2.1 Introduction

Generalized Laplacian distribution corresponds to the distribution of differences of independently and identically distributed (i.i.d.) gamma random variables. Mathai (1993a,b,c), Mathai et al. (2006) discussed various generalizations of Laplace distribution and their applications in different contexts such as input-output processes, growth-decay mechanism, formation of sand dunes, growth of melatonin in human body, formation of solar neutrinos etc. Another variant namely, the skew Laplace distribution was introduced and studied in detail by Kozubowski and Podgórski (2008a,b). Skew Laplace distribution arises as a mix-

---

Some results included in this chapter form part of papers Jose and Manu (2009) and Jose and Manu (2011).
ture of normal distributions with stochastic variance having gamma distribution, and hence it is also called Variance Gamma Model. The tails of the variance gamma distribution decrease more slowly than the normal distribution. Due to the simplicity, flexibility and an excellent fit to empirical data, the variance gamma model is recently very popular among financial modelers.

In the present chapter, we consider the generalized Laplacian distribution and a more general case namely bilateral gamma distribution. We develop first order autoregressive processes with generalized Laplacian marginals and the generation of the process. The regression behaviour and simulation of sample paths are discussed. We consider the distribution of sums and the covariance structure. We introduce the geometric generalized Laplacian distribution and study its properties. Geometric generalized Laplacian processes are discussed. The $k^{th}$ order autoregressive processes are described. Bilateral gamma distribution and its estimation of parameters and a numerical illustration is also carried out.

### 2.2 Generalized Laplacian distribution and its properties

Mathai (1993a,b,c) introduced a generalized Laplacian distribution with parameters $\alpha$ and $\beta$, which has the characteristic function (c.f.),

$$
\phi(t) = \frac{1}{(1 + \beta^2t^2)^\alpha}, \quad \alpha > 0, \quad \beta > 0.
$$

Kotz et al. (2001) give a more general form of the generalized Laplacian distribution with more parameters and we concentrate on this form. The c.f is as follows

$$
\phi(t) = \frac{e^{i\theta t}}{(1 + i\sigma^2t^2 - i\mu t)^r}, \quad -\infty < t < \infty, \quad \sigma > 0, \quad -\infty < \mu < \infty.
$$
This c.f. can be factored as
\[ \phi(t) = e^{i \theta t} \left( \frac{1}{1 + \frac{i \sigma}{\sqrt{2} \kappa t}} \right)^\tau \left( \frac{1}{1 - \frac{i \sigma}{\sqrt{2} \kappa t}} \right)^\tau, \quad (2.2.1) \]

where \( \kappa > 0, \mu = \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right) \).

The probability density function (p.d.f.) of generalized Laplace distribution \( GL(\theta, \kappa, \sigma, \tau) \) is
\[ h(x) = \sqrt{2} e^{\frac{\sqrt{2} \sqrt{x - \theta}}{\kappa + \frac{1}{\kappa}}} \frac{2^{\frac{\tau}{2}} \Gamma(\frac{\tau}{2})}{\sqrt{\pi \sigma^{\tau + \frac{1}{2}}} \tau} K_{\tau - \frac{1}{2}} \left( \frac{\sqrt{2} (\kappa + \frac{1}{\kappa}) |x - \theta|}{2 \sigma} \right) \quad \text{for } x \neq \theta, \quad (2.2.2) \]

where \( K_\lambda \) is the modified Bessel function of the third kind with the index \( \lambda \), see Kotz et al. (2001). There are some special cases associated with this distribution. A standard GL density is obtained for \( \theta = 0 \) and \( \sigma = 1 \). For \( \tau = 1 \), we have an asymmetric Laplace, and for \( \kappa = 1 \) and \( \theta = 0 \), we obtain a symmetric Laplace distribution. The p.d.f. (2.2.2) can also be written as
\[ f(x) = (\alpha \beta)^\tau \exp \left( -\frac{(\alpha - \beta)x}{2} \right) \left( \frac{|x|}{\alpha + \beta} \right)^{\tau - 1/2} K_{\tau - 1/2} \left( \frac{\alpha + \beta}{2} |x| \right), \]

where \( \alpha \) and \( \beta \) describe the left and right-tail behavior and \( \tau \) is a parameter relating to the peakedness of the pdf (for details see Wu (2008)).

The mean value, variance, coefficients of skewness and kurtosis of generalized Laplacian distribution are obtained as

Mean value = \( \tau \mu \), Variance = \( \tau (\mu^2 + \sigma^2) \).

Coefficient of skewness = \( \frac{\mu (2 \mu^2 + 3 \sigma^2)}{\sqrt{\tau (\mu^2 + \sigma^2)^3/2.}} \).

Coefficient of kurtosis = \( 3 + \frac{3(2 \mu^4 + 4 \sigma^2 \mu^2 + 3 \sigma^4)}{\tau (\mu^2 + \sigma^2)^2} \).
From the factorization of the c.f. of the generalized Laplacian distribution, it can be noted that a variable $Y$ following $GL(\theta, \kappa, \sigma, \tau)$ can be represented as $Y \overset{d}{=} \theta + \frac{\sigma}{\sqrt{2}} (G_1 - \kappa G_2)$, where $G_1$ and $G_2$ are i.i.d. gamma random variables following one parameter gamma distribution with p.d.f.,

$$f_{G_i}(x) = \frac{1}{\Gamma(\tau)} e^{-\frac{x}{\tau}} x^{\tau-1}, \quad x > 0, \quad \tau > 0, \quad i = 1, 2.$$ 

The generalized Laplacian random variable $Y$ also admits the representation of the form $Y \overset{d}{=} \theta + \mu W + \sigma \sqrt{W} Z$, where $Z$ is a standard normal variate and $W$ is a gamma variate with parameters $(\tau, 1)$. Figure 2.1 gives the histogram and frequency curve corresponding to $GL(0, 2, 1, \frac{1}{2})$ obtained as the result of a simulation study.

### 2.3 Generalized Laplacian autoregressive processes

Consider the usual linear, additive first order autoregressive model given by

$$X_n = a X_{n-1} + \varepsilon_n, \quad 0 < a < 1, \quad n = 0, \pm 1, \pm 2, \ldots, \quad (2.3.1)$$
where \( \{\varepsilon_n\} \) is the innovation sequence of i.i.d. random variables. The basic problem is to find the distribution of \( \{\varepsilon_n\} \) such that \( \{X_n\} \) has the generalized Laplacian distribution \( GL(\theta, \kappa, \sigma, \tau) \) as the stationary marginal distribution. The following theorem summarizes the results in this context.

**Theorem 2.3.1.** A necessary and sufficient condition that an AR\((1)\) process of the form \( X_n = aX_{n-1} + \varepsilon_n \) is stationary with \( GL(\theta, \kappa, \sigma, \tau) \) marginals is that the innovations \( \{\varepsilon_n\} \) are i.i.d. as a convolution of the form \( X + Y_1 - Y_2 \) as in (2.3.3) provided \( \varepsilon_0 \overset{d}{=} X_0 \).

**Proof.** The necessary part can be proved as follows. In terms of the c.f., we can rewrite (2.3.1) in the form \( \phi_{X_n}(t) = \phi_{X_{n-1}}(at)\phi_{\varepsilon_n}(t) \).

Under stationarity this yields \( \phi_{\varepsilon}(t) = \frac{\phi_{X}(t)}{\phi_{X(at)}} \).

Substituting the c.f. given by (2.2.1) we get,

\[
\phi_{\varepsilon}(t) = \frac{e^{i\theta t} \left(1 + i\frac{\sigma}{\sqrt{2\kappa}} at\right)^{\tau} \left(1 - i\frac{\sigma}{\sqrt{2\kappa}} at\right)^{\tau}}{e^{i\theta at} \left(1 + i\frac{\sigma}{\sqrt{2\kappa}} at\right)^{\tau} \left(1 - i\frac{\sigma}{\sqrt{2\kappa}} t\right)^{\tau}} = e^{i\theta(1-a)t} \left[a + (1-a)\frac{1}{(1+i\frac{\sigma}{\sqrt{2\kappa}} at)}\right]^{\tau} \left[a + (1-a)\frac{1}{(1-i\frac{\sigma}{\sqrt{2\kappa}} t)}\right]^{\tau} \quad \text{(2.3.2)}
\]

This implies that \( \varepsilon \) has a convolution structure of the following form.

\[
\varepsilon \overset{d}{=} X + Y_1 - Y_2, \quad \text{(2.3.3)}
\]

where \( X \) is a degenerate random variable taking value \( \theta(1 - a) \) with probability one and \( Y_1 \) and \( Y_2 \) are \( \tau \)-fold convolutions of \( T_1 \) and \( T_2 \) where,

\[
T_1 = \begin{cases} 
0, & \text{with probability } a \\
E_1, & \text{with probability } 1-a
\end{cases}
\]

\[
T_2 = \begin{cases} 
0, & \text{with probability } a \\
E_2, & \text{with probability } 1-a
\end{cases}
\]
where $E_1$ and $E_2$ are exponential random variables with means $\frac{\sigma}{\sqrt{2}\kappa}$ and $\frac{\sigma}{\sqrt{2}\kappa}$ respectively.

When $\tau$ is integer valued, $Y_1$ and $Y_2$ can be easily generated as gamma convolutions.

The sufficiency part can be proved by the method of induction. We assume that $X_{n-1}$ follows $GL(\theta, \kappa, \sigma, \tau)$ with c.f. (2.2.1).

$$
\phi_{X_n}(t) = \phi_{X_{n-1}}(at)\phi_{\varepsilon_0}(t) = \left[ e^{i\theta at} \left( 1 + i \frac{\sigma}{\sqrt{2}\kappa}at \right)^\tau \left( 1 - i \frac{\sigma}{\sqrt{2}\kappa}at \right)^\tau \right] \left[ e^{i\theta(1-a)t} \left( 1 + i \frac{\sigma}{\sqrt{2}\kappa}at \right)^\tau \left( 1 - i \frac{\sigma}{\sqrt{2}\kappa}at \right)^\tau \right] = e^{i\theta t} \left( 1 + i \frac{\sigma}{\sqrt{2}\kappa}t \right)^\tau \left( 1 - i \frac{\sigma}{\sqrt{2}\kappa}t \right)^\tau,
$$

which is the same as the generalized Laplacian c.f. This shows that the process $\{X_n\}$ is strictly stationary with generalized Laplacian marginals, provided $\varepsilon_0 \overset{d}= X_0$ which follows $GL(\theta, \kappa, \sigma, \tau)$.

### 2.3.1 Generation of the process

Irrespective of whether $\tau$ is integer-valued or not, the process can be generated using a compound Poisson representation. In (2.3.2), the expression $\left[ a + (1-a) \frac{1}{(1-i\frac{\sigma}{\sqrt{2}\kappa}t)^\tau} \right]$ can be obtained as the c.f. of a compound Poisson distribution, see Lawrance (1982). In particular, this is the c.f. of the random sum $\sum_{i=1}^{N} a^U_i V_i$, where $\{V_i\}$ are gamma(1, $\frac{\sigma}{\sqrt{2}\kappa}$) random variables, $\{U_i\}$ are i.i.d. random variables uniformly distributed on (0,1), $N$ is a Poisson random variable of mean $\lambda = \tau \ln a$ and all these random variables are independent. This can be proved as follows.

Let $\eta = \sum_{i=1}^{N} a^U_i V_i$, where $\{V_i\}$ are gamma(1, $\frac{\sigma}{\sqrt{2}\kappa}$) random variables, $\{U_i\}$ are i.i.d. random variables uniformly distributed on (0,1), $N$ is a Poisson random variable of mean
−τ \ln a. Then the c.f. of \( \eta \) can be obtained as

\[
\phi_\eta(t) = \sum_{n=0}^{\infty} \left[ \phi_V(a^U t) \right]^n P[N = n] = \sum_{n=0}^{\infty} \left[ \phi_V(a^U t) \right]^n e^{\tau \ln a (-\tau \ln a)^n / n!} = \exp \left( \tau \ln a \left( 1 - \phi_V(a^U t) \right) \right).
\]

Similarly we can generate the other gamma process also and taking the difference we can obtain a generalized Laplacian process.

**Remark 2.3.1.** If \( X_0 \) is distributed arbitrary, then also the process is asymptotically stationary with generalized Laplacian marginal distribution.

**Proof.** We have

\[
X_n = aX_{n-1} + \varepsilon_n = a^n X_0 + \sum_{l=0}^{n-1} a^l \varepsilon_{n-l}.
\]

\[
\phi_{X_n}(t) = \phi_{X_0}(a^n t) \prod_{l=0}^{n-1} \phi_\varepsilon(a^l t) = \phi_{X_0}(a^n t) \{ \exp(i \theta t (1 - a) \sum_{l=0}^{n-1} a^l) \prod_{l=0}^{n-1} \left[ \frac{(1 + i \sigma \sqrt{2 \kappa} a^{l+1} t)^T (1 - i \sigma \sqrt{2 \kappa} a^{l+1} t)^T}{(1 + i \sigma \sqrt{2 \kappa} a^l t)^T (1 - i \sigma \sqrt{2 \kappa} a^l t)^T} \right] \}
\]

\[
\rightarrow \frac{e^{i \theta t}}{(1 + i \sigma \sqrt{2 \kappa} t)^T (1 - i \sigma \sqrt{2 \kappa} t)^T} \text{ as } n \to \infty.
\]
2.3.2 Regression behaviour and sample path properties

The forward regression equation for the process is given by

\[ E(X_n|X_{n-1} = x) = ax + (1 - a)[\theta + \tau \mu]. \]

Similarly the conditional variance is given by

\[ \text{Var}(X_n|X_{n-1} = x) = (1 - a^2)\tau(\mu^2 + \sigma^2) \quad \text{where} \quad \mu = \frac{\sigma}{\sqrt{2}} \left( \frac{1}{\kappa} - \kappa \right). \]

In the backward direction, the conditional distribution of \( X_n \) given \( X_{n+1} = x \) has non-linear regression and non-constant conditional variance. In this backward case we will proceed from the joint characteristic function of \( X_n \) and \( X_{n+1} \) following the steps described in Lawrance (1978). Then we have,

\[ \phi_{X_n,X_{n+1}}(t_1, t_2) = \frac{\phi_X(t_1 + at_2)\phi_X(t_2)}{\phi_X(at_2)}. \]

Differentiating this with respect to \( t_1 \) and setting \( t_1 = 0, t_2 = t \);

\[ i E[e^{itX_{n+1}}E(X_n|X_{n+1})] = \frac{\phi_X'(at)\phi_X(t)}{\phi_X(at)} = \phi_X'(at)\phi_c(t), \quad (2.3.4) \]

where \( \phi_c(t) \) is as defined in (2.3.2). Lawrance (1978) showed that linear backward regression is only possible in the Gaussian case and that non-normal autoregressive processes are not time reversible as a result due to Weiss (1977). In this case, \( E(X_n|X_{n-1}) \neq E(X_n|X_{n+1}) \) so that the process is not time reversible.

2.3.3 Observed sample path properties

The sample path properties of the process are studied by generating 100 observations each from the generalized Laplacian process with various parameter \( (\theta, \kappa, \sigma, \tau) \) combina-
Figure 2.2: (a) Sample path of the process for $a = 0.8$ and $(\theta, \kappa, \sigma, \tau) = (0, 1, 1, 2)$ (b) Sample path of the process for $a = 0.3$ and $(\theta, \kappa, \sigma, \tau) = (0, 1, 1, 2)$

Figure 2.3: (a) Sample path of the process for $a = 0.3$ and $(\theta, \kappa, \sigma, \tau) = (0, 1.3, 1, 2)$ (b) Sample path of the process for $a = 0.3$ and $(\theta, \kappa, \sigma, \tau) = (0, 1.2, 1.9, 2)$

In Figure 2.2(a), we take $a = 0.8$ and the values of $(\theta, \kappa, \sigma, \tau)$ as $(0, 1, 1, 2)$ and in Figures 2.2(b), 2.3(a) and 2.3(b) we take $a = 0.3$ and parameter values as $(0, 1, 1, 2), (0, 1.3, 1, 2)$ and $(0, 1.2, 1.9, 2)$ respectively. The process exhibits both positive and negative values with upward as well as downward runs as evident from the figures. These figures point out the rich variety of contexts where the newly developed time series models can be applied. These observations can be verified by referring to the table showing $P(X_n < X_{n-1})$. These probabilities are obtained by a simulation procedure. Sequences of 10,000 observations from generalized Laplacian process are generated repeatedly for ten times, and from each sequence the probability is estimated. Table 2.1 provides the
Table 2.1: \( P(X_n < X_{n-1}) \) for the generalized Laplacian process.

average of such probabilities from the ten trials along with an estimate of variance for different values of \( a \) and for different combinations of \((\theta, \kappa, \sigma, \tau)\).

### 2.3.4 Distribution of sums and joint distribution of \((X_n, X_{n+1})\)

When a stationary sequence \( X_n \) is used, the distribution of sums \( T_r = X_n + X_{n+1} + \cdots + X_{n+r-1} \) is important. We have

\[
X_{n+j} = a^j X_n + a^{j-1} \epsilon_{n+1} + a^{j-2} \epsilon_{n+2} + \cdots + \epsilon_{n+j}.
\]
Hence

\[
T_r = X_n + X_{n+1} + \cdots + X_{n+r-1} \\
= \sum_{j=0}^{r-1} [a^j X_n + a^{j-1} \varepsilon_{n+1} + \cdots + \varepsilon_{n+j}] \\
= X_n \left( \frac{1 - a^r}{1 - a} \right) + \sum_{j=1}^{r-1} \varepsilon_{n+j} \left( \frac{1 - a^{r-j}}{1 - a} \right).
\]

The c.f. of \( T_r \) is given by

\[
\phi_{T_r}(t) = \phi_{X_n} \left( t \frac{1 - a^r}{1 - a} \right) \prod_{j=1}^{r-1} \phi_{\varepsilon} \left( t \frac{1 - a^{r-j}}{1 - a} \right).
\]

\[
\phi_{T_r}(t) = \frac{e^{i\theta(t \frac{1 - a^r}{1 - a})}}{\left[ 1 + i \sigma \sqrt{2\kappa} \left( t \frac{1 - a^r}{1 - a} \right) \right]^{r}} \prod_{j=1}^{r-1} e^{i\theta(1 - a^{r-j})} \left[ a + (1 - a) \frac{1}{1 + i \sigma \sqrt{2\kappa} t \left( \frac{1 - a^{r-j}}{1 - a} \right)} \right]^{r} \times \left[ a + (1 - a) \frac{1}{1 - i \sigma \sqrt{2\kappa} t \left( \frac{1 - a^{r-j}}{1 - a} \right)} \right]^{r}.
\]

The distribution of \( T_r \) can be obtained by inverting the c.f. as the c.f. uniquely determines the distribution of a random variable. Now the joint distribution of \((X_n, X_{n+1})\) can be given
in terms of c.f. as

\[
\begin{align*}
\phi_{X_n, X_{n+1}}(t_1, t_2) &= E \left[ e^{i(t_1 X_n + i t_2 X_{n+1})} \right] \\
&= E \left[ e^{i(t_1 X_n + i t_2 (a X_n + \varepsilon_{n+1}))} \right] \\
&= E \left[ e^{i((t_1 + at_2) X_n + i t_2 \varepsilon_{n+1})} \right] \\
&= \phi_{X_n}(t_1 + at_2) \phi_{\varepsilon_{n+1}}(t_2) \\
&= e^{i\theta(t_1 + t_2)} \left[ a + (1 - a) \frac{1}{1 + i \frac{\sigma^2}{2} \kappa t_2} \right]^{\tau} \left[ a + (1 - a) \frac{1 - i \frac{\sigma^2}{2 \kappa} t_2}{1 - i \frac{\sigma^2}{2 \kappa} (t_1 + at_2)} \right]^{\tau}.
\end{align*}
\]

The above c.f. is not symmetric in \( t_1 \) and \( t_2 \), demonstrating once again that the process is not time reversible.

**Remark 2.3.2.** The covariance between \( X_n \) and \( X_{n-1} \) is

\[
\text{Cov}(X_n, X_{n-1}) = \theta^2 + (1 - a) \tau^2 \mu^2 - \theta \tau \mu.
\]

The correlation coefficient \( \rho \) is

\[
\rho = \frac{\theta^2 + (1 - a) \tau^2 \mu^2 - \theta \tau \mu}{(1 - a^2) \tau (\mu^2 + \sigma^2)}.
\]

When \( \theta = 0 \),

\[
\rho = \frac{\tau \mu^2}{(1 + a) (\mu^2 + \sigma^2)}.
\]

### 2.4 Geometric generalized Laplacian distribution

Here we introduce a new distribution namely the geometric generalized Laplacian distribution (\( GGLD(\theta, \mu, \sigma, \tau) \)). Its c.f. is given by

\[
\psi(t) = \frac{1}{1 + \tau \ln(1 + \frac{\tau}{2} \sigma^2 t^2 - i \mu t) - i \theta e}.
\]
The above c.f. \( \psi(t) \) can be written in the form \( \phi(t) = \exp \left( 1 - \frac{1}{\psi(t)} \right) \), where \( \phi(t) \) is the c.f. of the generalized Laplacian distribution, which is infinitely divisible. Therefore the geometric generalized Laplacian distribution is geometrically infinitely divisible (see Klebanov et al. (1984), Pillai and Sandhya (1990), Seethalekshmi and Jose (2004a,b; 2006)).

**Theorem 2.4.1.** Let \( X_1, X_2, \ldots \) be i.i.d. \( \text{GGLD}(0, \mu, \sigma, \tau) \) random variables and \( N \) be geometric with mean \( \frac{1}{p} \), such that \( P[N = k] = p(1 - p)^{k-1}, k = 1, 2, \ldots, 0 < p < 1 \). Then \( Y = X_1 + X_2 + \cdots + X_N \overset{d}{=} \text{GGLD}(0, \mu, \sigma, \tau/p) \).

**Proof.** The c.f. of \( Y \) is

\[
\phi_Y(t) = \sum_{k=1}^{\infty} [\phi_X(t)]^k p(1 - p)^{k-1} = \frac{p/(1 + \tau \ln(1 + t^2 \sigma^2 / 2 - i\mu t))}{1 - ((1 - p)/(1 + \tau \ln(1 + t^2 \sigma^2 / 2 - i\mu t)))} = \frac{1}{1 + (\tau/p) \ln(1 + t^2 \sigma^2 / 2 - i\mu t)}.
\]

Hence \( Y \overset{d}{=} \text{GGLD}(0, \mu, \sigma, \tau/p) \).

**Theorem 2.4.2.** Geometric generalized Laplacian distribution \( \text{GGLD}(0, \mu, \sigma, \tau) \) is the limit distribution of geometric sum of random variables following \( \text{GL}(0, \mu, \sigma, \tau/n) \).

**Proof.** We have \( [1 + t^2 \sigma^2 / 2 - i\mu t]^{-\tau/n} = \{1 + [1 + t^2 \sigma^2 / 2 - i\mu t]^{\tau/n} - 1\}^{-1} \) is the c.f. of a probability distribution since generalized Laplacian distribution is infinitely divisible. Hence by Lemma 3.2 of Pillai (1990),

\[
\phi_n(t) = \{1 + n[(1 + t^2 \sigma^2 / 2 - i\mu t)^{\tau/n} - 1]\}^{-1}
\]

is the c.f. of geometric sum of i.i.d. generalized Laplacian random variables. Taking limit
as \( n \to \infty \).

\[
\phi(t) = \lim_{n \to \infty} \phi_n(t) \\
= \{1 + \lim_{n \to \infty} n[(1 + \frac{t^2\sigma^2}{2} - i\mu t)^{\tau/n} - 1]\}^{-1} \\
= [1 + \tau \ln(1 + \frac{t^2\sigma^2}{2} - i\mu t)]^{-1}.
\]

### 2.4.1 Geometric generalized Laplacian processes

Here we develop a first order autoregressive process with geometric generalized Laplacian marginals. Consider an autoregressive structure given by

\[
X_n = \begin{cases} 
\varepsilon_n, & \text{with probability } p \\
X_{n-1} + \varepsilon_n, & \text{with probability } 1-p
\end{cases} \tag{2.4.2}
\]

where \( \{X_n\} \) and \( \{\varepsilon_n\} \) are independently distributed and \( 0 < p < 1 \). Now we shall construct an AR(1) process with stationary marginal distribution as \( \text{GGLD}(0, \mu, \sigma, \tau) \).

**Theorem 2.4.3.** Consider a stationary autoregressive process \( \{X_n\} \) with the structure as in (2.4.2). Then a necessary and sufficient condition that \( \{X_n\} \) is stationary with \( \text{GGLD}(0, \mu, \sigma, \tau) \) marginal distribution, is that \( \{\varepsilon_n\} \) follows \( \text{GGLD}(0, \mu, \sigma, p\tau) \).

**Proof.** In terms of c.f.s, we can write the above model as

\[
\phi_{X_n}(t) = p\phi_{\varepsilon_n}(t) + (1-p)\phi_{X_{n-1}}(t)\phi_{\varepsilon_n}(t).
\]

Assuming stationarity it reduces to the form

\[
\phi_X(t) = p\phi_{\varepsilon}(t) + (1-p)\phi_X(t)\phi_{\varepsilon}(t).
\]
Therefore \( \phi_\varepsilon(t) = \frac{\phi_X(t)}{p + (1 - p)\phi_X(t)} \).

Substituting,

\[
\phi_X(t) = \frac{1}{1 + \tau \ln(1 + \frac{1}{2}\sigma^2t^2 - i\mu t)}.
\]

We get

\[
\phi_\varepsilon(t) = \frac{1}{1 + p\tau \ln(1 + \frac{1}{2}\sigma^2t^2 - i\mu t)}.
\] (2.4.3)

Hence \( \varepsilon \sim GGLD(0, \mu, \sigma, p\tau) \).

Conversely assume that \( \{\varepsilon_n\} \) are identically distributed as given by (2.4.3), so that

\[
\phi_\varepsilon(t) = \frac{1}{1 + p\tau \ln(1 + \frac{1}{2}\sigma^2t^2 - i\mu t)}.
\]

Then

\[
\phi_X(t) = \frac{p\phi_\varepsilon(t)}{1 - (1 - p)\phi_\varepsilon(t)} = \frac{1}{1 + \tau \ln(1 + \frac{1}{2}\sigma^2t^2 - i\mu t)}.
\]

Hence it follows that \( X_n \) is distributed as \( GGLD(0, \mu, \sigma, \tau) \).

### 2.4.2 Extension to higher order processes

Consider a \( k^{th} \) order autoregressive process having the following structure,

\[
X_n = \begin{cases} 
\varepsilon_n, & \text{with probability } p, \\
a_1X_{n-1} + \varepsilon_n, & \text{with probability } p_1, \\
a_2X_{n-2} + \varepsilon_n, & \text{with probability } p_2, \\
\vdots & \vdots & \vdots \\
a_kX_{n-k} + \varepsilon_n, & \text{with probability } p_k,
\end{cases}
\]

where \( p_1 + p_2 + \cdots + p_k = 1 - p, \ 0 \leq p_i, p \leq 1, i = 1, 2, \ldots, k \) and \( \{\varepsilon_n\} \) is sequence of independently and identically distributed random variables.

When \( a_1 = a_2 = \cdots = a_k = a; \ 0 < a < 1 \), we can write the above model in terms of
characteristic functions as

$$\phi_{X_n}(t) = p\phi_{\varepsilon_n}(t) + p_1\phi_{X_{n-1}}(at)\phi_{\varepsilon_n}(t) + \cdots + p_k\phi_{X_{n-k}}(at)\phi_{\varepsilon_n}(t).$$

If \( \{X_n\} \) is stationary, then

$$\phi_X(t) = \phi_\varepsilon(t)[p + (1-p)\phi_X(at)].$$

That is

$$\phi_\varepsilon(t) = \frac{\phi_X(t)}{p + (1-p)\phi_X(at)}.$$ 

This shows that the innovation structure can be obtained as in the first order case.

### 2.5 Bilateral gamma distribution and bilateral gamma processes

Küchler and Tappe (2008a,b) introduced and studied bilateral gamma distribution and its various properties and also developed an associated Lévy process namely bilateral gamma processes which have applications in modelling financial market fluctuations. A bilateral gamma distribution with parameters \( \alpha_1, \lambda_1, \alpha_2, \lambda_2 > 0 \) (BG(\( \alpha_1, \lambda_1, \alpha_2, \lambda_2 \))) is defined as the distribution of the difference of two independent gamma variables with parameters (\( \alpha_1, \lambda_1 \)) and (\( \alpha_2, \lambda_2 \)). The characteristic function of a bilateral gamma distribution is

$$\varphi(t) = \left( \frac{\lambda_1}{\lambda_1 - it} \right)^{\alpha_1} \left( \frac{\lambda_2}{\lambda_2 + it} \right)^{\alpha_2}, \quad t \in \mathbb{R}. \quad (2.5.1)$$

The p.d.f. is

$$f(x) = \frac{\lambda_1^{\alpha_1} \lambda_2^{\alpha_2}}{(\lambda_1 + \lambda_2)^{\alpha_2} \Gamma(\alpha_1) \Gamma(\alpha_2)} e^{-\lambda_1 x} \int_0^\infty v^{\lambda_2-1} \left( x + \frac{v}{\lambda_1 + \lambda_2} \right)^{\alpha_1-1} e^{-v} dv, \quad x > 0.$$
The density function can also be expressed by means of Whittaker function

\[
f(x) = \frac{\lambda_1^{\alpha_1} \lambda_2^{\alpha_2}}{(\lambda_1 + \lambda_2)^{\frac{1}{2}(\alpha_1 + \alpha_2)}} \frac{1}{\Gamma(\alpha_2)} \times W_{\frac{1}{2}(\alpha_1 - \alpha_2), \frac{1}{2}(\alpha_1 + \alpha_2 - 1)}(x(\lambda_1 + \lambda_2)), \quad x > 0,
\]

(2.5.2)

where

\[
W_{\lambda,\mu}(z) = \frac{z^{\lambda} e^{\frac{z}{2}}}{\Gamma(\mu - \lambda + \frac{1}{2})} \int_0^\infty t^{\mu - \lambda - \frac{1}{2}} e^{-t} \left(1 + \frac{t}{z}\right)^{\mu + \lambda - \frac{1}{2}} dt \quad \text{for} \quad \mu - \lambda > -\frac{1}{2}.
\]

### 2.5.1 Geometric bilateral gamma distribution

Here, we introduce a new distribution namely the geometric bilateral gamma distribution (GBG(\(\alpha_1, \lambda_1, \alpha_2, \lambda_2\))). Its characteristic function is given by

\[
\psi(t) = \frac{1}{1 + \alpha_1 \ln(1 - \frac{it}{\lambda_1}) + \alpha_2 \ln(1 + \frac{it}{\lambda_2})}.
\]

(2.5.3)

The above characteristic function \(\psi(t)\) can be written in the form \(\varphi(t) = \exp\left(1 - \frac{1}{\psi(t)}\right)\), where \(\varphi(t)\) is the characteristic function of the bilateral gamma distribution, which is infinitely divisible. Therefore the geometric bilateral gamma distribution is geometrically infinitely divisible, see Klebanov et al. (1984).

**Theorem 2.5.1.** Let \(X_1, X_2, \ldots\) are i.i.d. GBG(\(\alpha_1, \lambda_1, \alpha_2, \lambda_2\)) random variables and \(N\) be geometric with mean \(\frac{1}{p}\), such that \(P[N = k] = p(1-p)^{k-1}, k = 1, 2, \ldots, 0 < p < 1\). Then \(Y = X_1 + X_2 + \cdots + X_N \overset{d}{=} \text{GBG}(\alpha_1/p, \lambda_1, \alpha_2/p, \lambda_2)\).
Proof. The characteristic function of $Y$ is

$$
\psi_Y(t) = \sum_{k=1}^{\infty} [\psi_X(t)]^k p(1 - p)^{k-1} = \frac{p/(1 + \alpha_1 \ln(1 - \frac{it}{\lambda_1}) + \alpha_2 \ln(1 + \frac{it}{\lambda_2}))}{1 - ((1 - p)/(1 + \alpha_1 \ln(1 - \frac{it}{\lambda_1}) + \alpha_2 \ln(1 + \frac{it}{\lambda_2})))} = \frac{1}{1 + (\alpha_1/p) \ln(1 - \frac{it}{\lambda_1}) + (\alpha_2/p) \ln(1 + \frac{it}{\lambda_2})}. 
$$

Hence $Y \overset{d}{=} \text{GBG}(\alpha_1/p, \lambda_1, \alpha_2/p, \lambda_2)$.

### 2.5.2 Geometric bilateral gamma processes

Here, we develop a first-order autoregressive process with geometric bilateral gamma marginals. Consider an autoregressive structure given by

$$
X_n = \begin{cases} 
\epsilon_n, & \text{with probability } p \\
X_{n-1} + \epsilon_n, & \text{with probability } 1 - p 
\end{cases} \quad (2.5.4)
$$

where $\{X_n\}$ and $\{\epsilon_n\}$ are independently distributed and $0 < p < 1$.

Now we shall construct an $AR(1)$ process with stationary marginal distribution as $\text{GBG}(\alpha_1, \lambda_1, \alpha_2, \lambda_2)$. The innovation structure can be derived by considering the characteristic functions on both sides of (2.5.4), which will yield

$$
\psi_{X_n}(t) = p\psi_{\epsilon_n}(t) + (1 - p)\psi_{X_{n-1}}(t)\psi_{\epsilon_n}(t). 
$$

Assuming stationarity it reduces to the form

$$
\psi_X(t) = p\psi_{\epsilon}(t) + (1 - p)\psi_X(t)\psi_{\epsilon}(t). 
$$

42
Therefore \( \psi_x(t) = \frac{\psi_X(t)}{p + (1-p)\phi_X(t)} \).

On substitution, we get
\[
\psi_x(t) = \frac{1}{1 + p\alpha_1 \ln(1 - \frac{\mu}{\lambda_1}) + p\alpha_2 \ln(1 + \frac{\mu}{\lambda_2})}.
\]

Hence \( \varepsilon_n \sim \text{GBG}(p\alpha_1, \lambda_1, p\alpha_2, \lambda_2) \).

### 2.5.3 Estimation of parameters

Bilateral gamma distributions are absolutely continuous with respect to the Lebesgue measure, because they are the convolution of two gamma distributions. In order to perform a maximum likelihood estimation, the density in terms of Whittaker function given in (2.5.2) is used. We take a sample of \( n \) observations. The logarithm of the likelihood function for \( \Theta = (\alpha_1, \alpha_2, \lambda_1, \lambda_2) \) is given by
\[
\ln L(\Theta) = n \left( \alpha_1 \ln \lambda_1 + \alpha_2 \ln \lambda_2 - \frac{\alpha_1 + \alpha_2}{2} \ln(\lambda_1 + \lambda_2) - \ln \Gamma(\alpha_2) \right)
+ \left( \frac{\alpha_1 + \alpha_2}{2} - 1 \right) \left( \sum_{i=1}^{n} \ln |x_i| \right) - \frac{\lambda_1 - \lambda_2}{2} \left( \sum_{i=1}^{n} x_i \right)
+ \sum_{i=1}^{n} \ln \left( W_{\frac{1}{2}}(\frac{\alpha_1 - \alpha_2}{2}(\frac{1}{2}(\lambda_1 + \lambda_2)) \right) (|x_i| (\lambda_1 + \lambda_2)) \right) \quad (2.5.5)
\]

The vector \( \hat{\Theta}_0 \) obtained from the method of moments is taken as starting point for an algorithm, for example the Hooke-Jeeves algorithm (see Quarteroni et al. (2002)), which maximizes the logarithmic likelihood function (2.5.5) numerically. This gives a maximum likelihood estimate \( \hat{\Theta} \) of the parameters. For more details see Küchler and Tappe (2008b).

Now we consider the estimation of parameters. We use the conditional least square approach to estimate the four parameters of bilateral gamma distribution. The conditional least square estimators are strongly consistent under certain conditions, see Klimko and Nelson (1978). Let \( x_0, x_1, ..., x_k \) be a given set of observations of an AR(1) process \( \{X_n\} \).
with bilateral gamma marginals. Then the conditional estimators of the parameters can be obtained by minimizing

\[ E = \sum_{n=1}^{k} [x_n - E(X_n|X_{n-1} = x_{n-1})]^2 = \sum_{n=1}^{k} \left[ x_n - ax_{n-1} - (1-a) \left( \frac{\alpha_1}{\lambda_1} - \frac{\alpha_2}{\lambda_2} \right) \right]^2. \]

We get the estimates of the parameters by solving the following equations recursively.

\[
\hat{a} = \frac{\sum_{n=1}^{k} [x_n - \left( \frac{\alpha_1}{\lambda_1} - \frac{\alpha_2}{\lambda_2} \right)] [x_{n-1} - \left( \frac{\alpha_1}{\lambda_1} - \frac{\alpha_2}{\lambda_2} \right)]}{\sum_{n=1}^{k} [x_{n-1} - \left( \frac{\alpha_1}{\lambda_1} - \frac{\alpha_2}{\lambda_2} \right)]^2}
\]

\[
\hat{\alpha}_1 = \frac{\lambda_1}{(1-a)k} \left[ \sum_{n=1}^{k} x_n - a \sum_{n=1}^{k} x_{n-1} \right] + \frac{\lambda_1}{\lambda_2} \alpha_2
\]

\[
\hat{\alpha}_2 = \frac{\lambda_2}{(1-a)k} \left[ a \sum_{n=1}^{k} x_{n-1} - \sum_{n=1}^{k} x_n \right] + \frac{\lambda_2}{\lambda_1} \alpha_1
\]

\[
\hat{\lambda}_1 = \left\{ \frac{1}{(1-a)\alpha_1} \left[ \sum_{n=1}^{k} x_n - a \sum_{n=1}^{k} x_{n-1} \right] + \frac{\alpha_2}{\alpha_1} \frac{1}{\lambda_1} \right\}^{-1}
\]

\[
\hat{\lambda}_2 = \left\{ \frac{1}{(1-a)\alpha_2} \left[ \sum_{n=1}^{k} x_{n-1} - a \sum_{n=1}^{k} x_n \right] + \frac{\alpha_1}{\alpha_2} \frac{1}{\lambda_2} \right\}^{-1}
\]

These equations can be easily solved using standard software packages like Matlab, Maple etc.

Tjetjep and Seneta (2006) discussed the generalized method of moments to estimate the parameters in situations where the method of moments is not easily tractable. This resembles the method of minimum chi-squared estimation but applied to moments. This is based on the quantity:

\[ M = \min_{\text{parameters}} \sum_{i} M^{(i)}, \]

where

\[ M^{(i)} = \left( \frac{M_P(i) - M_E(i)}{M_E(i)} \right)^2, \]
where $M_{P}(i)$ is the moment expressed in terms of parameters and $M_{E}(i)$ is the sample moment. When the value of $M = 0$ it is clear that the totality of estimating equations for the model are consistent with each other and that there is a solution for the parameter estimates.

2.5.4 Application to a financial data

Now we apply the model to a data on daily exchange rates of Pound to Indian rupee during the period 2004 to 2008. The data is available in ‘Hand Book of Statistics on the Indian Economy’, published by Reserve Bank of India.

From the data, the summary statistics obtained are as follows.

\[
\text{Mean value} = -1.8543 \times 10^{-5}, \quad \text{Variance} = 5.3628 \times 10^{-6},
\]

\[
\text{Coefficient of skewness} = -0.1139, \quad \text{Coefficient of kurtosis} = 4.0465.
\]
The parameter values are obtained by the generalized method of moments as

\[
\alpha_1 = 0.003927, \quad \alpha_2 = 0.007339, \quad \lambda_1 = 84.9434, \quad \lambda_2 = 101.0374.
\]

Here these estimated values make \( M \) equal to zero, showing that the model is consistent with the data. The histogram of the log transformed data is given in Figure 2.4(a) and the Q-Q plot is given in Figure 2.4(b). The Q-Q plot confirms that the distribution is a good fit for the data.

### 2.6 Conclusion

In this chapter the generalized Laplacian distribution is considered and its properties are studied. We developed first order autoregressive processes with generalized Laplacian marginals and the generation of the process. The regression behaviour and simulation of sample paths are discussed. We consider the distribution of sums and the covariance structure. We introduced the geometric generalized Laplacian distribution and its properties are studied. Geometric generalized Laplacian processes are discussed. The \( k^{th} \) order autoregressive processes are described. A more general case namely bilateral gamma distribution is also considered. Its parameters are estimated using conditional least squares and generalized method of moments and a numerical illustration is also carried out.

### References


