Chapter 5

Fuzzy optimization problem under generalized convexity

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5.1 Introduction

A. A. K. Majumdar has proved the sufficient optimality conditions for multi objective optimization problems using the concept of convexity and generalized convexity in his paper [47]. Wu has proved the sufficient optimality conditions for an optimization problem with fuzzy-valued objective function and real constraints in [80] using pseudoconvexity of objective function. Using the approach of [47], in this chapter, we prove the sufficient optimality conditions for a non-dominated solution of a fuzzy optimization problem with fuzzy-valued objective function and fuzzy constraints, under the concept of convexity and generalized convexity of fuzzy-valued functions.
5.2 Problem and its solution

We consider the (NCFOP) defined in Chapter 4.

\[
\begin{align*}
\text{Minimize} & \quad \tilde{f}(x) = \tilde{f}(x_1, \ldots, x_n) \\
\text{Subject to} & \quad \tilde{g}_j(x) \preceq \tilde{0}, \quad j = 1, \ldots, m, \\
& \quad x \in X \subseteq \mathbb{R}^n.
\end{align*}
\]

where $X$ is an open set and $\tilde{f}$ and $\tilde{g}_j, j = 1, \ldots, m,$ are fuzzy-valued functions defined on $X$. Here we recall the definition of a (weak) non-dominated solution from Chapter 4.

**Definition 5.2.1.** Let $x_0 \in X_1 = \{x \in X : \tilde{g}_j(x) \preceq \tilde{0}, j = 1, \ldots, m\}$. We say that an $x_0$ is a non-dominated solution of (NCFOP) if there exists no $x_1(\not= x_0) \in X_1$ such that $\tilde{f}(x_1) \preceq \tilde{f}(x_0)$. It is said to be a weak non-dominated solution if there exists no $x_1 \in X_1$ such that $\tilde{f}(x_1) \preceq \tilde{f}(x_0)$.

To establish the sufficient optimality conditions for (NCFOP), we need the following theorem of alternatives.

**Theorem 5.2.1.** [T4] (Tucker’s theorem of alternatives) Let $A$ and $B$ be matrices of dimension $n$ by $m$ and $n$ by $p$ respectively, and let $x, y, u$ be column vectors of dimensions $m, p, n$ respectively. Then exactly one of the following system has a solution:

**System 1:** $A^t u \leq 0, \ A^t u \not= 0 , \ B^t u \leq 0$ for some $u$

**System 2:** $Ax + By = 0$ for some $x > 0, y \geq 0$.

5.3 Sufficient optimality conditions

Using the concept of convexity and generalized convexity of a fuzzy-valued function defined in Chapter 2, we prove the sufficient optimality conditions for $x_0$ to be a (weak) non-dominated solution of (NCFOP).

**Theorem 5.3.1.** (Sufficiency condition 1 for a weak non-dominated solution). Assume that an $x_0 \in X_1$ satisfies the following conditions (i)-(iii):

(i) $\tilde{f}(x), \tilde{g}_j(x), j = 1, \ldots, m,$ are H-differentiable at $x = x_0 \in X_1$;
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(ii) \( \hat{f}(x), \hat{g}_j(x), j = 1, ..., m \), are convex at \( x = \bar{x}_0 \in X_1 \);

(iii) there exist \( 0 \leq \mu_j \in \mathbb{R}, j = 1, ..., m \), such that

- \( \nabla \hat{f}^L_\alpha(x_0) + \nabla \hat{f}^U_\alpha(x_0) + \sum_{j=1}^{m} \nabla \hat{g}^U_{j0}(x_0) \cdot \mu_j = 0 \), for all \( \alpha \in [0,1] \);

- \( \mu_j \cdot \hat{g}^U_{j0}(x_0) = 0 \), for all \( j = 1, \ldots, m \).

Then, \( \bar{x}_0 \) is a weak non-dominated solution of \( \text{(NCFOP)} \).

Proof. Suppose that \( \bar{x}_0 \in X_1 \) is not weak non-dominated solution. Then there exists \( \bar{x}_1 \in X_1 \) such that \( f(\bar{x}_1) \prec f(\bar{x}_0) \). That is, there exists \( \bar{x}_1 \in X_1 \) such that

\[
\begin{align*}
\left\{
\hat{f}^L_\alpha(\bar{x}_1) < \hat{f}^L_\alpha(\bar{x}_0)
\right\} \text{ or } \left\{
\hat{f}^U_\alpha(\bar{x}_1) < \hat{f}^U_\alpha(\bar{x}_0)
\right\}
\end{align*}
\]

for all \( \alpha \in [0,1] \). Therefore

\[
\hat{f}^L_\alpha(\bar{x}_1) + \hat{f}^U_\alpha(\bar{x}_1) < \hat{f}^L_\alpha(\bar{x}_0) + \hat{f}^U_\alpha(\bar{x}_0),
\]

for all \( \alpha \in [0,1] \). That is,

\[
\hat{F}_\alpha(\bar{x}_1) - \hat{F}_\alpha(\bar{x}_0) < 0, \quad (5.3.1)
\]

where \( \hat{F}_\alpha(x) = \hat{f}^L_\alpha(x) + \hat{f}^U_\alpha(x) \), for all \( \alpha \in [0,1] \). By definition of partial ordering, we have

\[
X_1 = \{ \bar{x} \in X \subset \mathbb{R}^n : \hat{g}_j(\bar{x}) \preceq 0, j = 1, ..., m \}
\]

\[
= \{ \bar{x} \in X \subset \mathbb{R}^n : \hat{g}^U_{j0}(\bar{x}) \leq 0, \hat{g}^L_{j0}(\bar{x}) \leq 0, j = 1, ..., m \text{ and } \alpha \in [0,1] \}
\]

\[
= \{ \bar{x} \in X \subset \mathbb{R}^n : \hat{g}^U_{j0}(\bar{x}) \leq 0, \hat{g}^L_{j0}(\bar{x}) \leq 0, j = 1, ..., m \}
\]

Let \( \mathcal{J} = \{ j : \hat{g}^U_{j0}(x_0) = 0 \} \) is an index set of active constraints at \( x = \bar{x}_0 \). Since \( x_0, \bar{x}_1 \in X_1 \), for \( j \in \mathcal{J} \), we have

\[
\hat{g}^U_{j0}(\bar{x}_1) - \hat{g}^U_{j0}(x_0) \leq 0. \quad (5.3.2)
\]

Now using hypothesis (ii) of the Theorem and Theorem 2.6.1 from Preliminaries for (5.3.1) and (5.3.2), we have
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\[ \nabla F_\alpha(x_0)(x_1 - x_0) < 0, \ \nabla \hat{g}^U_\alpha(x_0)(x_1 - x_0) \leq 0, \ j \in J \text{ and for all } \alpha \in [0, 1]. \]

Thus, the following system of inequalities

\[ \nabla F_\alpha(x_0)z < 0, \text{ for all } \alpha \in [0, 1], \ \nabla \hat{g}^U_\alpha(x_0)z \leq 0 \text{ possess a solution } z = x_1 - x_0. \]

Therefore, by Tucker’s theorem of alternatives (refer Theorem 5.2.1), there exist no \( \lambda > 0 \) and \( \mu_j \geq 0 \) such that

\[ \nabla F_\alpha(x_0)\lambda + \sum_{j \in J} \nabla \hat{g}^U_\alpha(x_0) \cdot \mu_j = 0, \]

for all \( \alpha \in [0, 1] \). That is,

\[ \nabla f^L_\alpha(x_0) + \nabla f^U_\alpha(x_0) + \sum_{j \in J} \nabla \hat{g}^U_\alpha(x_0) \cdot \mu_j = 0, \]

where \( \mu_j = \mu_j' / \lambda \) and \( F_\alpha(x_0) = \hat{f}^L_\alpha(x_0) + \hat{f}^U_\alpha(x_0), \) for all \( \alpha \in [0, 1] \). Taking \( \mu_j = 0 \) for \( j = \{1, ..., m\} - J, \) we can still say that there exist no \( \mu_j \geq 0 \) for all \( j \in J \) such that

\[ \nabla f^L_\alpha(x_0) + \nabla f^U_\alpha(x_0) + \sum_{j=1}^{\infty} \nabla \hat{g}^U_\alpha(x_0) \cdot \mu_j = 0, \]

for all \( \alpha \in [0, 1] \). Therefore, we can say that there exist no \( \mu_j \geq 0, j = 1, ..., m \) such that

\[ \nabla f^L_\alpha(x_0) + \nabla f^U_\alpha(x_0) + \sum_{j=1}^{m} \nabla \hat{g}^U_\alpha(x_0) \cdot \mu_j = 0, \]

for all \( \alpha \in [0, 1] \), and \( \mu_j \cdot \hat{g}^U_\alpha(x_0) = 0 \) for \( j = 1, ..., m \). This contradicts to hypothesis (iii) of the Theorem. Hence, \( x_0 \) is a weak non-dominated solution of (NCFOP). \( \square \)

**Theorem 5.3.2.** (Sufficiency condition for a non-dominated solution). Assume that an \( \bar{x}_0 \in X_1 \) satisfies the following conditions (i)-(iii):

(i) \( \hat{f}(x), \hat{g}_j(x), j = 1, ..., m, \) are strictly pseudoconvex at \( x = x_0 \in X_1; \)

(ii) there exist \( 0 \leq \mu_j \in \mathbb{R}, j = 1, ..., m, \) such that
5.3. SUFFICIENT OPTIMALITY CONDITIONS

(a) \( \nabla \check{f}_\alpha(x_0) + \nabla \check{f}_\beta(x_0) + \sum_{j=1}^{m} \nabla \check{g}_{j0}(x_0) \cdot \mu_j = 0 \), for all \( \alpha \in [0, 1] \);

(b) \( \mu_j \cdot \check{g}_{j0}(x_0) = 0 \), for all \( j = 1, \ldots, m \).

Then, \( \bar{x}_0 \) is a non-dominated solution of (NCFOP).

Proof. Suppose \( \bar{x}_0 \) is not non-dominated solution, then there exists an \( \bar{x}_1(\neq \bar{x}_0) \in X_1 \) such that \( \bar{f}(x_1) \preceq \bar{f}(x_0) \).

i.e., \( \bar{f}_\alpha(x_1) \leq \bar{f}_\alpha(x_0) \) and \( \bar{f}_\beta(x_1) \leq \bar{f}_\beta(x_0) \), for all \( \alpha \in [0, 1] \).

By assumption of strictly pseudoconvexity of the function \( \bar{f}(x) \) at \( x = x_0 \), we have \( \bar{f}_\alpha(x) \) and \( \bar{f}_\beta(x) \) are also strictly pseudoconvex functions. Using the above inequalities, we obtain

\[
\nabla \bar{f}_\alpha(x_0)^t(x_1 - \bar{x}_0) < 0 \text{ and } \nabla \bar{f}_\beta(x_0)^t(x_1 - \bar{x}_0) < 0, \text{ for all } \alpha \in [0, 1].
\]

Furthermore, we have

\[
\check{g}_{j0}(x_1) - \check{g}_{j0}(x_0) \leq 0
\]

where \( j \in J = \{ j : \check{g}_{j0}(x_0) = 0 \} \) is an index set of active constraints at \( x = \bar{x}_0 \). Therefore, we have

\[
\nabla F_\alpha(x_0)^t(x_1 - \bar{x}_0) < 0
\]

and

\[
\nabla \check{g}_{j0}(x_0)^t(x_1 - x_0) < 0
\]

where \( F_\alpha(x_0) = \bar{f}_\alpha(x_0) + \bar{f}_\beta(x_0) \), for all \( \alpha \in [0, 1] \). Thus, the following system of inequalities

\[
\nabla F_\alpha(x_0)^t z < 0, \text{ for } \alpha \in [0, 1] \text{ and } \nabla \check{g}_{j0}(x_0)^t z < 0\]

possess a solution \( z = \bar{x}_1 - \bar{x}_0 \).

Therefore, by the Tucker’s theorem of alternatives (refer Theorem 5.2.1), there exist no \( \lambda > 0 \) and \( 0 \leq \mu_j' \in \mathbb{R}, j \in J \), such that

\[
\nabla F_\alpha(x_0)^t \lambda + \sum_{j \in J} \nabla \check{g}_{j0}(x_0) \cdot \mu_j' = 0,
\]

for all \( \alpha \in [0, 1] \). Using the similar arguments in the proof of Theorem 5.3.1, there exist
no $0 \leq \mu_j \in \mathbb{R}$, $j = 1, \ldots, m$ such that

$$\nabla f^L_\alpha(x_0) + \nabla f^U_\alpha(x_0) + \sum_{j=1}^m \nabla \tilde{g}^U_{j0}(x_0) \cdot \mu_j = 0,$$

for all $\alpha \in [0,1]$ and $\mu_j \cdot \tilde{g}^U_{j0}(x_0) = 0$ for $j = 1, \ldots, m$, violating hypothesis (ii) of the theorem. Hence, $x_0$ is a non-dominated solution of (NCFOP).

\[ \square \]

**Theorem 5.3.3. (Sufficiency condition 2 for a weak non-dominated solution). Assume that an $\bar{x}_0 \in X$ satisfies the following conditions (i)-(iii):

(i) $\tilde{f}(x)$ is pseudoconvex at $\overline{x} = x_0 \in X$;

(ii) $\tilde{g}_j(x)$ are quasiconvex and $H$-differentiable at $\overline{x}_0$, for $j = 1, \ldots, m$;

(iii) there exist $0 \leq \mu_j \in \mathbb{R}$, $j = 1, \ldots, m$, such that

(a) $\nabla f^L_\alpha(x_0) + \nabla f^U_\alpha(x_0) + \sum_{j=1}^m \nabla \tilde{g}^U_{j0}(x_0) \cdot \mu_j = 0$, for all $\alpha \in [0,1]$;

(b) $\mu_j \cdot \tilde{g}^U_{j0}(x_0) = 0$, $j = 1, \ldots, m$.

Then, $x_0$ is a weak non-dominated solution of (NCFOP).

**Proof.** Suppose that $x_0 \in X$ is not weak non-dominated solution. Then there exists $x_1 \in X$ such that $f(x_1) < f(x_0)$. That is, there exists $x_1 \in X$ such that

$$\begin{align*}
\left\{ \begin{array}{l}
\tilde{f}^L_\alpha(x_1) < \tilde{f}^L_\alpha(x_0) \\
\tilde{f}^U_\alpha(x_1) \leq \tilde{f}^U_\alpha(x_0)
\end{array} \right. \text{ or } \left\{ \begin{array}{l}
\tilde{f}^L_\alpha(x_1) \leq \tilde{f}^L_\alpha(x_0) \\
\tilde{f}^U_\alpha(x_1) < \tilde{f}^U_\alpha(x_0)
\end{array} \right. \text{ or } \left\{ \begin{array}{l}
\tilde{f}^L_\alpha(x_1) < \tilde{f}^L_\alpha(x_0) \\
\tilde{f}^U_\alpha(x_1) < \tilde{f}^U_\alpha(x_0)
\end{array} \right.$$

for all $\alpha \in [0,1]$. Therefore

$$\tilde{f}^L_\alpha(x_1) + \tilde{f}^U_\alpha(x_1) < \tilde{f}^L_\alpha(x_0) + \tilde{f}^U_\alpha(x_0),$$

for all $\alpha \in [0,1]$. That is,

$$F_\alpha(x_1) - F_\alpha(x_0) < 0,$$

(5.3.3)
where \( F_\alpha(x) = \int_0^1 f(x) \, d\alpha + \int_0^1 g(x) \, d\alpha \), for all \( \alpha \in [0, 1] \). By definition of partial ordering, we have

\[
X_1 = \{ \bar{x} \in X : g_j(x) \leq \bar{0}, j = 1, \ldots, m \} = \{ \bar{x} \in X : g_j^L(x) \leq 0 \text{ and } g_j^U(x) \leq 0, j = 1, \ldots, m \} = \{ \bar{x} \in X : g_j^L(x) \leq 0, j = 1, \ldots, m \} = \{ \bar{x} \in X : g_j^U(x) \leq 0, j = 1, \ldots, m \}
\]

Let \( J = \{ j : g_j^U(x_0) = 0 \} \) is an index set of active constraints at \( x = x_0 \). Since \( x_0, x_1 \in X_1 \), for \( j \in J \), we have

\[
\bar{g}_j^U(x_1) - \bar{g}_j^U(x_0) \leq 0. \tag{5.3.4}
\]

Now using hypothesis (i) and (ii) of the Theorem, from (5.3.3) and (5.3.4),

\[
\nabla F_\alpha(x_0)(x_1 - x_0) < 0, \quad \nabla \bar{g}_j^U(x_0)(x_1 - x_0) \leq 0, j \in J \text{ and for all } \alpha \in [0, 1].
\]

Thus, the following system of inequalities

\[
\nabla F_\alpha(x_0) z < 0, \text{ for all } \alpha \in [0, 1], \quad \nabla \bar{g}_j^U(x_0) z \leq 0 \text{ possess a solution } z = x_1 - x_0.
\]

Therefore, by Tucker's theorem of alternatives (refer Theorem 5.2.1), there exist no \( \lambda > 0 \) and \( \nu_j \geq 0 \) such that

\[
\nabla F_\alpha(x_0) \lambda + \sum_{j \in J} \nabla \bar{g}_j^U(x_0) \cdot \nu_j = 0,
\]

for all \( \alpha \in [0, 1] \). Using the similar arguments in the proof of Theorem 5.3.1, there exist no \( 0 \leq \nu_j \in R, j = 1, \ldots, m \) such that

\[
\nabla f^L(x_0) + \nabla f^U(x_0) + \sum_{j=1}^m \nabla g_j^U(x_0) \cdot \mu_j = 0,
\]

for all \( \alpha \in [0, 1] \) and \( \mu_j \cdot \bar{g}_j^U(x_0) = 0 \) for \( j = 1, \ldots, m \), violating hypothesis (iii) of the theorem. Hence, \( x_0 \) is a weak non-dominated solution of \( (NCFOP) \).

Now we prove sufficient optimality condition for \( (NCFOP) \) under quasiconvexity of a fuzzy-
valued objective function. We quote the following Theorem of quasiconvex functions from [30].

**Theorem 5.3.4.** [30] Let \( X^0 \subseteq \mathbb{R}^n \) be open. If a differentiable function \( f : X^0 \to \mathbb{R} \) is quasiconvex at a point \( x \in X^0 \), where \( \nabla f(x) \neq 0 \), then it is pseudoconvex at \( x \).

**Theorem 5.3.5.** (Sufficiency condition \( \beta \) for a weak non-dominated solution). Assume that, for \( x_0 \in X_1 = \{ \overline{x} \in X : \bar{g}_j(x) \leq 0, j = 1, ..., m \} \)

(i) \( \bar{f}, \bar{g}_j \; , \; j = 1, ..., m \) are \( H \)-differentiable at \( \bar{x}_0 \) and \( \nabla \bar{f}^L_\alpha(x_0) \neq 0 \) and \( \nabla \bar{f}^U_\alpha(x_0) \neq 0 \), for all \( \alpha \in [0,1] \).

(ii) \( \bar{f} \) and \( \bar{g}_j \; , \; j = 1, ..., m \) are quasiconvex functions at \( \bar{x}_0 \).

(iii) Let \( 0 \leq \mu_j \in \mathbb{R}, \; j = 1, ..., m \) and \( \bar{x}_0 \in X_1 \) satisfies the following conditions:

(a) \( \nabla \bar{f}^L_{\alpha}(x_0) + \nabla \bar{f}^U_{\alpha}(x_0) + \sum_{j=1}^m \nabla \bar{g}_{j0}(x_0) \cdot \mu_j = 0 \), for all \( \alpha \in [0,1] \);

(b) \( \mu_j \cdot \bar{g}_{j0}(x_0) = 0, \; j = 1, ..., m \).

Then, \( x_0 \) is a weak non-dominated solution of (NCFOP).

**Proof.** Suppose that \( x_1 \in X_1 \) is not weak non-dominated solution. Then there exists \( \bar{x}_1 \in X_1 \) such that \( f(x_1) < f(x_0) \). That is, there exists \( x_1 \in X_1 \) such that

\[
\begin{align*}
\begin{cases}
\bar{f}_{\alpha}(x_1) < \bar{f}_{\alpha}(x_0) \\
\bar{g}_{\alpha}(x_1) \leq \bar{g}_{\alpha}(x_0)
\end{cases}
\quad \text{or} \quad
\begin{cases}
\bar{f}_{\alpha}(x_1) \leq \bar{f}_{\alpha}(x_0) \\
\bar{g}_{\alpha}(x_1) < \bar{g}_{\alpha}(x_0)
\end{cases}
\quad \text{or} \quad
\begin{cases}
\bar{f}_{\alpha}(x_1) < \bar{f}_{\alpha}(x_0) \\
\bar{g}_{\alpha}(x_1) < \bar{g}_{\alpha}(x_0)
\end{cases}
\]

for all \( \alpha \in [0,1] \). Therefore

\[
\bar{f}_{\alpha}(x_1) + \bar{g}_{\alpha}(x_1) < \bar{f}_{\alpha}(x_0) + \bar{g}_{\alpha}(x_0),
\]

for all \( \alpha \in [0,1] \). That is,

\[
F_{\alpha}(x_1) - F_{\alpha}(x_0) < 0,
\]

where \( F_{\alpha}(x) = \bar{f}_{\alpha}(x) + \bar{g}_{\alpha}(x) \), for all \( \alpha \in [0,1] \). By definition of partial ordering, we have

\[
X_1 = \{ \overline{x} \in X : \bar{g}_j(x) \leq 0, j = 1, ..., m \}
= \{ \overline{x} \in X : \bar{g}^L_{j\alpha}(x) \leq 0 \ \text{and} \ \bar{g}^U_{j\alpha}(x) \leq 0, j = 1, ..., m \ \text{and} \ \alpha \in [0,1] \}
= \{ \overline{x} \in X : \bar{g}^L_{j\alpha}(x) \leq 0, j = 1, ..., m \}
= \{ \overline{x} \in X : \bar{g}^U_{j\alpha}(x) \leq 0, j = 1, ..., m \}
\]
Let $J = \{ j : \hat{g}_0^U(x_0) = 0 \}$ be an index set of active constraints at $x = x_0$. Since $x_0, x_1 \in X_1$, for $j \in J$, we have

$$\hat{g}_0^U(x_1) - \hat{g}_0^U(x_0) \leq 0. \quad (5.3.6)$$

Now using hypothesis (i) and (ii) of the Theorem, $\hat{f}_\alpha^L(x)$ and $\hat{f}_\alpha^R(x)$ are pseudocovex functions at $x_0$ (ref. Theorem 5.3.4), for all $\alpha \in [0, 1]$. Therefore, from (5.3.5) and (5.3.6),

$$\nabla F_\alpha(x_0)(x_1 - x_0) < 0, \quad \nabla \hat{g}_0^U(x_0)(x_1 - x_0) \leq 0, \quad j \in J \text{ and for all } \alpha \in [0, 1].$$

Thus, the following system of inequalities

$$\nabla F_\alpha(x_0)z < 0, \quad \forall \alpha \in [0, 1], \quad \nabla \hat{g}_0^U(x_0)z \leq 0$$

possess a solution $z = x_1 - x_0$.

Therefore, by Tucker’s theorem of alternatives (refer Theorem 5.2.1), there exist no $\lambda > 0$ and $\mu_j \geq 0$ such that

$$\nabla F_\alpha(x_0)\lambda + \sum_{j \in J} \nabla \hat{g}_0^U(x_0) \cdot \mu_j = 0,$$

for all $\alpha \in [0, 1]$. Using the similar arguments in the proof of Theorem 5.3.1, there exist no $0 \leq \mu_j \in \mathbb{R}$, $j = 1, \ldots, m$ such that

$$\nabla \hat{f}_\alpha^L(x_0) + \nabla \hat{f}_\alpha^R(x_0) + \sum_{j=1}^{m} \nabla \hat{g}_0^U(x_0) \cdot \mu_j = 0,$$

for all $\alpha \in [0, 1]$ and $\mu_j \cdot \hat{g}_0^U(x_0) = 0$ for $j = 1, \ldots, m$, violating assumption (iii) of the theorem. Hence, $x_0$ is a weak non-dominated solution of (NCFOP). \qed

### 5.4 Illustrations

Here we provide two examples to show the effect of fuzzy modeling of the following crisp type optimization problem.

#### Example 5.4.1.

Minimize $f(x_1, x_2) = 2 \cdot x_1^2 + 2 \cdot x_2^2$
Subject to: \( g(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 2)^2 \leq 3 \)

has the minimum point \((x_1^*, x_2^*) = (2 - \sqrt{3}/\sqrt{2}, 2 - \sqrt{3}/\sqrt{2})\) and minimum value is 
\( f(x_1^*, x_2^*) = 6.419. \)

Now we consider a fuzzy optimization problem having fuzzy coefficients and find the non-dominated solution using the optimality conditions.

**Example 5.4.2.** We consider the following fuzzy optimization problem

Minimize \( \hat{f}(x_1, x_2) = (\hat{2} \odot x_1^2) \oplus (\hat{2} \odot x_2^2) \)

Subject to: \( \hat{g}(x_1, x_2) = (\hat{1} \odot (x_1 - 2)^2) \oplus (\hat{1} \odot (x_2 - 2)^2) \leq \hat{3} \)

where \( \hat{2} = (0, 2, 3), \hat{1} = (-1, 1, 2) \) and \( \hat{3} = (2, 3, 4) \) are triangular fuzzy numbers as shown in Figure 5.1.

![Figure 5.1: Membership functions of triangular fuzzy numbers \( \hat{1} = (-1, 1, 2), \hat{2} = (0, 2, 3) \) and \( \hat{3} = (2, 3, 4) \)](image)

By arithmetics of fuzzy numbers, we obtain

\[
\hat{f}_L^L(x_1, x_2) = 2\alpha x_1^2 + 2\alpha x_2^2
\]

\[
\hat{f}_L^U(x_1, x_2) = (3 - \alpha)x_1^2 + (3 - \alpha)x_2^2 \text{ and}
\]

\[
\hat{g}_L^L(x_1, x_2) = (2 - \alpha)(x_1 - 2)^2 + (2 - \alpha)(x_2 - 2)^2 \leq (4 - \alpha).
\]
We also obtain

\[
\nabla \tilde{f}^L_\alpha(x_1, x_2) = \begin{pmatrix}
4\alpha x_1 \\
4\alpha x_2 
\end{pmatrix},
\]

\[
\nabla \tilde{f}^U_\alpha(x_1, x_2) = \begin{pmatrix}
2(3 - \alpha)x_1 \\
2(3 - \alpha)x_2 
\end{pmatrix}
\]

and

\[
\nabla \tilde{g}_0(x_1, x_2) = \begin{pmatrix}
2(x_1 - 2) \\
2(x_2 - 2) 
\end{pmatrix}.
\]

For checking the conditions (a) and (b) in Theorem 5.3.1., we need to solve

the following system of equations:

\[
\begin{align*}
\alpha x_1 + 3x_1 + 2\mu x_1 - 4\mu &= 0 \\
\alpha x_2 + 3x_2 + 2\mu x_2 - 4\mu &= 0 \\
\mu \cdot ((x_1 - 2)^2 + (x_2 - 2)^2 - 2) &= 0.
\end{align*}
\]

Then we get \((x_1, x_2) = (1, 1)\) and \(\mu = (\alpha + 3)/2\).

We see that \((x_1, x_2) = (1, 1)\) is the feasible solution to the given (NCFOP).

For any fixed \(\alpha \in [0, 1]\), we see that \(\tilde{f}^L_\alpha(x)\), \(\tilde{f}^U_\alpha(x)\) and \(\tilde{g}_0(x)\) are strictly convex functions at \(x = (1, 1)\), therefore from Theorem 5.3.1, we say that \((x_1^*, x_2^*) = (1, 1)\) is a weak nondominated solution to the given (NCFOP) and minimum value of the fuzzy-valued objective function is \(\tilde{\lambda} = (0, 4, 6)\) having \(\tilde{\lambda}_3^* = [4\alpha, 6 - 2\alpha]\). We defuzzify the minimum value using the center of area method given in [29] as 3.3333. If we compare with this solution with a solution to crisp type of optimization problem in Example 5.4.1 which is 6.419. We observe that by approximating coefficients as fuzzy numbers we get better minimum value.

Remark 5.4.1. In the above example, we have solved fuzzy optimization problem having fuzzy coefficients are non symmetric left spread triangular fuzzy numbers. If we spread non symmetric triangular fuzzy numbers on right side as shown in the following Figure
5.2. then weak non-dominated solution of the same fuzzy optimization problem is given
by \( (x_1, x_2) = (2 - 2/\sqrt{6}, 2 - 2/\sqrt{6}) \) and minimum value is \( \hat{f}(x_1, x_2) = (2.8, 5.6, 11.2) \). Its
defuzzified value is 6.533.

![Graph 1](image1.png)

Figure 5.2: Membership functions of triangular fuzzy numbers \( ̑ = (0, 1, 3) \) and \( \hat{\u0311} = (1, 2, 4) \)

If we consider fuzzy coefficients are symmetric triangular fuzzy numbers as shown in Figure
5.3, then the non-dominated solution will be \( (x_1, x_2) = (1, 1) \) and \( \mu = 4 \). In this case,
minimum value is \( \hat{\u0311} = (2, 4, 6) \) and its defuzzified value is \( \frac{3}{2} \).

![Graph 2](image2.png)

![Graph 3](image3.png)

Figure 5.3: Membership functions of triangular fuzzy numbers \( ̑ = (0, 1, 2) \) and \( \hat{\u0311} = (1, 2, 3) \)

Thus fuzzification of the parameters representing coefficients of the \( x_1^2 \) and \( x_2^2 \) in \( f \) and
coefficients of \( (x_1 - 2)^2, (x_2 - 2)^2 \) in \( g \) has a significant effect on the non-dominated solution
and the defuzzified value of the objective function.

Example 5.4.3. Consider the fuzzy optimization problem
Minimize \( \hat{f}(x_1, x_2) = (\hat{1} \circ x_1^2) \oplus (\hat{2} \circ x_2^3) \oplus (\hat{-3} \circ x_1) \oplus (\hat{-3} \circ x_2) \)

Subject to constraint: \( \hat{g}(x_1, x_2) = (\hat{3} \circ x_1) \oplus (\hat{5} \circ x_2) \oplus (\hat{-7}) \leq \hat{0} \)

where \( \hat{0} = (0,0,0) \) and \( \hat{1} = (0,1,2), \hat{\hat{3}} = (-4,-3,-2), \hat{\hat{3}} = (2,3,4), \hat{\hat{5}} = (4,5,6) \) and \( \hat{\hat{\hat{7}}} = (-8,-7,-6) \) are triangular fuzzy numbers.

Using arithmetics of fuzzy numbers, we obtain

\[
\hat{f}_\alpha(x_1, x_2) = \alpha x_1^2 + (1 + \alpha)x_2^2 + (-1 + \alpha)x_1 + (-1 + \alpha)x_2 \quad \text{and} \\
\hat{f}_\alpha^U(x_1, x_2) = (2 - \alpha)x_1^2 + (3 - \alpha)x_2^2 + (-2 - \alpha)x_1 + (-2 - \alpha)x_2.
\]

And

\[
\hat{g}_\alpha(x_1, x_2) = (2 + \alpha)x_1 + (4 + \alpha)x_2 + (-8 + \alpha) \quad \text{and} \\
\hat{g}_\alpha^U(x_1, x_2) = (4 - \alpha)x_1 + (6 - \alpha)x_2 + (-6 - \alpha).
\]

Now we have

\[
\nabla \hat{f}_\alpha(x_1, x_2) = \begin{pmatrix} 2x_1 + (-1 + \alpha) \\ 2(1 + \alpha)x_2 + (-1 + \alpha) \end{pmatrix},
\]

\[
\nabla \hat{f}_\alpha^U(x_1, x_2) = \begin{pmatrix} 2(2 - \alpha)x_1 - 2 - \alpha \\ 2(3 - \alpha)x_2 - 2 - \alpha \end{pmatrix},
\]

\[
\nabla \hat{g}_\alpha^U(x_1, x_2) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}.
\]

For checking the conditions (a) and (b) in Theorem 5.3.1, we solve the following system of equations:

\[
4x_1 - 6 + 4\mu = 0 \\
8x_2 - 6 + 6\mu = 0 \\
\mu \cdot (4x_1 + 6x_2 - 6) = 0.
\]

Then, we get \( (x_1, x_2) = (\frac{33}{31}, \frac{6}{17}) \) and \( \mu = \frac{9}{17} \) which is feasible solution to the given (NC-
FOP).

For any fixed $\alpha \in [0, 1]$, we see that $\hat{f}^L_\alpha(x)$, $\hat{f}^U_\alpha(x)$ are strictly convex, and $\hat{g}^U_\alpha(x)$ are convex functions at $(x) = \left(\frac{33}{34}, \frac{6}{17}\right)$. Therefore, by Theorem 5.3.1 we say that $(x_1, x_2) = \left(\frac{33}{34}, \frac{6}{17}\right)$ is a weak non-dominated solution.

Now we consider one more example which illustrate the Theorem 5.3.3.

**Example 5.4.4.** Consider the fuzzy optimization problem

\[
\text{Minimize } \hat{f}(x) = \hat{2} \circ x^3 \oplus \hat{-2} \circ x \\
\text{subject to constraint: } \hat{g}(x) = \hat{2} \circ x^3 \oplus \hat{-2} \leq \hat{0}
\]

where $\hat{2} = (1, 2, 3)$, $\hat{-2} = (-3, -2, -1)$ and $\hat{0} = (0, 0, 0)$ are triangular fuzzy numbers.

Using arithmetics of fuzzy numbers, we obtain

\[
\hat{f}^L_\alpha(x) = (1 + \alpha)x^3 + (\alpha - 3)x, \\
\hat{f}^U_\alpha(x) = (3 - \alpha)x^3 + (1 - \alpha)x.
\]

And

\[
\hat{g}^U_\alpha(x) = (3 - \alpha)x^3 + (1 - \alpha).
\]

Now we have

\[
D\hat{f}^L_\alpha(x) = 3(1 + \alpha)x^2 + (\alpha - 3), \\
D\hat{f}^U_\alpha(x) = 3(3 - \alpha)x^2 - (1 - \alpha), \\
D\hat{g}^U_\alpha(x) = 6x^2.
\]
5.5. CONCLUSIONS

By the conditions (a) and (b) in Theorem 5.3.3, we have the following system of equations:

\[3x^2 - 3 + 9x^2 - 4 + \mu t = 0\]
\[\mu \cdot (3x^2 - 4) = 0.

Solving the system, we get \(x_0 = -1/\sqrt{3}\) and \(\mu = 0\) feasible solution to the given (NCFOP).

For any fixed \(\alpha \in [0, 1]\), we see that \(\tilde{f}_U(x) = (1 + \alpha)x^3 + (3 + \alpha)x\), \(\tilde{f}_V(x) = (3 - \alpha)x^3 + (-1 - \alpha)x\) are pseudoconvex functions at \(x_0\) (refer Example 2.6.8 in Preliminaries), and \(\tilde{g}_0 = 3x^3 - 1\) is quasiconvex at \(x_0\) (refer Property 2.6.3 in Preliminaries).

Therefore, by Theorem 5.3.3 we say that \(-1/\sqrt{3}\) is a weak non-dominated solution.

5.5 Conclusions

In the current chapter, we have proposed the sufficient optimality conditions for obtaining a non-dominated solution of a constrained fuzzy optimization problem. The optimality conditions have been proved based on the assumptions of convexity and generalized convexity- pseudoconvexity and quasiconvexity of fuzzy-valued objective function and fuzzy constraints. We have worked out some examples of fuzzy optimization problems using these optimality conditions which shows the effect of fuzzy modeling also.