Chapter 5

CONSTRUCTION OF

$H$-SUPERMAGIC GRAPHS

5.1 Introduction

It is interesting to note that some families of graphs can be constructed from simple graphs. For example the windmill $W(r,k)$ is the onepoint union of $k(\geq 2)$ disjoint copies of the cycle $C_r$, generalised books are the edge amalgamation of a finite collection of cycles of same length. In this chapter we construct some $H$-supermagic graphs for a given graph $H$.

5.2 $H$-supermagic labeling of chain graph

In this section, we define a new graph called chain graph using a finite number of copies of a given graph and we exhibit an $H$-supermagic labeling of a chain graph.

Let $H_1, H_2, \cdots, H_n$ be copies of a given graph $H$. Let $u_i$ and $v_i$ be two distinct vertices of $H_i$ for $i = 1, 2, \cdots, n$. The chain graph $\mathcal{H}n$ of $H$ of length $n$ is
the graph obtained by identifying two vertices $u_i$ and $v_{i+1}$ for $i = 1, 2, \cdots, n - 1$.

The following theorem gives the necessary conditions for a chain of any 2-connected simple graph $H$ to be $H$-supermagic.

**Theorem 5.2.1.** Let $H$ be a 2-connected $(p, q)$ simple graph. Then $\mathcal{H}n$ is $H$-supermagic if any one of the following conditions is satisfied.

(i) $p + q$ is even.

(ii) $p + q + n$ is even.

**Proof:** Let $\mathcal{H}n = (V, E)$ be a chain of $n$ copies of $H$. Let us denote the $i^{th}$ copy of $H$ in $\mathcal{H}n$ by $H_i = (V_i, E_i)$. Note that $|V| = np - n + 1$ and $|E| = nq$. Moreover, we remark that by $H$ is a 2-connected graph, $\mathcal{H}n$ does not contain a subgraph $H$ other than $H_i$.

Let $v_i$ be the vertex in common with $H_i$ and $H_{i+1}$ for $1 \leq i \leq n - 1$. Let $v_0$ and $v_n$ be any two vertices in $H_1$ and $H_n$ respectively so that $v_0 \neq v_1$ and $v_n \neq v_{n-1}$.

Let $V'_i = V_i - \{v_{i-1}, v_i\}$ for $1 \leq i \leq n$.

**Case (1):** $p + q$ is even.

Suppose $p$ and $q$ are odd.

As $p - 2$ is odd, by Lemma 3.3.4, there exists an $n$-equipartition $P'_1 = \{X'_1, X'_2, \cdots X'_n\}$ of $[1, n(p-2)]$ such that

$$\sum P'_1 = \frac{(p-3)(np-n+1)}{2} + [1, n].$$

Adding $n+1$ to every integer of the set $[1, n(p-2)]$, we get an $n$-equipartition $P_1 = \{X_1, X_2, \cdots X_n\}$ of $[n+2, np-n+1]$ such that
\[ \sum P_1 = (p - 2)(n + 1) + \frac{(p - 3)(np - n + 1)}{2} + [1, n]. \]

Similarly, since \( q \) is odd, there exists an \( n \)-equipartition \( P_2 = \{Y_1, Y_2, \cdots Y_n\} \) of 
\[ (np - n + 1) + [1, nq] \] such that 
\[ \sum P_2 = q(np - n + 1) + \frac{(q - 1)(nq + n + 1)}{2} + [1, n]. \]

Define a total labeling \( f : V \cup E \to \{1, 2, 3, \cdots , np + nq - n + 1\} \) as follows:

- \( f(v_i) = i + 1 \) for \( 0 \leq i \leq n \).
- \( f(V_i') = X_{n-i+1} \) for \( 1 \leq i \leq n \).
- \( f(E_i) = Y_{n-i+1} \) for \( 1 \leq i \leq n \).

Then, for \( 1 \leq i \leq n \),

\[ f(H_i) = f(v_{i-1}) + f(v_i) + \sum f(V_i') + \sum f(E_i) \]
\[ = f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1} \]
\[ = \frac{n(p + q)^2 + 3(p + q) - 2n(p + q) + 2n - 2}{2}. \]

As \( H_i \cong H \) for \( 1 \leq i \leq n \), \( Hn \) is \( H \)-supermagic.

Suppose that both \( p \) and \( q \) are even.

As \( p \) is even, by Lemma 3.3.14, there exists an \( n \)-equipartition \( P'_1 = \{X'_1, X'_2, \cdots X'_n\} \) of \([1, n(p - 2)]\) such that \( \sum P'_1 \) is arithmetic progression of difference 2 and

\[ \sum P'_1 = \left\{ \frac{n ((p - 2)^2 - 2) + p - 4}{2} + 2r : 1 \leq r \leq n \right\}. \]
Adding \(n+1\) to every integer of the set \([1, n(p - 2)]\), we get an \(n\)-equipartition
\[
P_1 = \{X_1, X_2, \cdots X_n\} \text{ of } [n + 2, np - n + 1] \text{ such that}
\]
\[
\sum P_1 = \left\{(p - 2)(n + 1) + \frac{n[(p - 2)^2 - 2] + p - 4}{2} + 2i : 1 \leq i \leq n\right\}.
\]

As \(q\) is even, by Lemma 3.3.8, there exists an \(n\)-equipartition \(P'_2 = \{Y'_1, Y'_2, \cdots Y'_n\}\) of \([1, nq]\) such that
\[
\sum P'_2 = \left\{\frac{q(nq + 1)}{2}\right\}.
\]

Adding \(np - n + 1\) to every integer of the set \([1, nq]\) there exists an \(n\)-equipartition \(P_2 = \{Y_1, Y_2, \cdots Y_n\}\) of \((np - n + 1) + [1, nq]\) such that
\[
\sum P_2 = \left\{q(np - n + 1) + \frac{q(nq + 1)}{2}\right\}.
\]

Define a total labeling \(f : V \cup E \to \{1, 2, 3, \cdots , np + nq - n + 1\}\) as follows:
\[
f(v_i) = i + 1 \text{ for } 0 \leq i \leq n.
\]
\[
f(V'_i) = X_{n-i+1} \text{ for } 1 \leq i \leq n.
\]
\[
f(E_i) = Y_{n-i+1} \text{ for } 1 \leq i \leq n.
\]

Then, for \(1 \leq i \leq n\),
\[
f(H_i) = f(v_{i-1}) + f(v_i) + \sum f(V'_i) + \sum f(E_i)
\]
\[
= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1}
\]
\[
= \frac{n(p + q)^2 + 3(p + q) - 2n(p + q) + 2n - 2}{2}.
\]

As \(H_i \cong H\) for \(1 \leq i \leq n\), \(Hn\) is \(H\)-supermagic.
**Case (2):** $p + q + n$ is even.

Suppose $p$ is odd, $q$ is even and $n$ is odd.

Since $p$ is odd as in the proof of Case (1), there exists an $n$-equipartition $P_1 = \{X_1, X_2, \ldots X_n\}$ of $[n + 2, np - n + 1]$ such that

$$\sum P_1 = (p - 2)(n + 1) + \frac{(p - 3)(np - n + 1)}{2} + [1, n].$$

Since $q$ is even and $n$ is odd, by Lemma 3.3.6, there exists an $n$-equipartition $P_2' = \{Y'_1, Y'_2, \ldots Y'_n\}$ of $[1, nq]$ such that

$$\sum P_2' = \frac{(q - 1)(nq + n + 1)}{2} + [1, n].$$

Adding $np - n + 1$ to every integer of the set $[1, nq]$ there exists an $n$-equipartition $P_2 = \{Y_1, Y_2, \ldots Y_n\}$ of $(np - n + 1) + [1, nq]$ such that

$$\sum P_2 = q(np - n + 1) + \frac{(q - 1)(nq + n + 1)}{2} + [1, n].$$

Define a total labeling $f : V \cup E \to \{1, 2, 3, \ldots, np + nq - n + 1\}$ as follows:

- $f(v_i) = i + 1$ for $0 \leq i \leq n$.
- $f(V'_i) = X_{n-i+1}$ for $1 \leq i \leq n$.
- $f(E_i) = Y_{n-i+1}$ for $1 \leq i \leq n$.

Then for $1 \leq i \leq n$,

$$f(H_i) = f(v_{i-1}) + f(v_i) + \sum f(V'_i) + \sum f(E_i)$$

$$= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1}$$
As $H_i \cong H$ for $1 \leq i \leq n$, $\mathcal{H}n$ is $H$-supermagic.

Suppose $p$ is even, $q$ is odd and $n$ is odd.

Since $p - 2$ is even and $n$ is odd, by Lemma 3.3.6, there exists an $n$-equipartition $P'_1 = \{X'_1, X'_2, \cdots X'_n\}$ of $[1, n(p - 2)]$ such that

$$\sum P'_1 = \frac{(p - 3)[n(p - 2) + n + 1]}{2} + [1, n].$$

Adding $n + 1$ to every integer of the set $[1, n(p - 2)]$, we get an $n$-equipartition $P_1 = \{X_1, X_2, \cdots X_n\}$ of $[n + 2, np - n + 1]$ such that

$$\sum P_1 = (p - 2)(n + 1) + \frac{(p - 3)[n(p - 2) + n + 1]}{2} + [1, n].$$

Since $q$ is odd, as in Case (1), there exists an $n$-equipartition $P_2 = \{Y_1, Y_2, \cdots Y_n\}$ of $(np - n + 1) + [1, nq]$ such that

$$\sum P_2 = q(np - n + 1) + \frac{(q - 1)(nq + n + 1)}{2} + [1, n].$$

Define a total labeling $f : V \cup E \to \{1, 2, 3, \cdots, np + nq - n + 1\}$ as follows:

$f(v_i) = i + 1$ for $0 \leq i \leq n$.

$f(V'_i) = X_{n-i+1}$ for $1 \leq i \leq n$.

$f(E_i) = Y_{n-i+1}$ for $1 \leq i \leq n$.

Then, for $1 \leq i \leq n$,

$$f(H_i) = f(v_{i-1}) + f(v_i) + \sum f(V'_i) + \sum f(E_i).$$
\[ f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1} \]
\[ = \frac{n(p + q)^2 + 3(p + q) - 2n(p + q) + 2n - 2}{2}. \]

As \( H_i \cong H \) for \( 1 \leq i \leq n \), \( Hn \) is \( H \)-supermagic with supermagic sum

\[ s(f) = \frac{n(p + q)^2 + 3(p + q) - 2n(p + q) + 2n - 2}{2}. \]

\[ \square \]

**Corollary 5.2.2.** \( k \)-polygonal snake of length \( n \) is \( C_k \)-supermagic.

**Proof:** A \( k \)-polygonal snake of length \( n \) is nothing but the chain of the \( k \)-cycle \( C_k \) of length \( n \) in which the two vertices chosen in each copy of \( C_k \) for joining two copies are adjacent. Hence, \( k \)-polygonal snake of length \( n \) is \( C_k \)-supermagic. \( \square \)

**Example 5.2.3.** A \( H \)-supermagic labeling of a chain of a 2-connected \((5,7)\) simple graph \( H \) of length 5 with supermagic sum \( s(f) = 322 \) is shown in Figure 5.1.

![Figure 5.1: H-supermagic labeling of a chain Hs of a 2-connected (5,7) graph H.](image)
Example 5.2.4. A $H$-supermagic labeling of a chain of a 2-connected $(4,4)$ simple graph $H$ of length 4 with supermagic sum $s(f) = 111$ is shown in Figure 5.2.

![Figure 5.2: $C_4$-supermagic labeling of a chain of $C_4$ of length 4.](image)

Example 5.2.5. A $H$-supermagic labeling of a chain of a 2-connected $(5,6)$ simple graph $H$ of length 5 with supermagic sum $s(f) = 268$ is shown in Figure 5.3.

![Figure 5.3: $H$-supermagic labeling of the chain $H5$ of a 2-connected $(5,6)$ graph $H$.](image)
Example 5.2.6. A $H$-supermagic labeling of a chain of a 2-connected (6,9) simple graph $H$ of length 3 with supermagic sum $s(f) = 317$ is shown in Figure 5.4.

![Figure 5.4: $H$-supermagic labeling of a chain $H$ of a 2-connected (6,9) graph $H$.](image)

5.3 Edge amalgamation of a finite collection of graphs

For any finite collection $(G_i, u_i v_i)$ of graphs $G_i$, $1 \leq i \leq n$ each with a fixed edge $u_i v_i$, Carlson [13] defined the **edge amalgamation** $\text{Edgeamal}[(G_i, u_i v_i)]_{i=1}^n$ as the graph obtained by taking the union of all the $G_i$’s and identifying their fixed edges.

If all the $G_i$’s are cycles then $\text{Edgeamal}[(G_i, u_i v_i)]_{i=1}^n$ is called a generalized book. That is, if $n$ and $m$ be any positive integers with $n \geq 1$ and $m \geq 3$, then $n$ copies of the cycle $C_m$ with an edge in common is called a generalised book and we call it as a book with $n$ number of $m$-gon pages.

**Theorem 5.3.1.** Let $H$ be a 2-connected $(p, q)$ simple graph and $H_1, H_2, \ldots, H_n$ be $n$ graphs, each of which is isomorphic to $H$. Let $u_i v_i$ be the fixed edge of $H_i$ for $i = 1, 2, \ldots, n$. Then the edge amalgamation $\text{Edgeamal}[(H_i, u_i v_i)]_{i=1}^n$ is $H$-supermagic.
Proof: Let $H$ be a 2-connected simple graph with $p$ vertices and $q$ edges. Let $H_1, H_2, \ldots, H_n$ be $n$ graphs isomorphic to $H$. Let $u_iv_i$ be the fixed edge of $H_i$ for $i = 1, 2, \ldots, n$.

Let $G = \text{Edgeamal}((H_i, u_iv_i))_{i=1}^n$. Let $V = V(G)$ and $E = E(G)$. Then $|V| = n(p - 2) + 2$ and $|E| = n(q - 1) + 1$.

Let $H_i = (V_i, E_i)$ for $1 \leq i \leq n$. Label the common edge of $G$ as $e = w_1w_2$. Let $V'_i = V_i - \{w_1, w_2\}$ and $E'_i = E_i - \{e\}$ for $1 \leq i \leq n$.

Case 1: $n$ is odd.

Subcase (i): $p$ is even and $q$ is odd.

Since $p - 2$ and $q - 1$ are even, by Lemma 3.3.8, there exist $n$-equipartitions $\mathbb{P}'_1 = \{X'_1, X'_2, \ldots, X'_n\}$ of $[1, (p - 2)n]$ and $\mathbb{P}'_2 = \{Y'_1, Y'_2, \ldots, Y'_n\}$ of $[1, (q - 1)n]$ such that $\sum X'_i = \frac{(p-2)(pn-2n+1)}{2}$ and $\sum Y'_i = \frac{(q-1)(qn-n+1)}{2}$.

Add 2 to every integer of the set $[1, (p - 2)n]$ and $(p - 2)n + 3$ to every integer of the set $[1, (q - 1)n]$. We get $n$-equipartitions $\mathbb{P}_1 = \{X_1, X_2, \ldots, X_n\}$ of $[3, pn - 2n + 2]$ and $\mathbb{P}_2 = \{Y_1, Y_2, \ldots, Y_n\}$ of $[pn - 2n + 4, (p + q - 3)n + 3]$ such that $\sum X_i = 2(p - 2) + \frac{(p-2)(pn-2n+1)}{2}$ and $\sum Y_i = (q - 1)(pn - 2n + 3) + \frac{(q-1)(qn-n+1)}{2}$ for $1 \leq i \leq n$.

Define a total labeling $f : V \cup E \to [1, (p + q - 3)n + 3]$ as follows:

$f(w_1) = 1$ and $f(w_2) = 2$.

$f(e) = pn - 2n + 3$.

$f(V'_i) = X_i$ for $1 \leq i \leq n$.

$f(E'_i) = Y_{n-i+1}$ for $1 \leq i \leq n$.

Then for $1 \leq i \leq n$,
\[
f(H_i) = f(w_1) + f(w_2) + f(e) + \sum f(V'_i) + \sum f(E'_i)
\]
\[
= f(w_1) + f(w_2) + f(e) + \sum X'_i + \sum Y'_{n-i+1}
\]
\[
= \frac{n(p + q)^2 + p + q + 5(n - 1)}{2} - (n - 1)(2p + 3q)
\]
= constant.

Since \( H_i \equiv H \) for \( 1 \leq i \leq n \), \( G \) is \( H \)-supermagic.

**Sub case (ii):** \( p \) is odd and \( q \) is even.

Since \( p - 2 \) and \( q - 1 \) are odd, by Lemma 3.3.4, there exist \( n \)-equipartitions \( \mathbb{P}'_1 = \{X'_1, X'_2, \ldots, X'_n\} \) of \([1, (p - 2)n]\) and \( \mathbb{P}'_2 = \{Y'_1, Y'_2, \ldots, Y'_n\} \) of \([1, (q - 1)n]\) such that
\[
\sum X'_i = \frac{(p-3)(pn-n+1)}{2} + i \text{ and } \sum Y'_i = \frac{(q-2)(qn+1)}{2} + i \text{ for } 1 \leq i \leq n.
\]

Add 2 to every integer of the set \([1, (p - 2)n]\) and \((p - 2)n + 3\) to every integer of the set \([1, (q - 1)n]\). We get \( n \)-equipartitions \( \mathbb{P}_1 = \{X_1, X_2, \ldots, X_n\} \) of \([3, pn - 2n + 2]\) and \( \mathbb{P}_2 = \{Y_1, Y_2, \ldots, Y_n\} \) of \([pn - 2n + 4, (p + q - 3)n + 3]\) such that
\[
\sum X_i = 2(p - 2) + \frac{(p-3)(pn-n+1)}{2} + i \text{ and } \sum Y_i = (q - 1)(pn - 2n + 3) + \frac{(q-2)(qn+1)}{2} + i \text{ for } 1 \leq i \leq n.
\]

Define a total labeling \( f : V \cup E \to [1, (p + q - 3)n + 3] \) as follows:
\[
f(w_1) = 1 \quad \text{and} \quad f(w_2) = 2.
\]
\[
f(e) = pn - 2n + 3.
\]
\[
f(V'_i) = X_i \quad \text{for} \quad 1 \leq i \leq n.
\]
\[
f(E'_i) = Y_{n-i+1} \quad \text{for} \quad 1 \leq i \leq n.
\]

Then for \( 1 \leq i \leq n \),

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\[ f(H_i) = f(w_1) + f(w_2) + f(e) + \sum f(V'_i) + \sum f(E'_i) \]
\[ = f(w_1) + f(w_2) + f(e) + \sum X'_i + \sum Y'_{n-i+1} \]
\[ = \frac{n(p + q)^2 + p + q + 5(n - 1)}{2} - (n - 1)(2p + 3q) \]
\[ = \text{constant}. \]

Since \( H_i \equiv H \) for \( 1 \leq i \leq n \), \( G \) is \( H \)-supermagic.

**Sub case (iii):** \( p \) and \( q \) are odd.

Since \( p - 2 \) is odd, by Lemma 3.3.4, there exists an \( n \)-equipartition \( \mathbb{P}'_1 = \{X'_1, X'_2, \ldots, X'_n\} \) of \([1, (p - 2)n]\) such that \( \sum X'_i = \frac{(p-3)(pn-n+1)}{2} + i \) for \( 1 \leq i \leq n \).

Since \( q - 1 \) is even and \( n \) is odd, by Lemma 3.3.6, there exists an \( n \)-equipartition \( \mathbb{P}'_2 = \{Y'_1, Y'_2, \ldots, Y'_n\} \) of \([1, (q - 1)n]\) such that \( \sum Y'_i = \frac{(q-2)(qn+1)}{2} + i \) for \( 1 \leq i \leq n \).

Add 2 to every integer of the set \([1, (p - 2)n]\) and \((p - 2)n + 3\) to every integer of the set \([1, (q - 1)n]\). We get \( n \)-equipartitions \( \mathbb{P}_1 = \{X_1, X_2, \ldots, X_n\} \) of \([3, pn - 2n + 2]\) and \( \mathbb{P}_2 = \{Y_1, Y_2, \ldots, Y_n\} \) of \([pn - 2n + 4, (p + q - 3)n + 3]\) such that \( \sum X_i = 2(p - 2) + \frac{(p-3)(np-n+1)}{2} + i \) and \( \sum Y_i = (q - 1)(pn - 2n + 3) + \frac{(q-2)(qn+1)}{2} + i \) for \( 1 \leq i \leq n \).

Define a total labeling \( f : V \cup E \to [1, (p + q - 3)n + 3] \) as follows:
\[ f(w_1) = 1 \quad \text{and} \quad f(w_2) = 2. \]
\[ f(e) = pn - 2n + 3. \]
\[ f(V'_i) = X_i \quad \text{for} \quad 1 \leq i \leq n. \]
\[ f(E_i') = Y_{n-i+1} \quad \text{for} \quad 1 \leq i \leq n. \]

Then for \( 1 \leq i \leq n \),

\[
\begin{align*}
f(H_i) &= f(w_1) + f(w_2) + f(e) + \sum f(V_i') + \sum f(E_i') \\
&= f(w_1) + f(w_2) + f(e) + \sum X_i' + \sum Y_{n-i+1}' \\
&= \frac{n(p + q)^2 + p + q + 5(n - 1)}{2} - (n - 1)(2p + 3q) \\
&= \text{constant}.
\end{align*}
\]

Since \( H_i \cong H \) for \( 1 \leq i \leq n \), \( G \) is \( H \)-supermagic.

**Sub case (iv):** \( p \) and \( q \) are even.

Since \( p - 2 \) is even and \( n \) is odd, by Lemma 3.3.6, there exists an \( n \)-equipartition \( \mathbb{P}_1' = \{X_1', X_2', \ldots, X_n'\} \) of \( [1, (p - 2)n] \) such that \( \sum X_i' = \frac{(p - 3)(pn - n + 1)}{2} + i \) for \( 1 \leq i \leq n \).

Since \( q - 1 \) is odd, by Lemma 3.3.4, there exists an \( n \)-equipartition \( \mathbb{P}_2' = \{Y_1', Y_2', \ldots, Y_n'\} \) of \( [1, (q - 1)n] \) such that \( \sum Y_j' = \frac{(q - 2)(qn + 1)}{2} + i \) for \( 1 \leq i \leq n \).

Add 2 to every integer of the set \([1, (p - 2)n]\) and \((p - 2)n + 3\) to every integer of the set \([1, (q - 1)n]\). We get \( n \)-equipartitions \( \mathbb{P}_1 = \{X_1, X_2, \ldots, X_n\} \) of \([3, pn - 2n + 2]\) and \( \mathbb{P}_2 = \{Y_1, Y_2, \ldots, Y_n\} \) of \([pn - 2n + 4, (p + q - 3)n + 3]\) such that \( \sum X_i = 2(p - 2) + \frac{(p - 3)(pn - n + 1)}{2} + i \) and \( \sum Y_i = (q - 1)(pn - 2n + 3) + \frac{(q - 2)(qn + 1)}{2} + i \) for \( 1 \leq i \leq n \).

Define a total labeling \( f : V \cup E \to [1, (p + q - 3)n + 3] \) as follows:

\( f(w_1) = 1 \) and \( f(w_2) = 2. \)
\[ f(e) = pn - 2n + 3. \]

\[ f(V'_i) = X_i \quad \text{for} \quad 1 \leq i \leq n. \]

\[ f(E'_i) = Y_{n-i+1} \quad \text{for} \quad 1 \leq i \leq n. \]

Then for \( 1 \leq i \leq n \),

\[
\begin{align*}
f(H_i) &= f(w_1) + f(w_2) + f(e) + \sum f(V'_i) + \sum f(E'_i) \\
&= f(w_1) + f(w_2) + f(e) + \sum X'_i + \sum Y'_{n-i+1} \\
&= \frac{n(p + q)^2 + p + q + 5(n - 1)}{2} - (n - 1)(2p + 3q) \\
&= \text{constant}.
\end{align*}
\]

Since \( H_i \equiv H \) for \( 1 \leq i \leq n \), \( G \) is \( H \)-supermagic.

**Case 2:** \( n \) is even.

**Sub case (i):** \( p \) is even and \( q \) is odd.

A similar argument as in Sub case (i) of Case (1) shows that \( G \) is \( H \)-supermagic.

**Sub case (ii):** \( p \) is odd and \( q \) is even.

A similar argument as in Sub case (ii) of Case (1) shows that \( G \) is \( H \)-supermagic.

**Sub case (iii):** \( p \) and \( q \) are odd.

Since \( p - 2 \) is odd, by Lemma 3.3.4, there exists an \( n \)-equipartition \( P'_1 = \{X'_1, X'_2, \ldots, X'_n\} \) of \([1, (p - 2)n]\) such that \( \sum X'_i = \frac{(p-3)(pn-n+1)}{2} + i \) for \( 1 \leq i \leq n \).

Since \( q - 1 \) and \( n \) are even, by Lemma 3.3.10, there exists an \( n \)-equipartition \( P'_2 = \{Y'_1, Y'_2, \ldots, Y'_n\} \) of \([1, (q - 1)n+1]-\{\frac{n}{2}+1\}\) such that \( \sum Y'_i = \frac{(q-1)^2(n+3(q-1)-n-2)+i}{2} \).
for $1 \leq i \leq n$.

Add 2 to every integer of the set $[1, (p-2)n]$ and $(p-2)n+2$ to every integer of the set $[1, (q-1)n]$. We get $n$-equipartitions $\mathbb{P}_1 = \{X_1, X_2, \ldots, X_n\}$ of $[3, pn-2n+2]$ and $\mathbb{P}_2 = \{Y_1, Y_2, \ldots, Y_n\}$ of $[pn-2n+3, (p+q-3)n+3]-(p-2)n+\frac{n}{2}+3$ such that for $1 \leq i \leq n$,

$$
\sum X_i = 2(p-2) + \frac{(p-3)(pn-n+1)}{2} + i \quad \text{and} \quad \sum Y_i = (q-1)(pn-2n+2) + \frac{(q-1)^2n+3(q-1)-n-2}{2} + i.
$$

Define a total labeling $f : V \cup E \rightarrow [1, (p+q-3)n+3]$ as follows:

- $f(w_1) = 1$ and $f(w_2) = 2$.
- $f(e) = (p-2)n + \frac{n}{2} + 3$.
- $f(V'_i) = X_i$ for $1 \leq i \leq n$.
- $f(E'_i) = Y_{n-i+1}$ for $1 \leq i \leq n$.

Then for $1 \leq i \leq n$,

$$
f(H_i) = f(w_1) + f(w_2) + f(e) + \sum f(V'_i) + \sum f(E'_i)
$$

$$
= f(w_1) + f(w_2) + f(e) + \sum X'_i + \sum Y'_{n-i+1}
$$

$$
= \frac{n(p+q)^2 + p + q}{2} - (n-1)(2p+3q-3)
$$

$$
= \text{constant}.
$$

Since $H_i \cong H$ for $1 \leq i \leq n$, $G$ is $H$-supermagic.

**Sub case (iv):** $p$ and $q$ are even.

Since $p-2$ and $n$ are even, by Lemma 3.3.12, there exists an $n$-equipartition
\[ \mathbb{P}_1 = \{X_1, X_2, \cdots, X_n\} \text{ of } [1, (p-2)n+2]-\{1, \frac{n}{2}+2\} \text{ such that } \sum X_i = \frac{(p-2)^2n+5(p-2)-n-2}{2} + i \text{ for } 1 \leq i \leq n. \]

Since \( q - 1 \) is odd, by Lemma 3.3.4, there exists an \( n \)-equipartition \( \mathbb{P}'_2 = \{Y'_1, Y'_2, \cdots, Y'_n\} \) of \([1, (q-1)n]\) and \( \sum Y'_i = \frac{(q-2)(qn+1)}{2} + i \) for \( 1 \leq i \leq n \).

Add \((p-2)n+3\) to every integer of the set \([1, (q-1)n]\). We get an \( n \)-equipartition \( \mathbb{P}_2 = \{Y_1, Y_2, \cdots, Y_n\} \) of \([pn-2n+4, (p+q-3)n+3]\) such that
\[ \sum Y_i = (q-1)(pn-2n+3) + \frac{(q-2)(qn+1)}{2} + i \text{ for } 1 \leq i \leq n. \]

Define a total labeling \( f: V \cup E \to [1, (p+q-3)n+3] \) as follows:
\[ f(w_1) = 1 \text{ and } f(w_2) = \frac{n}{2} + 2. \]
\[ f(e) = pn-2n+3. \]
\[ f(V'_i) = X_i \text{ for } 1 \leq i \leq n. \]
\[ f(E'_i) = Y_{n-i+1} \text{ for } 1 \leq i \leq n. \]

Then for \( 1 \leq i \leq n \),
\[ f(H_i) = f(w_1) + f(w_2) + f(e) + \sum f(V'_i) + \sum f(E'_i) = f(w_1) + f(w_2) + f(e) + \sum X'_i + \sum Y'_{n-i+1} = \frac{n(p+q)^2 + p + q}{2} - (n-1)(2p + 3q - 3) = \text{constant.} \]

Since \( H_i \equiv H \) for \( 1 \leq i \leq n \), \( G \) is \( H \)-supermagic.

Hence, the edge amalgamation \( \text{Edgeamal}((H_i, u_iV_i))_{i=1}^n \) is \( H \)-supermagic. \( \square \)
Example 5.3.2. The edge amalgamation of three graphs isomorphic to the given 2-connected (4,5)-simple graph $H$ with supermagic sum $s(f)=85$ is given in Figure 5.5.

Figure 5.5: $H$-supermagic labeling of an edge amalgamation of 3 copies of a 2-connected (4,5) graph $H$.

Example 5.3.3. Let $H_1, H_2, H_3, H_4$ and $H_5$ be five graphs isomorphic to the wheel $W_4 = C_4+K_1$ with their fixed edges given by lines with an arrow mark in the middle. A $W_4$-supermagic labeling $f$ of the edge amalgamation graph $\text{Edgeamal}((H_i, u_i v_i))_{i=1}^5$ with supermagic sum $s(f)=303$ is given in Figure 5.6.

Figure 5.6: $W_4$-supermagic labeling of an edge amalgamation graph.
Example 5.3.4. The edge amalgamation of three copies of the complete graph $K_3$ with supermagic sum $s(f)=32$ is given in Figure 5.7.

![Figure 5.7](image)

Figure 5.7: $K_3$-supermagic labeling of an edge amalgamation of three copies of $K_3$.

Example 5.3.5. Edge amalgamation of three graphs isomorphic to a given 2-connected (6,8)-simple graph $H$ with supermagic sum $s(f)=234$ is given in Figure 5.8. The fixed edges are the edges with arrow mark in the middle.

![Figure 5.8](image)

Figure 5.8: $H$-supermagic labeling of an edge amalgamation of three copies of a 2-connected (6,8)-simple graph $H$. 
**Example 5.3.6.** The edge amalgamation of four graphs isomorphic to the given 2-connected $(4,5)$-simple graph $H$ with supermagic sum $s(f)=105$ is given in Figure 5.9.

![Figure 5.9](image)

Figure 5.9: $H$-supermagic labeling of an edge amalgamation of four copies of a 2-connected $(4,5)$ graph $H$.

**Example 5.3.7.** The edge amalgamation of four graphs isomorphic to the given 2-connected $(5,6)$-simple graph $H$ with supermagic sum $s(f)=171$ is given in Figure 5.10. The fixed edges are the edges with arrow mark in the middle.

![Figure 5.10](image)

Figure 5.10: $H$-supermagic labeling of an edge amalgamation of four copies of a 2-connected $(5,6)$ graph $H$. 
**Example 5.3.8.** The edge amalgamation of four graphs isomorphic to the given 2-connected (5,7)-simple graph $H$ with supermagic sum $s(f)=210$ is given in Figure 5.11. The fixed edges are the edges with arrow mark in the middle.

![Figure 5.11: $H$-supermagic labeling of an edge amalgamation of four copies of a 2-connected (5,7) graph $H$.](image1)

**Example 5.3.9.** The edge amalgamation of four graphs isomorphic to the given 2-connected (6,8)-simple graph $H$ with supermagic sum $s(f)=300$ is given in Figure 5.12. The fixed edges are the edges with arrow mark in the middle.

![Figure 5.12: $H$-supermagic labeling of an edge amalgamation of 4 copies of a 2-connected (6,8) graph $H$.](image2)
The following corollaries are immediate from Theorem 5.3.1.

**Corollary 5.3.10.** The books, $K_{1,n} \times P_2$ are $C_4$-supermagic.

**Corollary 5.3.11.** Generalised books are $C_4$-supermagic. In other words, books with $n$ number of $m$-gon pages are $C_m$-supermagic for every positive integers $n \geq 1$ and $m \geq 3$.

**Example 5.3.12.** A $C_4$-supermagic labeling $f$ of a book with 4 pages ($K_{1,4} \times P_2$) is given in Figure 5.13. The supermagic sum of the book is $s(f) = 81$.

![Figure 5.13: $C_4$-supermagic labeling of a book with 4 pages.](image-url)
Example 5.3.13. A $C_5$-supermagic labeling $f$ of a book with 3 hexagon pages is given in Figure 5.14. The supermagic sum of the book is $s(f)=167$.

Figure 5.14: $C_5$-supermagic labeling of a book with 3 hexagon pages.

5.4 Star supermagic graphs

In this section we construct two different families of star supermagic graphs for the given star.

Theorem 5.4.1. Let $H_1, H_2, \ldots, H_k$ be $k$ disjoint copies of the $k$-star $K_{1,k}$ with vertex set $V(H_i) = \{v_i, v_{ir} : 1 \leq r \leq k\}$ and the edge set $E(H_i) = \{v_i v_{ir} : 1 \leq r \leq k\}$ and $G$ be the graph obtained by joining a new vertex $v$ with $v_{11}, v_{21}, \ldots, v_{k1}$. Then $G$ is $K_{1,k}$-supermagic.

Proof: Let $V_i = \{v_i, v_{ir} : 1 \leq r \leq k\}$ and $E_i = \{v_i v_{ir} : 1 \leq r \leq k\}$ for $1 \leq i \leq k$.

Then the vertex and the edge set of $G = (V, E)$ are given by $V = \bigcup_{i=1}^{k} V_i \cup \{v\}$ and $E = \bigcup_{i=1}^{k} E_i \cup \{vv_{11}, vv_{21}, \ldots, vv_{k1}\}$. Clearly, $|V| = k^2 + k + 1$ and $|E| = k^2 + k$. 

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Let $V_{k+1} = \{v, v_{11}, v_{21}, \ldots, v_{1k}\}$ and $E_{k+1} = \{vv_{11}, vv_{21}, \ldots, vv_{k1}\}$ and $H_{k+1} = (V_{k+1}, E_{k+1})$ be the graph with vertex set $V_{k+1}$ and edge set $E_{k+1}$. Note that every edge of $G$ belongs to at least one of the subgraphs $H_i$ for $1 \leq i \leq k + 1$. Since $H_i \cong K_{1,k}$ for $1 \leq i \leq k + 1$, $G$ admits a $K_{1,k}$-covering.

**Case 1:** $k$ is odd.

Since $k + 1$ is even, by Lemma 3.3.6, there exists a $k$-equipartition $\mathcal{P} = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, (k + 1)k]$ such that

$$\sum X_i = \frac{k(k+1)^2}{2} + i \quad \text{for} \quad 1 \leq i \leq k \quad (5.1)$$

It can be easily verified by the definition of $X_r$ in Lemma 3.3.6 that

$$\left(\frac{k+1}{2} - 1\right) k + \sigma(r) \in X_r \quad \text{for} \quad 1 \leq r \leq k,$$

where $\sigma$ is the permutation of $\{1, 2, \ldots, k\}$ given by

$$\sigma(r) = \begin{cases} 
\frac{k-2r+1}{2} & \text{for} \quad 1 \leq r \leq \frac{k-1}{2} \\
\frac{3k-2r+1}{2} & \text{for} \quad \frac{k+1}{2} \leq r \leq k
\end{cases}$$

We construct a new set of integers $X_{k+1}$ by choosing one particular element from each $X_r$ for $r = 1, 2, \ldots, k$ together with $k^2 + k + 1$ as follows:

$$X_{k+1} = \left\{ \left(\frac{k+1}{2} - 1\right) k + \sigma(r) : 1 \leq r \leq k \right\} \cup \{k^2 + k + 1\},$$

$$\sum X_{k+1} = \sum_{r=1}^{k} \left[ \left(\frac{k+1}{2} - 1\right) k + \sigma(r) \right] + k^2 + k + 1$$

$$= \frac{k^2(k-1)}{2} + \frac{k(k+1)}{2} + k^2 + k + 1$$

$$= \frac{k(k+1)^2}{2} + k + 1 \quad (5.2)$$
From (5.1) and (5.2) we have

\[ \sum X_i = \frac{k(k+1)^2}{2} + i \quad \text{for } 1 \leq i \leq k + 1. \]

As \( k \) is odd, by Lemma 3.3.4, there exists a \((k+1)\)-equipartition \( Q' = \{Y'_1, Y'_2, \cdots, Y'_{k+1}\} \) of the set \( Y = [1, k(k+1)] \) such that \( \sum Y'_i = \frac{(k-1)(k(k+1)^2+1)}{2} + i \) for \( 1 \leq i \leq k + 1 \).

Adding \( k^2 + k + 1 \) to every integer of the set \([1, k(k+1)]\), we get a \((k+1)\)-equipartition \( Q = \{Y_1, Y_2, \cdots, Y_{k+1}\} \) of the set \( Y = [k^2 + k + 2, 2k^2 + 2k + 1] \) such that

\[ \sum Y_i = k(k^2 + k + 1) + \frac{(k-1)((k+1)^2+1)}{2} + i \quad \text{for } 1 \leq i \leq k + 1. \]

Define a total labeling \( f : V \cup E \rightarrow [1, 2k^2 + 2k + 1] \) as follows:

- \( f(v) = k^2 + k + 1 \).
- \( f(V_i) = X_i \) for \( 1 \leq i \leq k + 1 \) with \( f(v_{i1}) = \left( \frac{k+1}{2} - 1 \right) k + \sigma(i), i = 1, 2, \cdots k \).
- \( f(E_i) = Y_{k+2-i} \) for \( 1 \leq i \leq k + 1 \).

Then for \( 1 \leq i \leq k + 1 \),

\[ f(H_i) = \sum f(V_i) + \sum f(E_i) \]
\[ = \sum X_i + \sum Y_{k+2-i} \]
\[ = \frac{4k^3 + 5k^2 + 5k + 2}{2}, \quad \text{which is a constant.} \]

Since \( H_i \cong K_{1,k} \) for \( 1 \leq i \leq k + 1 \), \( G \) is \( K_{1,k} \)-supermagic.
Case 2: \(k\) is even.

Since \(k + 1\) is odd, by Lemma 3.3.4, there exists a \(k\)-equipartition \(P = \{X_1, X_2, \ldots, X_k\}\) of \(X = [1, (k + 1)k]\) such that

\[
\sum X_i = \frac{k(k + 1)^2}{2} + i \quad \text{for} \quad 1 \leq i \leq k
\]  

(5.3)

It can be easily verified by the definition of \(X_r\) in Lemma 3.3.4 that

\[
(k + 1) + k + 1 \in X_r \quad \text{for} \quad 1 \leq r \leq \frac{k}{2}, \quad \text{and} \quad \left(\frac{k}{2} \right) + k + 1 \in X_r \quad \text{for} \quad \frac{k}{2} + 1 \leq r \leq k.
\]

We construct a new set of integers \(X_{k+1}\) by choosing one particular element from each \(X_r\) for \(r = 1, 2, \ldots, k\) together with \(k^2 + k + 1\) as follows:

\[
X_{k+1} = \{\left(\frac{k + 2}{2} - 1\right)k + r : 1 \leq r \leq \frac{k}{2}\} \cup \left\{\left(\frac{k}{2} - 1\right)k + r : \frac{k}{2} + 1 \leq r \leq k\right\} \cup \{k^2 + k + 1\}.
\]

\[
\sum X_{k+1} = \sum_{r=1}^{\frac{k}{2}} \left\{\left(\frac{k + 2}{2} - 1\right)k + r\right\} + \sum_{r=1}^{\frac{k}{2}+1} \left\{\left(\frac{k}{2} - 1\right)k + r\right\} + k^2 + k + 1
\]

\[
= \frac{k^2(k - 1)}{2} + \frac{k(k + 1)}{2} + k^2 + k + 1
\]

\[
= \frac{k(k + 1)^2}{2} + k + 1. \quad \text{(5.4)}
\]

From (5.3) and (5.4) we have

\[
\sum X_i = \frac{k(k + 1)^2}{2} + i \quad \text{for} \quad 1 \leq i \leq k + 1.
\]

As \(k\) is even, by Lemma 3.3.6, there exists a \((k + 1)\)-equipartition \(Q' = \{Y'_1, Y'_2, \ldots, Y'_{k+1}\}\) of the set \(Y = [1, k(k + 1)]\) such that \(\sum Y'_i = \frac{(k-1)\left((k+1)^2 + 1\right)}{2} + i\) for
Adding \( k^2 + k + 1 \) to every integer of the set \([1, k(k+1)]\), we get a \((k+1)\)-equipartition \( Q = \{Y_1, Y_2, \cdots, Y_{k+1}\} \) of the set \( Y = [k^2 + k + 2, 2k^2 + 2k + 1] \) such that \( \sum Y_i = k(k^2 + k + 1) + \frac{(k-1)(k+1)^2 + 1}{2} + i \) for \( 1 \leq i \leq k + 1 \).

Define a total labeling \( f : V \cup E \rightarrow [1, 2k^2 + 2k + 1] \) as follows:

\[
f(v) = k^2 + k + 1.
\]

\[
f(V_i) = X_i \text{ for } 1 \leq i \leq k + 1 \text{ with } f(v_{11}) = \begin{cases} \left(\frac{k+2}{2} - 1\right)k + r & \text{for } 1 \leq i \leq \frac{k}{2} \\ \left(\frac{k}{2} - 1\right)k + r & \text{for } \frac{k}{2} + 1 \leq i \leq k \end{cases}
\]

\[
f(E_i) = Y_{k+2-i} \text{ for } 1 \leq i \leq k + 1.
\]

Then, for \( 1 \leq i \leq k + 1 \),

\[
f(H_i) = \sum f(V_i) + \sum f(E_i)
= \sum X_i + \sum Y_{k+2-i}
= \frac{4k^3 + 5k^2 + 5k + 2}{2}, \text{ which is a constant.}
\]

Since \( H_i \cong K_{1,k} \) for \( 1 \leq i \leq k + 1 \), \( G \) is \( K_{1,k} \)-supermagic.

From the above two cases we conclude that \( G \) is \( K_{1,k} \)-supermagic with supermagic sum \( s(f) = \frac{4k^3 + 5k^2 + 5k + 2}{2} \). \( \square \)
Example 5.4.2. A $K_{1,5}$-star supermagic labeling of a graph constructed from 5 copies of $K_{1,5}$ with supermagic sum $s(f)=326$ is given in Figure 5.15.

![Figure 5.15: $K_{1,5}$-star supermagic graph.](image)

Example 5.4.3. A $K_{1,6}$-star supermagic labeling of a graph constructed from 6 copies of $K_{1,6}$ with supermagic sum $s(f)=538$ is given in Figure 5.16.

![Figure 5.16: $K_{1,6}$-star supermagic graph.](image)
Theorem 5.4.4. Let $G_1, G_2, \cdots, G_{k+1}$ be disjoint copies of the $k$-star $K_{1,k}$. Let $V(G_i) = \{v_i, v_{ir} : 1 \leq r \leq k\}$ and $E(G_i) = \{v_i v_{ir} : 1 \leq r \leq k\}$ for $1 \leq i \leq k + 1$. Let $G$ be the graph obtained by joining a new vertex $v$ to the centre of each of the $k+1$ $k$-stars. Then $G$ is $K_{1,k+1}$-supermagic.

Proof: The vertex and the edge sets of $G_i$ for $1 \leq i \leq k + 1$ are $V(G_i) = \{v_i, v_{ir} : 1 \leq r \leq k\}$ and $E(G_i) = \{v_i v_{ir} : 1 \leq r \leq k\}$.

Then the vertex and the edge set of $G = (V, E)$ are given by $V = \bigcup_{i=1}^{k+1} V_i \cup \{v\}$ and $E = \bigcup_{i=1}^{k+1} E_i \cup \{vv_1, vv_2, \cdots, vv_{k+1}\}$. Clearly, $|V| = (k + 1)^2 + 1$ and $|E| = (k + 1)^2$.

Let $H_i = (V_i, E_i)$ be the $(k + 1)$-star $K_{1,k+1}$ with $V_i = V(G_i) \cup \{v\}$ and $E_i = E(G_i) \cup \{vv_i\}$ for $1 \leq i \leq k + 1$.

Let $V_{k+2} = \{v, v_1, v_2, \cdots, v_{k+1}\}$, $E_{k+2} = \{vv_1, vv_2, \cdots, vv_{k+1}\}$ and let $H_{k+2} = (V_{k+2}, E_{k+2})$ be the $(k + 1)$-star with vertex set $V_{k+2}$ and edge set $E_{k+2}$.

Clearly, every edge of $G$ belongs to at least one of the subgraphs $H_i$ for $1 \leq i \leq k + 2$. Since $H_i \cong K_{1,k+1}$ for $1 \leq i \leq k + 2$, $G$ admits $K_{1,k+1}$-covering.

Case 1: $k$ is even.

Since $k + 1$ is odd, by Lemma 3.3.4, there exists a $(k + 1)$-equipartition $\mathcal{P}' = \{X'_1, X'_2, \cdots, X'_{k+1}\}$ of $X = [1, (k + 1)^2]$ such that

$$\sum X'_i = \frac{k(k^2 + 3k + 3)}{2} + i \quad \text{for} \quad 1 \leq i \leq k + 1.$$ 

Adding 1 to every integer of the set $[1, (k+1)^2]$, we get a $(k+1)$-equipartition $\mathcal{P} = \{X_1, X_2, \cdots, X_{k+1}\}$ of $[2, (k + 1)^2 + 1]$ such that
\[ \sum X_i = (k + 1) + \frac{k(k^2 + 3k + 3)}{2} + i \quad \text{for } 1 \leq i \leq k + 1. \]

As \( k + 1 \) is odd, by Lemma 3.3.4, there exists a \((k + 1)\)-equipartition \( Q' = \{Y'_1, Y'_2, \ldots, Y'_{k+1}\} \) of the set \( Y = [1, (k + 1)^2] \) such that
\[ \sum Y'_i = \frac{k(k^2 + 3k + 3)}{2} + i \quad \text{for } 1 \leq i \leq k + 1. \]

Adding \((k + 1)^2 + 1\) to every integer of the set \([1, (k + 1)^2]\), we get a \((k + 1)\)-equipartition \( Q = \{Y_1, Y_2, \ldots, Y_{k+1}\} \) of the set \( Y = [(k + 1)^2 + 2, 2(k + 1)^2 + 1] \) such that
\[ \sum Y_i = (k + 1)[(k + 1)^2 + 1] + \frac{k(k^2 + 3k + 3)}{2} + i \quad \text{for } 1 \leq i \leq k + 1. \]

Define a total labeling \( f : V \cup E \rightarrow [1, 2(k + 1)^2 + 1] \) as follows:
\( f(v) = 1. \)
\( f(V(G_i)) = X_i \) with \( f(v_i) = i + 1 \) for \( 1 \leq i \leq k + 1. \)
\( f(E_i) = Y_{k+2-i} \) for \( 1 \leq i \leq k + 1 \) with \( f(vv_i) = 2k^2 + 3k + 2 + i \) for \( 1 \leq i \leq k + 1. \)

Now, for \( 1 \leq i \leq k + 1, \)
\[
\begin{align*}
\quad f(H_i) &= f(v) + \sum f(V(G_i)) + \sum f(E_i) \\
&= 1 + \sum X_i + \sum Y_{k+2-i} \\
&= 1 + (k + 1) + \frac{k(k^2 + 3k + 3)}{2} + \frac{k(k^2 + 3k + 3)}{2} + k + 2 - i \\
&= 2k^3 + 6k^2 + 9k + 6.
\end{align*}
\]
Also, \( f(H_{k+2}) = \sum f(V_{k+2}) + \sum f(E_{k+2}) \) where \( f(V_{k+2}) = \{1, 2, 3, \ldots, k + 2\} \) and \( f(E_{k+2}) = \{2k^2 + 3k + 2 + i : 1 \leq i \leq k + 1\} \).

Now,
\[
\sum f(V_{k+2}) = 1 + 2 + 3 + \cdots + (k + 2) = \frac{(k + 2)(k + 3)}{2}.
\]
\[
\sum f(E_{k+2}) = \sum_{i=1}^{k+1} [2k^2 + 3k + 2 + i] = (k + 1)(2k^2 + 3k + 2) + \frac{(k + 1)(k + 2)}{2} = \frac{(k + 1)(4k^2 + 7k + 6)}{2}.
\]

and hence
\[
f(H_{k+2}) = \sum f(V_{k+2}) + \sum f(E_{k+2}) = 2k^3 + 6k^2 + 9k + 6.
\]

Thus, \( f(H_i) = 2k^3 + 6k^2 + 9k + 6 \) for \( 1 \leq i \leq k + 2 \).

Since \( H_i \cong K_{1,k+1} \) for \( 1 \leq i \leq k + 2 \), \( G \) is \( K_{1,k+1} \)-supermagic.

**Case 2:** \( k \) is odd.

Since \( k + 1 \) is even, by Lemma 3.3.8, there exists a \((k + 1)\)-equipartition \( \mathcal{P}' = \{X'_1, X'_2, \ldots, X'_{k+1}\} \) of \( X = [1, (k + 1)^2] \) such that
\[
\sum X'_{il} = \frac{(k + 1)(k^2 + 2k + 2)}{2} \quad \text{for} \quad 1 \leq i \leq k + 1.
\]

Adding 1 to every integer of the set \([1, (k+1)^2]\), we get a \((k+1)\)-equipartition...
Let \( P = \{X_1, X_2, \ldots, X_{k+1}\} \) of \([2, (k + 1)^2 + 1]\) such that

\[
\sum X_i = (k + 1) + \frac{(k + 1)(k^2 + 2k + 2)}{2} \quad \text{for} \quad 1 \leq i \leq k + 1.
\]

As \( k + 1 \) is even, by Lemma 3.3.8, there exists a \((k + 1)\)-equipartition \( Q' = \{Y'_1, Y'_2, \ldots, Y'_{k+1}\}\) of the set \( Y = [1, (k + 1)^2] \) such that

\[
\sum Y'_i = \frac{(k + 1)(k^2 + 2k + 2)}{2} \quad \text{for} \quad 1 \leq i \leq k + 1.
\]

Adding \((k + 1)^2 + 1\) to every integer of the set \([1, (k + 1)^2]\), we get a \((k + 1)\)-equipartition \( Q = \{Y_1, Y_2, \ldots, Y_{k+1}\}\) of the set \( Y = [(k + 1)^2 + 2, 2(k + 1)^2 + 1]\) such that

\[
\sum Y_i = (k + 1)((k + 1)^2 + 1) + \frac{(k + 1)(k^2 + 2k + 2)}{2} \quad \text{for} \quad 1 \leq i \leq k + 1.
\]

Define a total labeling \( f : V \cup E \to [1, 2(k + 1)^2 + 1] \) as follows:

\( f(v) = 1. \)

\( f(V(G_i)) = X_i \) with \( f(v_i) = i + 1 \) for \( 1 \leq i \leq k + 1. \)

\( f(E_i) = Y_{k+2-i} \) for \( 1 \leq i \leq k + 1 \) with \( f(vv_i) = 2k^2 + 3k + 2 + i \) for \( 1 \leq i \leq k. \)

Now, for \( 1 \leq i \leq k + 1, \)

\[
f(H_i) = f(v) + \sum f(V(G_i)) + \sum f(E_i)
\]

\[
= 1 + \sum X_i + \sum Y_{k+2-i}
\]

\[
= 1 + (k + 1) + \frac{(k + 1)(k^2 + 2k + 2)}{2} + (k + 1)((k + 1)^2 + 1)
\]

\[
+ \frac{(k + 1)(k^2 + 2k + 2)}{2}
\]

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= 2k^3 + 6k^2 + 9k + 6.

Also \( f(H_{k+2}) = \sum f(V_{k+2}) + \sum f(E_{k+2}) \) where \( f(V_{k+2}) = \{1, 2, 3, \ldots, k + 2\} \) and \( f(E_{k+2}) = \{2k^2 + 3k + 2 + i : 1 \leq i \leq k + 1\} \).

Now,

\[
\sum f(V_{k+2}) = 1 + 2 + 3 + \cdots + (k + 2)
= \frac{(k + 2)(k + 3)}{2}.
\]

\[
\sum f(E_{k+2}) = \sum_{i=1}^{k+1}(2k^2 + 3k + 2 + i)
= (k + 1)(2k^2 + 3k + 2) + \frac{(k + 1)(k + 2)}{2}
= \frac{(k + 1)(4k^2 + 7k + 6)}{2}.
\]

Hence, \( f(H_{k+2}) = \sum f(V_{k+2}) + \sum f(E_{k+2}) = 2k^3 + 6k^2 + 9k + 6 \).

Thus, \( f(H_i) = 2k^3 + 6k^2 + 9k + 6 \) for \( 1 \leq i \leq k + 2 \).

Since \( H_i \cong K_{1,k+1} \) for \( 1 \leq i \leq k + 2 \), \( G \) is \( K_{1,k+1} \)-supermagic.

From the above two cases we conclude that \( G \) is \( K_{1,k+1} \)-supermagic with supermagic sum \( s(f) = 2k^3 + 6k^2 + 9k + 6 \). \( \Box \)
Example 5.4.5. A $K_{1,5}$-star supermagic labeling of a graph constructed from 5 copies of $K_{1,4}$ with supermagic sum $s(f) = 266$ is given in Figure 5.17.

![Figure 5.17: $K_{1,5}$-star supermagic graph.](image)

Example 5.4.6. A $K_{1,4}$-supermagic labeling of a graph constructed from 4 copies of $K_{1,3}$ with supermagic sum 141 is shown in Figure 5.18.

![Figure 5.18: $K_{1,4}$-star supermagic graph.](image)
5.5 \textbf{H-supermagic labeling of one point union of graphs}

In this section we prove that the one point union of finite number of copies of a given 2-connected simple graph $H$ is $H$-supermagic. Since the windmill $W(r, k)$ is the one point union of $k(\geq 2)$ disjoint copies of the cycle $C_r$ we observe that all windmills are cycle-supermagic.

One point union of finite number of connected graphs is obtained by identifying one vertex from each graph.

\textbf{Theorem 5.5.1.} One point union of $n$ copies of a 2-connected graph $H$ is $H$-supermagic.

\textbf{Proof:} Let $H$ be a simple 2-connected $(p, q)$ graph. Let $H_i = (V_i, E_i)$ for $1 \leq i \leq n$ be $n$ copies of $H$. Let $v_i \in V_i$ for $1 \leq i \leq n$ and construct a graph $G$ by identifying $v_1, v_2, \ldots, v_n$. Let us rename the identified vertex in $G$ by $w$ and let $V$ be the vertex set and $E$ be the edge set of $G$. Now, take $V_i' = V_i - \{v_i\}$ and $E_i' = E_i$ for $1 \leq i \leq n$. Then, $V = \bigcup_{i=1}^n V_i' \cup \{w\}$ and $E = \bigcup_{i=1}^n E_i$. Hence, $|V| = np - n + 1$ and $|E| = nq$.

\textbf{Case (1):} $n$ is odd.

\textbf{Sub case (i):} $p$ is odd and $q$ is even.

Since $p - 1$ and $q$ are even, by Lemma 3.3.8, there exist $n$-equipartitions $P_1' = \{X'_1, X'_2, \ldots, X'_n\}$ of $[1, (p - 1)n]$ and $P_2' = \{Y'_1, Y'_2, \ldots, Y'_n\}$ of $[1, qn]$ such that

$$\sum_{i=1}^n X'_i = \frac{(p - 1)(pn - n + 1)}{2}, \quad \text{and} \quad \sum_{i=1}^n Y'_i = \frac{q(qn + 1)}{2} \quad \text{for} \quad 1 \leq i \leq n.$$ 

Add 1 to every integer of the set $[1, (p - 1)n]$ and $(p - 1)n + 1$ to every integer of the set $[1, qn]$. We get $n$-equipartitions $P_1 = \{X_1, X_2, \ldots, X_n\}$ of $[2, pn - n + 1]$ and $P_2 = \{Y_1, Y_2, \ldots, Y_n\}$ of $[2, qn]$. Then, $P = P_1 \cup P_2$ is an $n$-equipartition of $V$. We assign labels $\lambda(X'_i) = X_i$ and $\lambda(Y'_i) = Y_i$ and $\lambda(w) = n$ to the vertex set $V$ so that

$$\sum_{i=1}^n \lambda(X'_i) = \frac{(p - 1)(pn - n + 1)}{2} \quad \text{and} \quad \sum_{i=1}^n \lambda(Y'_i) = \frac{q(qn + 1)}{2} \quad \text{for} \quad 1 \leq i \leq n.$$
and $P_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[pn - n + 2, (p + q)n - n + 1]$ such that

$$\sum X_i = (p - 1) + \frac{(p - 1)(pn - n + 1)}{2} \quad \text{and} \quad \sum Y_i = [(p - 1)n + 1]q + \frac{q(qn + 1)}{2} \quad \text{for} \ 1 \leq i \leq n.$$  

Define a total labeling $f : V \cup E \to [1, (p + q)n - n + 1]$ as follows:

$f(w) = 1$.

$f(V') = X_i$ for $1 \leq i \leq n$.

$f(E') = Y_i$ for $1 \leq i \leq n$.

Then for $1 \leq i \leq n$,

$$f(H_i) = f(w) + \sum f(V') + \sum f(E')$$

$$= f(w) + \sum X_i + \sum Y_i$$

$$= 1 + (p - 1) + \frac{(p - 1)(pn - n + 1)}{2} + [(p - 1)n + 1]q + \frac{q(qn + 1)}{2}$$

$$= \frac{n(p + q)^2 - (2n - 3)(p + q) + n - 1}{2}$$

$$= \text{constant.}$$

Since $H_i \cong H$ for $1 \leq i \leq n$, $G$ is $H$-supermagic.

**Sub case (ii):** $p$ is even and $q$ is odd.

Since $p - 1$ and $q$ are odd, by Lemma 3.3.4, there exist $n$-equipartitions $P'_1 = \{X'_1, X'_2, \cdots, X'_n\}$ of $[1, (p - 1)n]$ and $P'_2 = \{Y'_1, Y'_2, \cdots, Y'_n\}$ of $[1, qn]$ such that

$$\sum X'_i = \frac{(p - 2)(pn + 1)}{2} + i.$$
\[ \sum_{i=1}^{n} Y_i' = \frac{(q-1)(qn+n+1)}{2} + i \quad \text{for} \quad 1 \leq i \leq n. \]

Add 1 to every integer of the set \([1,(p-1)n]\) and \((p-1)n+1\) to every integer of the set \([1,(q-1)n]\). We get \(n\)-equipartitions \(P_1 = \{X_1, X_2, \cdots, X_n\}\) of \([2, pn-n+1]\) and \(P_2 = \{Y_1, Y_2, \cdots, Y_n\}\) of \([pn-n+2,(p+q)n-n+1]\) such that

\[ \sum_{i=1}^{n} X_i = (p-1) + \frac{(p-2)(pn+1)}{2} + i \]
\[ \sum_{i=1}^{n} Y_i = [(p-1)n+1]q + \frac{(q-1)(qn+n+1)}{2} + i \quad \text{for} \quad 1 \leq i \leq n. \]

Define a total labeling \(f : V \cup E \rightarrow [1,(p+q)n-n+1]\) as follows:

\(f(w) = 1\).

\(f(V_i') = X_i\) for \(1 \leq i \leq n\).

\(f(E_i') = Y_{n-i+1}\) for \(1 \leq i \leq n\).

Then for \(1 \leq i \leq n\),

\[ f(H_i) = f(w) + \sum f(V_i') + \sum f(E_i') \]
\[ = f(w) + \sum X_i + \sum Y_{n-i+1} \]
\[ = 1 + (p-1) + \frac{(p-2)(pn+1)}{2} + i + [(p-1)n+1]q \]
\[ + \frac{(q-1)(qn+n+1)}{2} + n-i+1 \]
\[ = \frac{n(p+q)^2 - (2n-3)(p+q) + n-1}{2} \]
\[ = \text{constant}. \]
Since $H_i \equiv H$ for $1 \leq i \leq n$, $G$ is $H$-supermagic.

**Sub case (iii):** $p$ and $q$ are even.

Since $p - 1$ is odd, by Lemma 3.3.4, there exists an $n$-equipartition $P'_1 = \{X'_1, X'_2, \cdots, X'_n\}$ of $[1, (p - 1)n]$ such that

$$\sum X'_i = \frac{(p - 2)(pn + 1)}{2} + i \text{ for } 1 \leq i \leq n.$$ 

Since $q$ is even and $n$ is odd, by Lemma 3.3.6, there exists an $n$-equipartition $P'_2 = \{Y'_1, Y'_2, \cdots, Y'_n\}$ of $[1, qn]$ such that

$$\sum Y'_i = \frac{(q - 1)(qn + n + 1)}{2} + i \text{ for } 1 \leq i \leq n.$$ 

Add 1 to every integer of the set $[1, (p - 1)n]$ and $(p - 1)n + 1$ to every integer of the set $[1, qn]$. We get $n$-equipartitions $P_1 = \{X_1, X_2, \cdots, X_n\}$ of $[2, pn - n + 1]$ and $P_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[pn - n + 2, (p + q)n - n + 1]$ such that

$$\sum X_i = (p - 1) + \frac{(p - 2)(pn + 1)}{2} + i$$

$$\sum Y_i = [(p - 1)n + 1]q + \frac{(q - 1)(qn + n + 1)}{2} + i \text{ for } 1 \leq i \leq n.$$ 

Define a total labeling $f : V \cup E \to [1, (p + q)n - n + 1]$ as follows:

$f(w) = 1$.

$f(V'_i) = X_i$ for $1 \leq i \leq n$.

$f(E'_i) = Y_{n-i+1}$ for $1 \leq i \leq n$.

Then for $1 \leq i \leq n$, ...
\[ f(H_i) = f(w) + \sum f(V'_i) + \sum f(E'_i) \]
\[ = f(w) + \sum X_i + \sum Y_{n-i+1} \]
\[ = 1 + (p-1) + \frac{(p-2)(pn+1)}{2} + i + [(p-1)n + 1]q \]
\[ + \frac{(q-1)(qn+n+1)}{2} + n - i + 1 \]
\[ = \frac{n(p+q)^2 - (2n-3)(p+q) + n - 1}{2} \]
\[ = \text{constant}. \]

Since \( H_i \cong H \) for \( 1 \leq i \leq n \), \( G \) is \( H \)-supermagic.

**Sub case (iv):** \( p \) and \( q \) are odd.

Since \( p - 1 \) is even and \( n \) is odd, by Lemma 3.3.6, there exists an \( n \)-equipartition \( P'_1 = \{X'_1, X'_2, \cdots, X'_n\} \) of \([1, (p-1)n]\) such that \( \sum X'_i = \frac{(p-2)(pn+1)}{2} + i \) for \( 1 \leq i \leq n \).

Since \( q \) is odd, by Lemma 3.3.4, there exists an \( n \)-equipartition \( P'_2 = \{Y'_1, Y'_2, \cdots, Y'_n\} \) of \([1, qn]\) such that
\[ \sum Y'_i = \frac{(q-1)(qn+n+1)}{2} + i \text{ for } 1 \leq i \leq n. \]

Add 1 to every integer of the set \([1, (p-1)n]\) and \((p-1)n+1\) to every integer of the set \([1, qn]\). We get \( n \)-equipartitions \( P_1 = \{X_1, X_2, \cdots, X_n\} \) of \([2, pn-n+1]\) and \( P_2 = \{Y_1, Y_2, \cdots, Y_n\} \) of \([pn-n+2, (p+q)n-n+1]\) such that
\[ \sum X_i = (p - 1) + \frac{(p - 2)(pn + 1)}{2} + i \]

\[ \sum Y_i = [(p - 1)n + 1]q + \frac{(q - 1)(qn + n + 1)}{2} + i \quad \text{for} \quad 1 \leq i \leq n. \]

Define a total labeling \( f : V \cup E \rightarrow [1, (p + q)n - n + 1] \) as follows:

\( f(w) = 1. \)

\( f(V'_i) = X_i \) for \( 1 \leq i \leq n. \)

\( f(E'_i) = Y_{n-i+1} \) for \( 1 \leq i \leq n. \)

Then for \( 1 \leq i \leq n, \)

\[ f(H'_i) = f(w) + \sum f(V'_i) + \sum f(E'_i) \]

\[ = f(w) + \sum X_i + \sum Y_{n-i+1} \]

\[ = 1 + (p - 1) + \frac{(p - 2)(pn + 1)}{2} + i + [(p - 1)n + 1]q \]

\[ + \frac{(q - 1)(qn + n + 1)}{2} + n - i + 1 \]

\[ = \frac{n(p + q)^2 - (2n - 3)(p + q) + n - 1}{2} \]

\[ = \text{constant}. \]

Since \( H_i \cong H \) for \( 1 \leq i \leq n, \) \( G \) is \( H \)-supermagic.

**Case (2):** \( n \) is even.

**Sub case (i):** \( p \) is odd and \( q \) is even.

A similar argument as in Sub case(i) of Case (1) shows that \( G \) is \( H \)-supermagic.

**Sub case (ii):** \( p \) is even and \( q \) is odd.
A similar argument as in Sub case(ii) of Case (1) shows that $G$ is $H$-
supermagic.

**Sub case (iii):** $p$ and $q$ are even.

Since $p - 1$ is odd and $n$ is even, by Lemma 3.3.18, there exists an $n$-
equipartition $\mathbb{P}_1 = \{X_1, X_2, \cdots, X_n\}$ of $[1, (p - 1)n + 1] - \{\frac{n}{2} + 1\}$ such that

\[
\sum X_i = \frac{n(p-1)^2 + 3p - 4}{2} \quad \text{for } 1 \leq i \leq n.
\]

Since $q$ is even, by Lemma 3.3.8, there exists an $n$-equipartition $\mathbb{P}_2' = \{Y_1', Y_2', \cdots, Y_n'\}$ of $[1, qn]$ such that

\[
\sum Y_i' = \frac{q(qn + 1)}{2} \quad \text{for } 1 \leq i \leq n.
\]

Add $(p - 1)n + 1$ to every integer of the set $[1, qn]$. We get an $n$-equipartition $\mathbb{P}_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[pn - n + 2, (p + q)n - n + 1]$ such that

\[
\sum Y_i = [(p - 1)n + 1]q + \frac{q(qn + 1)}{2} \quad \text{for } 1 \leq i \leq n.
\]

Define a total labeling $f: V \cup E \to [1, (p + q)n - n + 1]$ as follows:

\[
f(w) = \frac{n}{2} + 1.
\]

\[
f(V_i') = X_i \quad \text{for } 1 \leq i \leq n.
\]

\[
f(E_i') = Y_i \quad \text{for } 1 \leq i \leq n.
\]

Then for $1 \leq i \leq n$,

\[
f(H_i) = f(w) + \sum f(V_i') + \sum f(E_i')
\]
\[
= f(w) + \sum X_i + \sum Y_{n-i+1}
\]
\[
\frac{n}{2} + 1 + \frac{n(p - 1)^2 + 3p - 4}{2} + \frac{[(p - 1)n + 1]q + \frac{q(qn + 1)}{2}}{2}
\]
\[
= \frac{n(p + q)^2 - (2n - 3)(p + q) + 2n - 2}{2}
\]
\[
= \text{constant}.
\]

Since \( H_i \cong H \) for \( 1 \leq i \leq n \), \( G \) is \( H \)-supermagic.

**Sub case (iv):** \( p \) and \( q \) are odd.

Since \( p - 1 \) and \( n \) are even, by Lemma 3.3.10, there exists an \( n \)-equipartition
\[
\mathcal{P}_1 = \{X_1, X_2, \ldots, X_n\} \text{ of } [1, (p - 1)n + 1] - \{\frac{n}{2} + 1\} \text{ such that}
\]
\[
\sum X_i = \frac{n(p - 1)^2 + 3p - n - 5}{2} + i \text{ for } 1 \leq i \leq n.
\]

Since \( q \) is odd, by Lemma 3.3.4, there exists an \( n \)-equipartition \( \mathcal{P}'_2 = \{Y'_1, Y'_2, \ldots, Y'_n\} \) of \([1, qn]\) such that
\[
\sum Y'_i = \frac{(q - 1)(qn + n + 1)}{2} + i \text{ for } 1 \leq i \leq n.
\]

Add \((p - 1)n + 1\) to every integer of the set \([1, qn]\). We get an \( n \)-equipartition
\[
\mathcal{P}_2 = \{Y_1, Y_2, \ldots, Y_n\} \text{ of } \left[pn - 2, (p + q)n - n + 1\right] \text{ such that}
\]
\[
\sum Y_i = [(p - 1)n + 1]q + \frac{(q - 1)(qn + n + 1)}{2} + i \text{ for } 1 \leq i \leq n.
\]

Define a total labeling \( f : V \cup E \rightarrow [1, (p + q)n - n + 1] \) as follows:
\[
f(w) = \frac{n}{2} + 1.
\]
\[
f(V'_i) = X_i \text{ for } 1 \leq i \leq n.
\]
\[ f(E'_i) = Y_{n-i+1} \text{ for } 1 \leq i \leq n. \]

Then for \(1 \leq i \leq n\),

\[
\begin{align*}
f(H_i) &= f(w) + \sum f(V'_i) + \sum f(E'_i) \\
&= f(w) + \sum X_i + \sum Y_{n-i+1} \\
&= \frac{n}{2} + 1 + \frac{n(p-1)^2 + 3p - n - 5}{2} + i + [(p-1)n + 1]q \\
&\quad + \frac{(q-1)(qn + n + 1)}{2} + n - i + 1 \\
&= \frac{n(p + q)^2 - (2n - 3)(p + q) + 2n - 2}{2} \\
&= \text{constant.}
\end{align*}
\]

Since \(H_i \cong H\) for \(1 \leq i \leq n\), \(G\) is \(H\)-supermagic.

Hence, the one point union \(G\) of \(n\) copies of a 2-connected graph \(H\) is \(H\)-supermagic. \(\square\)

**Example 5.5.2.** A \((5,6)\)-graph \(H\) and a \(H\)-supermagic labeling of the one point union of 3 copies of \(H\) with supermagic sum \(s(f) = 166\) is shown in Figure 5.19.

![Figure 5.19: \(H\)-supermagic labeling of one point union of 3 copies of a 2-connected \((5,6)\) graph \(H.\)](image)

Since \(H_i \cong H\) for \(1 \leq i \leq n\), \(G\) is \(H\)-supermagic.

Hence, the one point union \(G\) of \(n\) copies of a 2-connected graph \(H\) is \(H\)-supermagic. \(\square\)

**Example 5.5.2.** A \((5,6)\)-graph \(H\) and a \(H\)-supermagic labeling of the one point union of 3 copies of \(H\) with supermagic sum \(s(f) = 166\) is shown in Figure 5.19.
Example 5.5.3. A \((4,5)\)-graph \(H\) and a \(H\)-supermagic labeling of the one point union of 5 copies of \(H\) with supermagic sum \(s(f)=173\) is shown in Figure 5.20.

![Graph](image)

Figure 5.20: \(H\)-supermagic labeling of one point union of 5 copies of a 2-connected \((4,5)\) graph \(H\).

Example 5.5.4. A \(K_4\)-supermagic labeling of the one point union of 3 copies of the complete graph \(K_4\) with supermagic sum 136 is shown in Figure 5.21.

![Graph](image)

Figure 5.21: \(K_4\)-supermagic labeling of one point union of 3 copies of \(K_4\).
**Example 5.5.5.** A $K_3$-supermagic labeling of the one point union of 5 copies of the complete graph $K_3$ with supermagic sum 71 is shown in Figure 5.22.

![Figure 5.22: $K_3$-supermagic labeling of one point union of 5 copies of $K_3.$](image)

**Example 5.5.6.** A $(5,6)$-graph $H$ and a $H$-supermagic labeling of the one point union of 4 copies of $H$ with supermagic sum 216 is shown in Figure 5.23.

![Figure 5.23: $H$-supermagic labeling of one point union of 4 copies of a 2-connected $(5,6)$ graph $H.$](image)
Example 5.5.7. A (6,7)-graph $H$ and a $H$-supermagic labeling of the one point union of 4 copies of $H$ with supermagic sum 307 is shown in Figure 5.24.

Figure 5.24: $H$-supermagic labeling of one point union of 4 copies of a 2-connected (6,7) graph $H$.

Example 5.5.8. A $K_4$-supermagic labeling of the one point union of 4 copies of the complete graph $K_4$ with supermagic sum 178 is shown in Figure 5.25.

Figure 5.25: $K_4$-supermagic labeling of one point union of 4 copies of $K_4$. 107
Example 5.5.9. A $K_3$-supermagic labeling of the one point union of 4 copies of the complete graph $K_3$ with supermagic sum 60 is shown in Figure 5.26.

![Figure 5.26: $K_3$-supermagic labeling of one point union of 4 copies of $K_3$.](image)

Let $C_r$ be a cycle of length $r \geq 3$. Then the graph obtained by identifying one vertex in each of $k \geq 2$ disjoint copies of the cycle $C_r$ is called the windmill and is denoted by $W(r,k)$.

The following corollary is immediate from Theorem 5.5.1.

Corollary 5.5.10. For any two integers $k \geq 2$ and $r \geq 3$, the windmill $W(r,k)$ is $C_r$-supermagic.
Example 5.5.11. A $C_4$-supermagic labeling of the windmill $W(4, 4)$ with supermagic sum 111 is given in Figure 5.27.

Figure 5.27: $C_4$-supermagic labeling of the windmill $W(4, 4)$.

### 5.6 $H$-supermagic labeling of garland graph

In this section we define a new family of graph called garland graph and find an $H$-supermagic labeling of that graph.

**Definition 5.6.1.** Let $H$ be a 2-connected $(p, q)$ graph with $q \geq 3$ and let \{$H_i : 1 \leq i \leq n$\} be $n$ copies of $H$. Let $e_i, e'_i$ be two distinct edges of $H_i$ for $i = 1, 2, \ldots, n$. Then the graph obtained by identifying the edge $e'_i$ of $H_i$ with $e_{i+1}$ of $H_{i+1}$ for $1 \leq i \leq n - 1$ is called the garland of $H$ of length $n$ and is denoted by $G_n(H)$. 
Example 5.6.2. A garland $G_4(H)$ of a given graph $H$ is given in Figure 5.28.

![Graph H](image)

Figure 5.28: Garland graph $G_4(H)$.

Definition 5.6.3. Let $H$ be a 2-connected $(p,q)$ graph with $p,q \geq 4$ and let $\{H_i : 1 \leq i \leq n\}$ be $n$ copies of $H$. Let $e_i, e'_i$ be two non-adjacent edges of $H_i$ for $i = 1, 2, \cdots, n$. Then the graph obtained by identifying the edge $e'_i$ of $H_i$ with $e_{i+1}$ of $H_{i+1}$ for $1 \leq i \leq n-1$ is called the Linear garland of $H$ of length $n$ and is denoted by $LG_n(H)$.

Theorem 5.6.4. Let $H$ be a 2-connected $(p,q)$-simple graph with $p,q \geq 4$. Then the linear garland $LG_n(H)$ of the graph $H$ of length $n$ is $H$-supermagic if either $n$ is odd or both $n$ and $p+q$ are even.

Proof: Let $H_i = (V_i, E_i), i = 1, 2, \cdots, n$ be $n$ copies of a given 2-connected $(p,q)$ graph $H$. Let $e_i, e'_i$ be two non-adjacent edges of $H_i$ for $i = 1, 2, \cdots, n$. Let $e_i = u_iv_i$ and $e'_i = u'_iv'_i$. Let $u'_i$ and $v'_i$ be identified with $u_{i+1}$ and $v_{i+1}$ respectively in the linear garland $LG_n(H)$.

Let $V'_i = V_i - \{u_i, v_i, u'_i, v'_i\}$ and $E'_i = E - \{e_i, e'_i\}$ for $i = 1, 2, \cdots, n$. 

Let $V$ be the vertex set and $E$ be the edge set of $\mathcal{L}G_n(H)$. Then we have,

$$V = \bigcup_{i=1}^{n} V'_i \cup \{u_i, v_i : 1 \leq i \leq n\} \cup \{u'_n, v'_n\}$$

and

$$E = \bigcup_{i=1}^{n} E'_i \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_n\}.$$

We can easily verify that $|V| = n(p-2)+2$ and $|E| = n(q-1)+1$.

Rename the vertices $u_i, v_i$ for $1 \leq i \leq n$ and $u'_n, v'_n$ using the mapping given below:

- $u_i \rightarrow w_{2i-1}$.
- $v_i \rightarrow w_{2i}$.
- $u'_n \rightarrow w_{2n+1}$ and $v'_n \rightarrow w_{2n+2}$.

Also rename the edges $e'_n$ with $e_{n+1}$.

Then, the following can be observed.

(i) $e_i = w_{2i-1}w_{2i}$.

(ii) $V_i = V'_i \cup \{w_{2i-1}, w_{2i}, w_{2i+1}, w_{2i+2}\}$ and $E_i = E'_i \cup \{e_i, e_{i+1}\}$.

(iii) $V = \bigcup_{i=1}^{n} V'_i \cup \{w_i : 1 \leq i \leq 2n+2\}$ and $E = \bigcup_{i=1}^{n} E'_i \cup \{e_i : 1 \leq i \leq n+1\}$.

For each $i$, $1 \leq i \leq 2n+2$, consider the decomposition $i = i_1 + 4i_2$, with $1 \leq i_1 \leq 4$ and $0 \leq i_2 \leq \frac{n}{4}$ and let $\alpha(i) = (i_1, i_2)$. Let $f_1$ be the lexicographic ordering of the pairs $\alpha(i)$.

Case (1): $n$ is odd.

Sub case (i): $p$ and $q$ are odd.

By Lemma 3.3.4, there exists an $n$-equipartition $\mathbb{P'} = \{X'_1, X'_2, \cdots, X'_n\}$ of $[1, n(p-4)]$ and an $n$-equipartition $\mathbb{Q'} = \{Y'_1, Y'_2, \cdots, Y'_n\}$ of $[1, n(q-2)]$ such that $\sum \mathbb{P'}$ and $\sum \mathbb{Q'}$ are two sets of consecutive integers.

Adding $2n + 2$ to every integer of the set $[1, n(p-4)]$ and $np - n + 3$ to every integer of the set $[1, n(q-2)]$, we get an $n$-equipartition $\mathbb{P} = \{X_1, X_2, \cdots, X_n\}$
of $[2n + 3, n(p - 2) + 2]$ and an $n$-equipartition $Q = \{Y_1, Y_2, \cdots, Y_n\}$ of $[np - n + 4, np + nq - 3n + 3]$ such that $\sum P$ and $\sum Q$ are sets of consecutive integers.

Define the total labeling $f : V \cup E \rightarrow [1, n(p + q - 3) + 3]$ as follows:

$f(w_i) = f_1(\alpha(i))$ for $n = 1, 2, \cdots, 2n + 2$.

$f(e_i) = n(p - 1) + 4 - i$ for $n = 1, 2, \cdots, n + 1$.

$f(V') = X_i$ for $n = 1, 2, \cdots, n$.

$f(E') = Y_{n-i+1}$ for $n = 1, 2, \cdots, n$.

$f(H_i) = \sum f(V_i) + \sum f(E_i) + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1})$

$= \sum X_i + \sum Y_{n-i+1} + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1})$

Now,

$f(H_{i+1}) - f(H_i) = \sum f(X_{i+1}) - \sum f(X_i) + \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) + f(w_{2i+3})$

$+ f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) + f(e_{i+2}) - f(e_i)$.

It can be noted that, if $\alpha(i) = (i_1, i_2)$ then $\alpha(i + 4) = (i_1, i_2 + 1)$ and hence

$f(w_{i+4}) - f(w_i) = 1$.

$f(w_{2i+3}) + f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) = f(w_{2i+3}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i})$

$= f(w_{(2i-1)+4}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i})$

$= 2$.

$f(e_{i+2}) - f(e_i) = [n(p - 1) + 2 - i] - [n(p - 1) + 4 - i] = -2$.

Also, $\sum f(X_{i+1}) - \sum f(X_i) = 1$ and $\sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) = -1$.

Hence, we have $f(H_{i+1}) - f(H_i) = 0$ which implies that $f(H_i)$ is constant
for \(1 \leq i \leq n - 1\).

Since each \(H_i\) is isomorphic to \(H\), \(\mathcal{L}G_n(H)\) is \(H\)-supermagic.

**Sub case (ii):** \(p\) and \(q\) are even.

Suppose \(p > 4\).

By Lemma 3.3.6, there exists an \(n\)-equipartition \(P' = \{X'_1, X'_2, \ldots, X'_n\}\) of \([1, n(p - 4)]\) and an \(n\)-equipartition \(Q' = \{Y'_1, Y'_2, \ldots, Y'_n\}\) of \([1, n(q - 2)]\) such that \(\sum P'\) and \(\sum Q'\) are two sets of consecutive integers.

Adding \(2n + 2\) to every integer of the set \([1, n(p - 4)]\) and \(np - n + 3\) to every integer of the set \([1, n(q - 2)]\), we get an \(n\)-equipartition \(P = \{X_1, X_2, \ldots, X_n\}\) of \([2n + 3, n(p - 2) + 2]\) and an \(n\)-equipartition \(Q = \{Y_1, Y_2, \ldots, Y_n\}\) of \([np - n + 4, np + nq - 3n + 3]\) such that \(\sum P\) and \(\sum Q\) are sets of consecutive integers.

Define the total labeling \(f : V \cup E \rightarrow [1, n(p + q - 3) + 3]\) as follows:

\[
f(w_i) = f_1(\alpha(i)) \quad \text{for} \quad i = 1, 2, \ldots, 2n + 2.
\]

\[
f(e_i) = n(p - 1) + 4 - i \quad \text{for} \quad i = 1, 2, \ldots, n + 1.
\]

\[
f(V'_i) = X_i \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

\[
f(E'_i) = Y_{n-i+1} \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

\[
f(H_i) = \sum f(V_i) + \sum f(E_i) + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1})
\]

\[
= \sum X_i + \sum Y_{n-i+1} + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1}).
\]

Now,

\[
f(H_{i+1}) - f(H_i) = \sum f(X_{i+1}) - \sum f(X_i) + \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) + f(w_{2i+3})
\]

\[
+ f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) + f(e_{i+2}) - f(e_i).
\]
It can be noted that, if \( \alpha(i) = (i_1, i_2) \) then \( \alpha(i + 4) = (i_1, i_2 + 1) \) and hence
\[
f(w_{i+4}) - f(w_i) = 1.
\]

\[
f(w_{2i+3}) + f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) = f(w_{2i+3}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i})
\]
\[
= f(w_{(2i-1)+4}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i})
\]
\[
= 2.
\]

\[
f(e_{i+2}) - f(e_i) = [n(p - 1) + 2 - i] - [n(p - 1) + 4 - i] = -2.
\]

Also, \( \sum f(X_{i+1}) - \sum f(X_i) = 1 \) and \( \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) = -1. \)

Hence we have \( f(H_{i+1}) - f(H_i) = 0 \) which implies that \( f(H_i) \) is constant for
\[1 \leq i \leq n - 1.\]

Since each \( H_i \) is isomorphic to \( H, \mathcal{LG}_n(H) \) is \( H \)-supermagic.

Suppose \( p = 4. \)

Since \( q \) is even, by Lemma 3.3.8, we get an \( n \)-equipartition \( Q' = \{Y'_1, Y'_2, \cdots, Y'_n\} \)
of \( [1, n(q - 2)] \) such that \( | \sum Q' | = 1. \) Adding \( 3n + 3 \) to every integer of the set
\( [1, n(q - 2)] \), we get an \( n \)-equipartition \( Q = \{Y_1, Y_2, \cdots, Y_n\} \) of \( [3n + 4, nq + n + 3] \)
such that \( | \sum Q | = 1. \)

Define the total labeling \( f : V \cup E \to [1, n(q + 1) + 3] \) as follows:
\[
f(w_i) = f_1(\alpha(i)) \text{ for } i = 1, 2, \cdots, 2n + 2.
\]
\[
f(e_i) = n(p - 1) + 4 - i \text{ for } i = 1, 2, \cdots, n + 1.
\]
\[
f(E'_i) = Y_{n-i+1} \text{ for } i = 1, 2, \cdots, n.
\]
\[
f(H_i) = \sum f(V_i) + \sum f(E_i) + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(e_i) + f(e_{i+1})
\]
\[
= \sum Y_{n-i+1} + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1}).
\]
Now,
\[
\begin{align*}
  f(H_{i+1}) - f(H_i) &= \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) + f(w_{2i+3}) \\
  &+ f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) - f(e_{i+2}) - f(e_i).
\end{align*}
\]

It can be noted that, if \( \alpha(i) = (i_1, i_2) \) then \( \alpha(i + 4) = (i_1, i_2 + 1) \) and hence
\[
  f(w_{i+4}) - f(w_i) = 1.
\]
\[
  f(w_{2i+3}) + f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) = f(w_{2i+3}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i})
  = f(w_{(2i-1)+4}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i})
  = 2.
\]
\[
  f(e_{i+2}) - f(e_i) = [n(p - 1) + 2 - i] - [n(p - 1) + 4 - i] = -2.
\]
Also, \( \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) = 0. \)

Hence we have \( f(H_{i+1}) - f(H_i) = 0 \) which implies that \( f(H_i) \) is constant for \( 1 \leq i \leq n - 1. \)

Since each \( H_i \) is isomorphic to \( H \), \( L\mathcal{G}_n(H) \) is \( H \)-supermagic.

**Sub case (iii):** \( p \) is odd and \( q \) is even.

Since \( p \) is odd, by Lemma 3.3.4, there exists an \( n \)-equipartition \( \mathcal{P}' = \{X'_1, X'_2, \ldots, X'_n\} \) of \([1, n(p-4)]\) and since \( q \) is even by Lemma 3.3.6, there exists an \( n \)-equipartition \( \mathcal{Q}' = \{Y'_1, Y'_2, \ldots, Y'_n\} \) of \([1, n(q-2)]\) such that \( \sum \mathcal{P}' \) and \( \sum \mathcal{Q}' \) are two sets of consecutive integers.

Adding \( 2n + 2 \) to every integer of the set \([1, n(p-4)]\) and \( np - n + 3 \) to every integer of the set \([1, n(q-2)]\), we get an \( n \)-equipartition \( \mathcal{P} = \{X_1, X_2, \ldots, X_n\} \) of \([2n+3, n(p-2)+2]\) and an \( n \)-equipartition \( \mathcal{Q} = \{Y_1, Y_2, \ldots, Y_n\} \) of \([np-n+3, np+n] \).
Consider the total labeling \( f : V \cup E \rightarrow [1, n(p + q - 3) + 3] \) defined as follows:

\[ f(w_i) = f_1(\alpha(i)) \text{ for } n = 1, 2, \cdots, 2n + 2. \]

\[ f(e_i) = n(p - 1) + 4 - i \text{ for } n = 1, 2, \cdots, n + 1. \]

\[ f(V'_i) = X_i \text{ for } n = 1, 2, \cdots, n. \]

\[ f(E'_i) = Y_{n-i+1} \text{ for } n = 1, 2, \cdots, n. \]

\[
f(H_i) = \sum f(V_i) + \sum f(E_i) + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1})
\]

\[ = \sum X_i + \sum Y_{n-i+1} + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1}). \]

Now,

\[
f(H_{i+1}) - f(H_i) = \sum f(X_{i+1}) - \sum f(X_i) + \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) + f(w_{2i+3})
\]

\[ + f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) + f(e_{i+2}) - f(e_i). \]

It can be noted that, if \( \alpha(i) = (i_1, i_2) \) then \( \alpha(i + 4) = (i_1, i_2 + 1) \) and hence

\[ f(w_{i+4}) - f(w_i) = 1. \]

\[ f(w_{2i+3}) + f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) = f(w_{2i+3}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i})
\]

\[ = f(w_{2(i-1)+4}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i})
\]

\[ = 2. \]

\[ f(e_{i+2}) - f(e_i) = [n(p - 1) + 2 - i] - [n(p - 1) + 4 - i] = -2. \]

Also, \( \sum f(X_{i+1}) - \sum f(X_i) = 1 \) and \( \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) = -1. \)

Hence we have \( f(H_{i+1}) - f(H_i) = 0 \) which implies that \( f(H_i) \) is constant for \( 1 \leq i \leq n - 1. \)

Since each \( H_i \) is isomorphic to \( H \), \( \mathcal{LG}_n(H) \) is \( H \)-supermagic.
Sub case (iv): $p$ is even and $q$ is odd.

Suppose $p > 4$.

Since $p$ is even and $n$ is odd, by Lemma 3.3.6, there exists an $n$-equipartition $\mathcal{P}' = \{X'_1, X'_2, \ldots, X'_n\}$ of $[1, n(p - 4)]$ and since $q$ is odd by Lemma 3.3.4, there exists an $n$-equipartition $\mathcal{Q}' = \{Y'_1, Y'_2, \ldots, Y'_n\}$ of $[1, n(q - 2)]$ such that $\sum \mathcal{P}'$ and $\sum \mathcal{Q}'$ are two sets of consecutive integers.

Adding $2n + 2$ to every integer of the set $[1, n(p - 4)]$ and $np - n + 3$ to every integer of the set $[1, n(q - 2)]$, we get an $n$-equipartition $\mathcal{P} = \{X_1, X_2, \ldots, X_n\}$ of $[2n + 3, n(p - 2) + 2]$ and an $n$-equipartition $\mathcal{Q} = \{Y_1, Y_2, \ldots, Y_n\}$ of $[np - n + 4, np + nq - 3n + 3]$ such that $\sum \mathcal{P}$ and $\sum \mathcal{Q}$ are sets of consecutive integers.

Consider the total labeling $f : V \cup E \rightarrow [1, n(p + q - 3) + 3]$ defined as follows:

$f(w_i) = f_1(\alpha(i))$ for $n = 1, 2, \ldots, 2n + 2$.

$f(e_i) = n(p - 1) + 4 - i$ for $n = 1, 2, \ldots, n + 1$.

$f(V'_i) = X_i$ for $n = 1, 2, \ldots, n$.

$f(E'_i) = Y_{n-i+1}$ for $n = 1, 2, \ldots, n$.

\[
f(H_i) = \sum f(V_i) + \sum f(E_i) + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1})
\]

\[
= \sum X_i + \sum Y_{n-i+1} + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1}).
\]

Now,

\[
f(H_{i+1}) - f(H_i) = \sum f(X_{i+1}) - \sum f(X_i) + \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) + f(w_{2i+3})
\]

\[
+ f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) + f(e_{i+2}) - f(e_i).
\]

It can be noted that, if $\alpha(i) = (i_1, i_2)$ then $\alpha(i + 4) = (i_1, i_2 + 1)$ and hence
Since each 

\[ f(w_{i+4}) - f(w_i) = 1. \]

\[
f(w_{2i+3}) + f(w_{2i+4}) - f(w_{2i-1}) = f(w_{2i+3}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i}) = f(w_{(2i-1)+4}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i}) = 2.
\]

\[
f(e_{i+2}) - f(e_i) = [n(p - 1) + 2 - i] - [n(p - 1) + 4 - i] = -2.
\]

Also, \( \sum f(X_{i+1}) - \sum f(X_i) = 1 \) and \( \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) = -1 \).

Hence we have \( f(H_{i+1}) - f(H_i) = 0 \) which implies that \( f(H_i) \) is constant for \( 1 \leq i \leq n - 1 \).

Since each \( H_i \) is isomorphic to \( H \), \( LG_n(H) \) is \( H \)-supermagic.

Suppose \( p = 4 \).

Since \( q \) and \( n \) are odd, by Lemma 3.3.16, we get an \( n \)-equi-partition \( Q' = \{Y'_1, Y'_2, \cdots, Y'_n\} \) of \([1, n(q - 2)]\) such that \( |\sum Q'| = 1 \). Adding \( 3n + 3 \) to every integer of the set \([1, n(q - 2)]\), we get an \( n \)-equi-partition \( Q = \{Y_1, Y_2, \cdots, Y_n\} \) of \([3n + 4, nq + n + 3]\) such that \( |\sum Q| = 1 \).

Define the total labeling \( f : V \cup E \rightarrow [1, n(q + 1) + 3] \) as follows:

\[
f(w_i) = f_1(\alpha(i)) \text{ for } i = 1, 2, \cdots, 2n + 2.
\]

\[
f(e_i) = n(p - 1) + 4 - i \text{ for } i = 1, 2, \cdots, n + 1.
\]

\[
f(E'_i) = Y_{n-i+1} \text{ for } i = 1, 2, \cdots, n.
\]

\[
f(H_i) = \sum f(V_i) + \sum f(E_i) + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(e_i) + f(e_{i+1})
\]

\[
= \sum Y_{n-i+1} + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1}).
\]
Now,
\[
f(H_{i+1}) - f(H_i) = \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) + f(w_{2i+3}) \\
+ f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) + f(e_{i+2}) - f(e_i).
\]

It can be noted that, if \( \alpha(i) = (i_1, i_2) \) then \( \alpha(i + 4) = (i_1, i_2 + 1) \) and hence
\[
f(w_{i+4}) - f(w_i) = 1.
\]
\[
f(w_{2i+3}) + f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) = f(w_{2i+3}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i})
\]
\[
= f(w_{(2i-1)+4}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i})
\]
\[
= 2.
\]
\[
f(e_{i+2}) - f(e_i) = [n(p - 1) + 2 - i] - [n(p - 1) + 4 - i] = -2.
\]

Also, \( \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) = 0 \).

Hence we have \( f(H_{i+1}) - f(H_i) = 0 \) which implies that \( f(H_i) \) is constant for \( 1 \leq i \leq n - 1 \).

Since each \( H_i \) is isomorphic to \( H \), \( \mathcal{L} \mathcal{G}_n(H) \) is \( H \)-supermagic.

**Case (2):** \( n \) is even.

**Sub case (i):** \( p \) and \( q \) are odd.

A similar argument as in Sub case(i) of Case (1) shows that \( \mathcal{L} \mathcal{G}_n(H) \) is \( H \)-supermagic.

**Sub case (ii):** \( p \) and \( q \) are even.

Suppose \( p > 4 \).

By Lemma 3.3.8, there exists an \( n \)-equipartition \( \mathcal{P}' = \{X'_1, X'_2, \ldots, X'_n\} \) of \([1, n(p - 4)]\) and an \( n \)-equipartition \( \mathcal{Q}' = \{Y'_1, Y'_2, \ldots, Y'_n\} \) of \([1, n(q - 2)]\) such that
| ∑ P' | = 1 and | ∑ Q' | = 1.

Adding 2n + 2 to every integer of the set [1, n(p − 4)] and np − n + 3 to every integer of the set [1, n(q − 2)], we get an n-equipartition \( P = \{X_1, X_2, \ldots, X_n\} \) of [2n + 3, n(p − 2) + 2] and an n-equipartition \( Q = \{Y_1, Y_2, \ldots, Y_n\} \) of [np − n + 4, np + nq − 3n + 3] such that | ∑ P | = 1 and | ∑ Q | = 1.

Consider the total labeling \( f : V \cup E \to [1, n(p + q − 3) + 3] \) defined as follows:

\[
f(w_i) = f_i(α(i)) \text{ for } n = 1, 2, \ldots, 2n + 2.
\]

\[
f(e_i) = n(p − 1) + 4 − i \text{ for } n = 1, 2, \ldots, n + 1.
\]

\[
f(V'_i) = X_i \text{ for } n = 1, 2, \ldots, n.
\]

\[
f(E'_i) = Y_i \text{ for } n = 1, 2, \ldots, n.
\]

\[
f(H) = \sum f(V_i) + \sum f(E_i) + f(w_{2i−1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1})
\]

\[
= \sum X_i + \sum Y_{n−i+1} + f(w_{2i−1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1}).
\]

Now,

\[
f(H_{i+1}) − f(H_i) = \sum f(X_{i+1}) − \sum f(X_i) + \sum f(Y_{n−i}) − \sum f(Y_{n−i+1}) + f(w_{2i+3})
\]

\[
+ f(w_{2i+4}) − f(w_{2i−1}) − f(w_{2i}) + f(e_{i+2}) − f(e_i).
\]

It can be noted that, if \( α(i) = (i_1, i_2) \) then \( α(i + 4) = (i_1, i_2 + 1) \) and hence

\[
f(w_{i+4}) − f(w_i) = 1.
\]

\[
f(w_{2i+3}) + f(w_{2i+4}) − f(w_{2i−1}) − f(w_{2i}) = f(w_{2i+3}) − f(w_{2i−1}) + f(w_{2i+4}) − f(w_{2i})
\]

\[
= f(w_{(2i−1)+4}) − f(w_{2i−1}) + f(w_{2i+4}) − f(w_{2i})
\]

\[
= 2.
\]

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\[ f(e_{i+2}) - f(e_i) = [n(p - 1) + 2 - i] - [n(p - 1) + 4 - i] = -2. \]

Also, \( \sum f(X_{i+1}) - \sum f(X_i) = 0 \) and \( \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) = 0. \)

Hence we have \( f(H_{i+1}) - f(H_i) = 0 \) which implies that \( f(H_i) \) is constant for \( 1 \leq i \leq n - 1. \)

Since each \( H_i \) is isomorphic to \( H \), \( \mathcal{LG}_n(H) \) is \( H \)-supermagic.

Suppose \( p = 4. \)

Since \( q \) is even, by Lemma 3.3.8, we get an \( n \)-equipartition \( Q' = \{ Y'_1, Y'_2, \ldots, Y'_n \} \) of \([1, n(q - 2)]\) such that \( |\sum Q'| = 1. \) Adding \( 3n + 3 \) to every integer of the set \([1, n(q - 2)]\), we get an \( n \)-equipartition \( Q = \{ Y_1, Y_2, \ldots, Y_n \} \) of \([3n + 4, nq + n + 3]\) such that \( |\sum Q| = 1. \)

Define the total labeling \( f : V \cup E \rightarrow [1, n(q + 1) + 3] \) as follows:

\[ f(w_i) = f_1(\alpha(i)) \text{ for } i = 1, 2, \ldots, 2n + 2. \]
\[ f(e_i) = n(p - 1) + 4 - i \text{ for } i = 1, 2, \ldots, n + 1. \]
\[ f(E'_i) = Y_{n-i+1} \text{ for } i = 1, 2, \ldots, n. \]

\[ f(H_i) = \sum f(V_i) + \sum f(E_i) + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1}) \]
\[ = \sum Y_{n-i+1} + f(w_{2i-1}) + f(w_{2i}) + f(w_{2i+1}) + f(w_{2i+2}) + f(e_i) + f(e_{i+1}). \]

Now,
\[ f(H_{i+1}) - f(H_i) = \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) + f(w_{2i+3}) \]
\[ + f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) + f(e_{i+2}) - f(e_i). \]
It can be noted that, if \( \alpha(i) = (i_1, i_2) \) then \( \alpha(i + 4) = (i_1, i_2 + 1) \) and hence \( f(w_{i+4}) - f(w_i) = 1. \)

\[
f(w_{2i+3}) + f(w_{2i+4}) - f(w_{2i-1}) - f(w_{2i}) = f(w_{2i+3}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i})
\]
\[
= f(w_{(2i-1)+4}) - f(w_{2i-1}) + f(w_{2i+4}) - f(w_{2i})
\]
\[
= 2.
\]

\[
f(e_{i+2}) - f(e_i) = [n(p - 1) + 2 - i] - [n(p - 1) + 4 - i] = -2.
\]

Also, \( \sum f(Y_{n-i}) - \sum f(Y_{n-i+1}) = 0. \)

Hence we have \( f(H_{i+1}) - f(H_i) = 0 \) which implies that \( f(H_i) \) is constant for \( 1 \leq i \leq n - 1. \)

Since each \( H_i \) is isomorphic to \( H \), \( \mathcal{L}G_n(H) \) is \( H \)-supermagic. \( \square \)

**Example 5.6.5.** A \( C_5 \)-supermagic labeling of a linear garland \( \mathcal{L}G_5(C_5) \) with supermagic sum 175 is given in Figure 5.29.

![Image](image.png)

Figure 5.29: \( C_5 \)-supermagic labeling of a linear garland \( \mathcal{L}G_5(C_5) \).
Example 5.6.6. A $H$-supermagic labeling of a linear garland $\mathcal{L}G_3(H)$ of a given $(6,8)$-graph $H$ with supermagic sum 239 is given in Figure 5.30.

![Figure 5.30](image)

Figure 5.30: $H$-supermagic labeling of a linear garland $\mathcal{L}G_3(H)$ of a 2-connected $(6,8)$ graph $H$.

Example 5.6.7. A $H$-supermagic labeling of a linear garland $\mathcal{L}G_3(H)$ of a given $(5,6)$-graph $H$ with supermagic sum 141 is given in Figure 5.31.

![Figure 5.31](image)

Figure 5.31: $H$-supermagic labeling of a linear garland $\mathcal{L}G_3(H)$ of a 2-connected $(5,6)$-graph $H$. 
Example 5.6.8. A $H$-supermagic labeling of a linear garland $\mathcal{L}G_5(H)$ of a given $(4,5)$-graph $H$ with supermagic sum 135 is given in Figure 5.32.

![Figure 5.32: $H$-supermagic labeling of a linear garland $\mathcal{L}G_5(H)$ of a 2-connected (4,5) graph $H$.](image)

Example 5.6.9. A $C_5$-supermagic labeling of a linear garland $\mathcal{L}G_4(C_5)$ with supermagic sum 147 is given in Figure 5.33.

![Figure 5.33: $C_5$-supermagic labeling of a linear garland $\mathcal{L}G_4(C_5)$.](image)
Example 5.6.10. A $C_6$-supermagic labeling of a linear garland $LG_4(C_6)$ with supermagic sum 221 is given in Figure 5.34.

![Figure 5.34: $C_6$-supermagic labeling of a linear garland $LG_4(C_6)$.](image)

We know that the ladder graph is defined as the cartesian product $P_2 \times P_n$ of a path on 2 vertices and another path on $n$ vertices. It can also be considered as a linear garland of $C_4$ of length $n$. Hence, the ladder $P_2 \times P_n = LG_n(C_4)$. The following corollary is the consequence of Theorem 5.6.4.

**Corollary 5.6.11.** The ladder graph $P_2 \times P_n$ is $C_4$-supermagic for all $n$.

Example 5.6.12. A $C_4$-supermagic labeling of the ladder $P_2 \times P_7$ with supermagic sum 123 is given in Figure 5.35.

![Figure 5.35: $C_4$-supermagic labeling of the ladder $P_2 \times P_7$.](image)