{pA}(f; I) + \frac{|f(0) - f(1/2^N)|^p}{\lambda_1}.
\]

It follows that \( V_{pA}(h; [0, 1]) = V_{pA}(f; I) + (|f(0) - f(1/2^N)|^p/\lambda_1) \). Since \( f \in \LambdaBV^{(p)}(I) \), we have \( h \in \LambdaBV^{(p)}[0, 1] \) and hence by Theorem 3.1.9

\[
\hat{h}(n) = O((1/\langle\theta_n\rangle)^{\frac{1}{2}}), \quad (3.8)
\]
where \( \theta_n = \sum_{j=1}^{n} 1/\lambda_j \), \( \forall n \in \mathbb{N} \). But

\[
\hat{h}(n_k) = \int_{0}^{1} h(x) \varphi_{nk}(x)dx \\
= \int_{0}^{1/2^N} f(x) \varphi_{nk}(x)dx + f(1/2^N) \int_{1/2^N}^{1} \varphi_{nk}(x)dx \\
= \int_{0}^{1/2^N} f(x) \varphi_{nk}(x)dx + f(1/2^N) \sum_{i=2^m-N+1}^{2^m} \int_{a_{i-1}}^{a_i} \varphi_{nk}(x)dx \\
= \int_{0}^{1/2^N} f(x) \varphi_{nk}(x)dx,
\]

in view of (3.6). Thus by (3.5) and (3.8), \( \hat{f}(n_k) = 2^N \hat{h}(n_k) = O(1/(\theta_{nk})^{1/p}) \) and hence the theorem is proved.

**Theorem 3.2.6.** Let \( f \) and \( I \) be as in Theorem 3.2.2. Then \( f \in \phi\LambdaBV(I) \) implies

\[
\hat{f}(n_k) = O\left( \phi^{-1}\left(1/(\sum_{j=1}^{n_k} 1/\lambda_j)\right)\right).
\]

**Proof.** Proceeding as in the proof of Theorem 3.2.2 we get (3.6). Now if \( f \in \phi\LambdaBV(I) \), we can see, in a similar way, that the function \( h \) considered in the proof of Theorem 3.2.5 is in \( \phi\LambdaBV[0,1] \). Thus by Theorem 3.1.10, \( \hat{h}(n) = O(\phi^{-1}(1/\theta_n)) \) and hence in view of (3.5),

\[
\hat{f}(n_k) = 2^N \hat{h}(n_k) = O(\phi^{-1}(1/\theta_{nk})).
\]

This completes the proof.

**3.3 Order of magnitude of multiple Walsh Fourier coefficients of functions of bounded p-variation**

In 1949, N. J. Fine [11] proved using the second mean value theorem that if \( f \) is of bounded variation on \([0,1]\) and if \( \hat{f}(n) \) denotes its (one dimensional) Walsh Fourier coefficient, then \( \hat{f}(n) = O(\frac{1}{n}) \), for all \( n \neq 0 \). In Section 3.1 we have studied the order of magnitude of Walsh Fourier coefficients of functions of various classes of generalized bounded variation and extended the result of Fine to these classes.
Further in Section 2.1, we have defined the notion of bounded $p$-variation ($p \geq 1$) for a complex-valued function on the rectangle $[a_1, b_1] \times \cdots \times [a_m, b_m]$ ($m$ being a positive integer) and studied the order of magnitude of trigonometric Fourier coefficients of such functions on $[0, 2\pi]^m$. Here we study the order of magnitude of Walsh Fourier coefficients for a function of bounded $p$-variation from $[0, 1]^m$ to $\mathbb{C}$ and obtain analogous results. For $m = 1$, our new results give our earlier result (see Theorem 3.1.1). Also, for $p = 1$, our results give the Walsh analogue of the results of Móricz [33] and Fülöp and Móricz [10] (see Theorems 1.1.32, 1.1.33 and 1.1.34), except possibly for the exact constant in their case. Multiple Walsh Fourier coefficient is defined as follows (refer, for example, [12]).

**Definition 3.3.1.** For a periodic $f = f(x_1, \ldots, x_m)$ with period 1 in each variable and Lebesgue integrable over the $m$-dimensional torus $I^m := [0, 1)^m$, in symbol $f \in L^1(I^m)$, its formal Walsh Fourier series is given by

$$f(x_1, \ldots, x_m) \sim \sum_{(n^{(1)}, \ldots, n^{(m)}) \in \mathbb{Z}^+^m} \hat{f}(n^{(1)}, \ldots, n^{(m)}) w_{n^{(1)}}(x_1) \ldots w_{n^{(m)}}(x_m)$$

where $\hat{f}(n^{(1)}, \ldots, n^{(m)}) \equiv \hat{f}(n)$ is the $n^{th}$ multiple Walsh Fourier coefficient of $f$ defined by

$$\hat{f}(n) = \int_{I^m} f(x_1, \ldots, x_m) w_{n^{(1)}}(x_1) \ldots w_{n^{(m)}}(x_m) dx_1 \ldots dx_m. \quad (3.9)$$

**Theorem 3.3.2.** Let $f : \mathbb{R}^m \to \mathbb{C}$ be 1-periodic in each variable. If $f$ belongs to $\text{BV}_V^{(p)}([0, 1]^m) \cap L^p(I^m)$ ($p \geq 1$) and $n = (n^{(1)}, \ldots, n^{(m)}) \in \mathbb{N}^m$, then

$$\hat{f}(n) = O\left( \frac{1}{\left( \prod_{j=1}^m n^{(j)} \right)^{1/p}} \right).$$

**Proof.** For the sake of simplicity in writing, we carry out the proof for $m = 2$, and we write $(x, y)$ and $(k, \ell)$ in place of $(x_1, x_2)$ and $(n^{(1)}, n^{(2)})$ respectively.

Let $n = (k, \ell) \in \mathbb{N}^2$. Let $s, t \in \mathbb{Z}^+$ be such that $2^s \leq k < 2^{s+1}$ and $2^t \leq \ell < 2^{t+1}$. For each $i = 0, 1, 2, 3, \ldots, 2^s$ and $j = 0, 1, 2, 3, \ldots, 2^t$ put $a_i = (i/2^s)$, $b_j = (j/2^t)$. Then by definition of Walsh functions, $\varphi_k$ takes the value 1 on one half of each of the intervals $(a_{i-1}, a_i)$ and the value $-1$ on the other half, and hence

$$\int_{a_{i-1}}^{a_i} \varphi_k(x) dx = 0, \quad (i = 1, 2, 3, \ldots, 2^s). \quad (3.10)$$
Similarly, the function \( \varphi_t \) takes the value 1 on one half of each of the intervals \((b_{j-1}, b_j)\) and the value \(-1\) on the other half, and hence
\[
\int_{b_{j-1}}^{b_j} \varphi_t(y)dy = 0, \quad (j = 1, 2, 3, \ldots, 2^t). \tag{3.11}
\]

Define three functions \( f_1, f_2, f_3 \) on \( \mathbb{R}^2 \) by setting
\[
\begin{align*}
f_1(x, y) &= f(a_{i-1}, y) \quad (a_{i-1} \leq x < a_i; \ 0 \leq y < 1) \text{ for } i = 1, 2, 3, \ldots, 2^s; \\
f_2(x, y) &= f(x, b_{j-1}) \quad (0 \leq x < 1; \ b_{j-1} \leq y < b_j) \text{ for } j = 1, 2, 3, \ldots, 2^t;
\end{align*}
\]
and
\[
f_3(x, y) = f(a_{i-1}, b_{j-1}) \quad (a_{i-1} \leq x < a_i; \ b_{j-1} \leq y < b_j)
\]
for \( i = 1, 2, 3, \ldots, 2^s; \ j = 1, 2, 3, \ldots, 2^t \). Then in view of Fubini’s theorem and relations (3.10) and (3.11) we have
\[
\begin{align*}
\int_0^1 \int_0^1 f_1(x, y)\varphi_k(x)\varphi_t(y)dxdy &= \int_0^1 \left[ \sum_{i=1}^{2^s} f(a_{i-1}, y) \int_{a_{i-1}}^{a_i} \varphi_k(x)dx \right] \varphi_t(y)dy = 0, \\
\int_0^1 \int_0^1 f_2(x, y)\varphi_k(x)\varphi_t(y)dxdy &= \int_0^1 \left[ \sum_{j=1}^{2^t} f(x, b_{j-1}) \int_{b_{j-1}}^{b_j} \varphi_t(y)dy \right] \varphi_k(x)dx = 0
\end{align*}
\]
and
\[
\begin{align*}
\int_0^1 \int_0^1 f_3(x, y)\varphi_k(x)\varphi_t(y)dxdy &= \sum_{i=1}^{2^s} \sum_{j=1}^{2^t} f(a_{i-1}, b_{j-1}) \left[ \int_{a_{i-1}}^{a_i} \varphi_k(x)dx \right] \left[ \int_{b_{j-1}}^{b_j} \varphi_t(y)dy \right] = 0.
\end{align*}
\]

Using these equations in the definition of \( \hat{f}(n) \) (see (3.9)) we get
\[
|\hat{f}(n)| = \left| \int_0^1 \int_0^1 f(x, y)\varphi_k(x)\varphi_t(y)dxdy \right|
\]
\[
= \left| \int_0^1 \int_0^1 [f(x, y) - f_1(x, y) - f_2(x, y) + f_3(x, y)]\varphi_k(x)\varphi_t(y)dxdy \right|
\]
\[
\leq \int_0^1 \int_0^1 |f(x, y) - f_1(x, y) - f_2(x, y) + f_3(x, y)|dxdy
\]
\[
\leq \left( \int_0^1 \int_0^1 |f(x, y) - f_1(x, y) - f_2(x, y) + f_3(x, y)|^p dxdy \right)^{1/p} \leq (1)^{2/q},
\]
in view of the H"older’s inequality (when $p > 1$) since $f - f_1 - f_2 + f_3 \in L^p(\mathbb{I}^2)$, where $q$ is such that $1/p + 1/q = 1$. Observe that when $p = 1$, we don’t use H"older’s inequality and in that case we consider the inequality except last step. In any case, it follows that

$$|\hat{f}(n)|^p \leq \int_0^1 \int_0^1 |f(x, y) - f_1(x, y) - f_2(x, y) + f_3(x, y)|^p dxdy$$

$$= \sum_{i=1}^{2^s} \sum_{j=1}^{2^t} \int_{a_{i-1}}^{a_i} \int_{b_{j-1}}^{b_j} |f(x, y) - f_1(x, y) - f_2(x, y) + f_3(x, y)|^p dxdy$$

$$= \sum_{i=1}^{2^s} \sum_{j=1}^{2^t} \int_{a_{i-1}}^{a_i} \int_{b_{j-1}}^{b_j} |f(x, y) - f(a_{i-1}, y) - f(x, b_{j-1}) + f(a_{i-1}, b_{j-1})|^p dxdy$$

$$\leq \sum_{i=1}^{2^s} \sum_{j=1}^{2^t} (V_p(f; [a_{i-1}, a_i] \times [b_{j-1}, b_j]))^p (a_i - a_{i-1})(b_j - b_{j-1})$$

$$\leq \frac{1}{2s2^t} (V_p(f; [0, 1]^2))^p \leq \frac{2^2}{k\ell} (V_p(f; [0, 1]^2))^p,$$

in view of Lemma 2.1.8. Thus we get

$$|\hat{f}(n)| \leq \frac{4^{1/p} \cdot V_p(f; [0, 1]^2)}{(k\ell)^{1/p}}. \quad (3.12)$$

This completes the proof. \qed

**Theorem 3.3.3.** Let $f : \mathbb{R}^m \to \mathbb{C}$ be 1-periodic in each variable. If $f$ belongs to $\text{BV}_H^{(p)}([0, 1]^m)$ ($p \geq 1$) then for any $\mathbf{0} \neq \mathbf{n} = (n^{(1)}, ..., n^{(m)}) \in (\mathbb{Z}^+)^m$,

$$\hat{f}(\mathbf{n}) = O \left( \frac{1}{\left( \prod_{j=1}^{m} n^{(j)} \right)^{1/p}} \right),$$

**Proof.** Here also we will carry out the proof for $m = 2$ and use notations as in the proof of Theorem 3.3.2. Since $f \in \text{BV}_H^{(p)}([0, 1]^2)$, in view of Lemma 2.1.11 (use Lemma 2.1.12 for general case), the discontinuities of $f$ lie on countable number of parallels to the axes and hence $f$ is measurable over $\mathbb{I}^2$ in the sense of Lebesgue. Further, by Lemma 2.1.6, $f$ is bounded over $[0, 1]^2$ and hence $f \in L^p(\mathbb{I}^2)$. As $\text{BV}_H^{(p)}([0, 1]^2) \subset \text{BV}_V^{(p)}([0, 1]^2)$, $f \in L^p(\mathbb{I}^2) \cap \text{BV}_V^{(p)}([0, 1]^2)$. Therefore if $\mathbf{n} = (k, \ell) \in \mathbb{N}^2$, by Theorem 3.3.2,

$$\hat{f}(\mathbf{n}) = O \left( \frac{1}{(k\ell)^{1/p}} \right).$$

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Next, let \( n = (k, \ell) \in (\mathbb{Z}^+)^2 \) be such that \( k \neq 0, \ell = 0 \) and let \( a_i \)'s and \( f_1 \) be as defined in the proof of Theorem 3.3.2. Then we have

\[
\int_0^1 \int_0^1 f_1(x, y) \varphi_k(x) dx dy = \int_0^1 \left( \sum_{i=1}^{2^*} f(a_{i-1}, y) \left[ \int_{a_{i-1}}^{a_i} \varphi_k(x) dx \right] \right) dy = 0,
\]

in view of Fubini’s theorem and (3.10); and,

\[
|\hat{f}(n)| = \left| \int_0^1 \int_0^1 [f(x, y) - f_1(x, y)] \varphi_k(x) dx dy \right| \\
\leq \left( \int_0^1 \int_0^1 |f(x, y) - f_1(x, y)|^p dx dy \right)^{1/p} (1)^{2/q},
\]

in view of Hölder’s inequality as in the proof of Theorem 3.3.2. Therefore

\[
|\hat{f}(n)|^p \leq \int_0^1 \left[ \sum_{i=1}^{2^*} \int_{a_{i-1}}^{a_i} |f(x, y) - f(a_{i-1}, y)|^p dx \right] dy \\
\leq \int_0^1 \left[ \sum_{i=1}^{2^*} (V_p(f(\cdot, y); [a_{i-1}, a_i])^p(a_i - a_{i-1}) \right] dy \\
\leq \frac{1}{2^*} \int_0^1 (V_p(f(\cdot, y); [0, 1]))^p dy \\
\leq \frac{2}{k} \int_0^1 2^p [(V_p(f; [0, 1]^2))^p + (V_p(f(\cdot, 0); [0, 1]))^p] dy \\
= \frac{2^{p+1} [(V_p(f; [0, 1]^2))^p + (V_p(f(\cdot, 0); [0, 1]))^p]}{k},
\]

in view of Lemma 2.1.8 (for a function of one variable) and Lemma 2.1.7. Thus we have

\[
\hat{f}(n) = \hat{f}(k, 0) = O \left( \frac{1}{k^{1/p}} \right). \tag{3.13}
\]

The case \( k = 0, \ell \neq 0 \), is similar to the above case and in this case we get

\[
\hat{f}(0, \ell) = O \left( \frac{1}{\ell^{1/p}} \right). \tag{3.14}
\]

This completes the proof.
Remark 3.3.4. Theorem 3.3.2 or Theorem 3.3.3 with \( m = 1 \) gives our earlier result (see Theorem 3.1.1). (3.12), (3.13) and (3.14) with \( p = 1 \) give Walsh analogues of the results of Móricz [33] (see Theorem 1.1.32) and Fülöp and Móricz [10, for \( n = 2 \)] (see Theorem 1.1.34), except possibly for the exact constant in their case.

### 3.4 Order of magnitude of multiple Walsh Fourier coefficients of series with small gaps for functions of bounded \( p \)-variation

In 1949, N. J. Fine [11] proved using the second mean value theorem that if \( f \) is of bounded variation on \([0,1]\) and if \( \hat{f}(n) \) denotes its (one dimensional) Walsh Fourier coefficient, then \( \hat{f}(n) = O\left(\frac{1}{n}\right) \), for all \( n \neq 0 \). In Section 3.1 we have studied the order of magnitude of Walsh Fourier coefficients of functions of various classes of generalized bounded variation and extended the result of Fine to these classes. The small gap analogue of results of Section 3.1 is given in Section 3.2. Further in Section 2.1, we have defined the notion of bounded \( p \)-variation (\( p \geq 1 \)) for a function from a rectangle \([a_1, b_1] \times \ldots \times [a_m, b_m]\) to \( \mathbb{C} \) and studied the order of magnitude of trigonometric Fourier coefficients of such functions from \([0, 2\pi]^m \) to \( \mathbb{C} \). We have also studied the order of magnitude of trigonometric Fourier coefficients of functions from \([0, 2\pi]^m \) to \( \mathbb{C} \) having lacunary Fourier series with certain gaps and are of bounded \( p \)-variation only locally in Section 2.2. In Section 3.3 we have studied the order of magnitude of Walsh Fourier coefficients for a function of bounded \( p \)-variation from \([0, 1]^m \) to \( \mathbb{C} \) having non-lacunary Fourier series. Here we study the order of magnitude of Walsh Fourier coefficients of functions from \([0, 1]^m \) to \( \mathbb{C} \) which are of bounded \( p \)-variation locally and having lacunary Walsh Fourier series having small gaps. Our new result generalizes and gives lacunary analogue of our earlier result (Theorem 3.3.2). For \( m = 1 \), our new result give our earlier result (Theorem 3.1.1).

Given a subset \( E \subset (\mathbb{Z}^+)^m \), a function \( f \in \text{L}^1(\mathbb{R}^m) \) is said to be \( E \)-spectral (or, said to have spectrum \( E \)) if and only if \( \hat{f}(n) = 0 \) for all \( n \) in \((\mathbb{Z}^+)^m \setminus E \). In what follows, we consider a set \( E \subset (\mathbb{Z}^+)^m \) described in the following way: For each \( j = 1, 2, \ldots, m \) consider sets \( E^{(j)} = \{n_0^{(j)}, n_1^{(j)}, n_2^{(j)}, \ldots\} \subset \mathbb{Z}^+ \) with \( \{n_k^{(j)}\}_{k=1}^\infty \) strictly
increasing for each \( j \) and satisfying the small gap conditions
\[
(n^{(j)}_{k+1} - n^{(j)}_k) \geq q \geq 1, \quad (k = 1, 2, \ldots; \ j = 1, 2, \ldots, m);
\] (3.15)
and then put \( E = \prod_{j=1}^{m} E^{(j)} \). \( \mathbf{n}_s = (n^{(1)}_{s_1}, n^{(2)}_{s_2}, \ldots, n^{(m)}_{s_m}) \) denotes the typical element of \( E \). When \( m = 1 \), \( E \) will be taken to be \( E^{(1)} \) with the superscript in \( n^{(1)}_k \)'s omitted.

**Theorem 3.4.1.** Let \( E \subset (\mathbb{Z}^+)^m \) be described as above and \( f: \mathbb{R}^m \to \mathbb{C} \) be 1-periodic in each variable. If \( f \in \text{BV}_V(p)(I) \cap L^p(I) \ (p \geq 1) \), where \( I \) is the rectangle \( I = [0, 2^{-N_1}] \times \ldots \times [0, 2^{-N_m}] \) in which \( 2^{-N_j} \geq 1/q \) for each \( j \); \( f \) is \( E \)-spectral and \( \mathbf{n}_k = (n^{(1)}_{k_1}, \ldots, n^{(m)}_{k_m}) \in (\mathbb{Z}^+)^m \) is such that \( n^{(j)}_{k_j} \) is sufficiently large for each \( j \), then
\[
\hat{f}(\mathbf{n}_k) = O \left( \frac{1}{|\prod_{j=1}^{m} n^{(j)}_{k_j}|^{1/p}} \right).
\]

**Proof.** For the sake of simplicity in writing, we carry out the proof for \( m = 2 \). For each \( j = 1, 2 \), consider the polynomial \( P_{N_j}(x_j) \) defined as follows: If \( N_j = 0 \), put \( P_{N_j} \equiv 1 \) and if \( N_j \in \mathbb{N} \) then put \( P_{N_j}(x_j) = \prod_{i=0}^{N_j-1}(1 + r_i(x_j)) \). Then as in the proof of Theorem 3.2.2, we have
\[
P_{N_j}(x_j) = \begin{cases} 
2^{N_j} & \text{if } x_j \in [0, 2^{-N_j}], \\
0 & \text{if } x_j \in [0, 1) \setminus [0, 2^{-N_j}).
\end{cases}
\]
Consider \( \mathbf{N} = (N_1, N_2) \) and put \( P_{\mathbf{N}}(x_1, x_2) = P_{N_1}(x_1)P_{N_2}(x_2) \). Then by the above property of \( P_{N_j} (j = 1, 2) \), we have
\[
P_{\mathbf{N}}(x_1, x_2) = \begin{cases} 
2^{N_1+N_2} & \text{if } (x_1, x_2) \in I, \\
0 & \text{if } (x_1, x_2) \in \mathbb{I}^2 \setminus I.
\end{cases}
\] (3.16)
We claim that if \( \mathbf{n}_k = (n^{(1)}_{k_1}, n^{(2)}_{k_2}) \in (\mathbb{Z}^+)^2 \) is such that \( \hat{f}(\mathbf{n}_k) \neq 0 \) then \( (fP_{\mathbf{N}})^\ast(\mathbf{n}_k) = \hat{f}(\mathbf{n}_k) \). In fact, writing \((x, y)\) in place of \((x_1, x_2)\), we have
\[
(fP_{\mathbf{N}})^\ast(\mathbf{n}_k) = \int_{I^2} f(x, y)P_{N_1}(x)P_{N_2}(y)\varphi^{(1)}_{n_{k_1}}(x)\varphi^{(2)}_{n_{k_2}}(y)dx\,dy
\]
\[
= \int_{I^2} f(x, y) \left( \prod_{i=0}^{N_1-1}(1 + r_i(x)) \right) \left( \prod_{j=0}^{N_2-1}(1 + r_j(y)) \right) \varphi_{n_{k_1}}^{(1)}(x)\varphi_{n_{k_2}}^{(2)}(y)dx\,dy
\]
\[
= \hat{f}(\mathbf{n}_k) + \sum_{i=0}^{N_1-1} \hat{f}(r_i\varphi_{n_k}) + \sum_{j=0}^{N_2-1} \hat{f}(r_j\varphi_{n_k}) + \sum_{i,j=0}^{N_1-1} \hat{f}(r_i r_j \varphi_{n_k}) + \sum_{i,j=0}^{N_2-1} \hat{f}(r_i r_j \varphi_{n_k}) + \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \hat{f}(r_i r_j \varphi_{n_k}) + \ldots + \hat{f}(r_0 \ldots r_{N_1-1} r_0 \ldots r_{N_2-1} \varphi_{n_k}). \tag{3.17}
\]
By our assumption the first term in the right hand side of (3.17) is nonzero. The characters appearing in the other terms in the right hand side of (3.17) are of the form \((\varphi_{n_{k_1}})(\psi_{n_{k_2}})\) where \(\varphi\) is (a function of \(x\) alone) such that \(\deg \varphi \leq N_1\) and \(\psi\) is (a function of \(y\) alone) such that \(\deg \psi \leq N_2\) and the degree of at least one of \(\varphi\) and \(\psi\) is nonzero. In view of the Payley ordering of Walsh characters, for each \(j \in \mathbb{N}\) there are totally \(2^j\) characters of degree \(j\), namely \(\varphi_{2^j - 1} = r_{j-1}, \varphi_{2^j - 1 + 1} = r^j_1\varphi_2, \ldots, \varphi_{2^j - 1} \equiv r_{j-1}\varphi_{2^j - 1} \equiv r_{j-2} \ldots r_1 r_0\). Consequently, total number of characters of positive degree \(\leq N\) is given by \(2^0 + 2^1 + 2^2 + \ldots + 2^{N-1} = 2^N - 1\); they are from \(\varphi_1\) to \(\varphi_2\). It follows that when \(\varphi_{n_{k_j}}\) is multiplied by any character of positive degree \(\leq N\) the resulting character \(\varphi_{m_j}\) is such that

\[
n_{k_j} < m_j \leq n_{k_j} + 2^{N_j} - 1 < n_{k_j} + 2^{N_j} \leq n_{k_j} + q \leq n_{k_{j+1}},
\]

in view of (3.15) and the fact that \(q \geq 2^{N_j}\). Since either \(\deg \varphi > 0\) or \(\deg \psi > 0\), either \(m_1 \notin E_1\) or \(m_2 \notin E_2\). Therefore \((m_1, m_2) \notin E\). Since \(f\) is \(E\)-spectral,

\[
\hat{f}\left((\varphi_{n_{k_1}})(\psi_{n_{k_2}})\right) = \hat{f}(\varphi_{m_1}\varphi_{m_2}) \equiv \hat{f}(m_1, m_2) = 0.
\]

Thus all the terms of the right hand side of (3.17) vanish except the first. This means that

\[
(fP_N)^{\cap}(n_k) = \hat{f}(n_k) \text{ if } \hat{f}(n_k) \neq 0. \tag{3.18}
\]

Now, let \(n_k = (n_{k_1}^{(1)}, n_{k_2}^{(2)})\) be such that \(n_{k_j}^{(i)}\) are large enough with \(\hat{f}(n_k) \neq 0\) and let \(m_j \in \mathbb{N}\) be such that \(2^{m_j} \leq n_{k_j}^{(i)} < 2^{m_j+1}\) with \(m_j > N_j\) for each \(j = 1, 2\). For simplicity in notation, let us write \(k, \ell, s\) and \(t\) for \(n_{k_1}^{(1)}, n_{k_2}^{(2)}, m_1\) and \(m_2\) respectively. Then \(2^s \leq k < 2^{s+1}, 2^\ell \leq \ell < 2^{\ell+1}\) and in view of (3.18) and (3.16) we have

\[
\hat{f}(n_k) = (fP_N)^{\cap}(n_k) = 2^{N_1+N_2} \int_0^{2^{-N_1}} \int_0^{2^{-N_2}} f(x, y) \varphi_k(x) \varphi_\ell(y) dxdy. \tag{3.19}
\]

Putting \(a_i = (i/2^s)\) for each \(i = 0, 1, 2, 3, \ldots, 2^s\) and \(b_j = (j/2^\ell)\) for each \(j = 0, 1, 2, 3, \ldots, 2^\ell\), as in the proof of Theorem 3.3.2, we get (3.10) and (3.11).

Next, define three functions \(f_1, f_2, f_3\) on \(I = [0, 2^{-N_1}) \times [0, 2^{-N_2})\) by setting

\[
f_1(x, y) = f(a_{i-1}, y) \quad (a_{i-1} \leq x < a_i; \ 0 \leq y < 2^{-N_2}) \quad \text{for} \ i = 1, 2, 3, \ldots, 2^{s-N_1};
\]
\[
f_2(x, y) = f(x, b_{j-1}) \quad (0 \leq x < 2^{-N_1}; \ b_{j-1} \leq y < b_j) \quad \text{for} \ j = 1, 2, 3, \ldots, 2^{\ell-N_2};
\]
and

\[
f_3(x, y) = f(a_{i-1}, b_{j-1}) \quad (a_{i-1} \leq x < a_i; \ b_{j-1} \leq y < b_j)
\]

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for \( i = 1, 2, 3, \ldots, 2^{s-N_1} \); \( j = 1, 2, 3, \ldots, 2^{t-N_2} \).

Then in view of Fubini’s theorem and relations (3.10) and (3.11) we have

\[
\int_0^{2^{-N_2}} \int_0^{2^{-N_1}} f_1(x, y) \varphi_k(x) \varphi_\ell(y) \, dx \, dy \\
= \int_0^{2^{-N_2}} \left[ \sum_{i=1}^{2^{s-N_1}} f(a_{i-1}, y) \int_{a_{i-1}}^{a_i} \varphi_k(x) \, dx \right] \varphi_\ell(y) \, dy = 0,
\]

\[
\int_0^{2^{-N_2}} \int_0^{2^{-N_1}} f_2(x, y) \varphi_k(x) \varphi_\ell(y) \, dx \, dy
\]

\[
= \int_0^{2^{-N_1}} \left[ \sum_{j=1}^{2^{t-N_2}} f(x, b_{j-1}) \int_{b_{j-1}}^{b_j} \varphi_\ell(y) \, dy \right] \varphi_k(x) \, dx = 0
\]

and

\[
\int_0^{2^{-N_2}} \int_0^{2^{-N_1}} f_3(x, y) \varphi_k(x) \varphi_\ell(y) \, dx \, dy
\]

\[
= \sum_{i=1}^{2^{s-N_1}} \sum_{j=1}^{2^{t-N_2}} f(a_{i-1}, b_{j-1}) \varphi_k(x) \left[ \int_{a_{i-1}}^{a_i} \varphi_\ell(y) \, dy \right] \left[ \int_{b_{j-1}}^{b_j} \varphi_\ell(y) \, dy \right] = 0.
\]

Using these equations in (3.19) we get

\[
|\hat{f}(n_k)| = 2^{N_1+N_2} \left| \int_0^{2^{-N_2}} \int_0^{2^{-N_1}} f(x, y) \varphi_k(x) \varphi_\ell(y) \, dx \, dy \right|
\]

\[
= 2^{N_1+N_2} \left| \int_0^{2^{-N_2}} \int_0^{2^{-N_1}} (f - f_1 - f_2 + f_3)(x, y) \varphi_k(x) \varphi_\ell(y) \, dx \, dy \right|
\]

\[
\leq 2^{N_1+N_2} \int_0^{2^{-N_2}} \int_0^{2^{-N_1}} |(f - f_1 - f_2 + f_3)(x, y)| \, dx \, dy
\]

\[
\leq 2^{N_1+N_2} \left( \int_0^{2^{-N_2}} \int_0^{2^{-N_1}} |(f - f_1 - f_2 + f_3)(x, y)|^p \, dx \, dy \right)^{1/p} \left(2^{-(N_1+N_2)}\right)^{1/q},
\]

in view of the Hölder’s inequality (when \( p > 1 \)) since \( f - f_1 - f_2 + f_3 \in L^p(I) \), where \( q \) is such that \( 1/p + 1/q = 1 \). Observe that when \( p = 1 \), we don’t use Hölder’s inequality and in that case we consider the inequality except last step.
In any case, it follows that

\[ |\hat{\hat{f}}(n_k)|^p \leq 2^{N_1+N_2} \int_0^{2^{-N_2}} \int_0^{2^{-N_1}} |f(x, y) - f_1(x, y) - f_2(x, y) + f_3(x, y)|^p \, dx \, dy \]

\[ = 2^{N_1+N_2} \sum_{i=1}^{2^{s-N_1}} \sum_{j=1}^{2^{s-N_2}} \int_{b_{j-1}}^{b_j} \int_{a_{i-1}}^{a_i} |f(x, y) - f_1(x, y) - f_2(x, y) + f_3(x, y)|^p \, dx \, dy \]

\[ = 2^{N_1+N_2} \sum_{i=1}^{2^{s-N_1}} \sum_{j=1}^{2^{s-N_2}} \int_{b_{j-1}}^{b_j} \int_{a_{i-1}}^{a_i} |f(x, y) - f(a_{i-1}, y) - f(x, b_{j-1}) + f(a_{i-1}, b_{j-1})|^p \, dx \, dy \]

\[ \leq 2^{N_1+N_2} \sum_{i=1}^{2^{s-N_1}} \sum_{j=1}^{2^{s-N_2}} (V_p(f; [a_{i-1}, a_i] \times [b_{j-1}, b_j]))^p (a_i - a_{i-1})(b_j - b_{j-1}) \]

\[ \leq \frac{2^{N_1+N_2+2}}{2s2^{t}} (V_p(f; I))^p \leq \frac{2^{N_1+N_2+2}}{k\ell} (V_p(f; I))^p, \]

in view of Lemma 2.1.8. Thus we get

\[ |\hat{\hat{f}}(n_k)| \leq \frac{2^{(N_1+N_2+2)/p}}{(k\ell)^{1/p}} V_p(f; I). \quad (3.20) \]

This completes the proof. \qed

**Remark 3.4.2.** Theorem 3.4.1 gives lacunary analogue of our earlier result Theorem 3.3.2. Since \( BV_{H}^{(p)}(I) \subset BV_{V}^{(p)}(I) \cap L^p(I) \) (in view of Lemma 2.1.12), Theorem 3.4.1 is true if we replace the assumption “\( f \in BV_{V}^{(p)}(I) \cap L^p(I) \)” by “\( f \in BV_{H}^{(p)}(I) \)”.

In that case it gives lacunary analogue of our earlier result Theorem 3.3.3 and simultaneously Walsh analogue of our earlier result Theorem 2.2.3.