CHAPTER 6

TWO DIMENSIONAL FUZZY DECISION MAKING

6.1 INTRODUCTION

The decision making is a crucial problem for success. Let $M_1$, $M_2$, $\cdots$, $M_m$ be $m$ alternatives available and $C_1$, $C_2$, $\cdots$, $C_n$ be $n$ criteria involved in the measurement of alternatives. Let $A_{ij}$ be the performance of alternative $M_i$ with respect to criteria $C_j$ and $w_j$ be the relative importance of criteria. Then the decision making problem is the selection of the best alternative with respect to criteria.

In classical decision making problems $A_{ij}$ and $w_j$ are real numbers. In reality, $A_{ij}$ and $w_j$ are not necessarily real numbers. They may be linguistic terms like good, poor which are fuzzy in nature. A fuzzy version of Analytic Hierarchy Process (AHP) Saaty (1990) was studied by Buckley (1985), Evangelos Triantaphyllou and Stuart H. Mann (1995), Laarhoven and Pedrycz (1983) using triangular fuzzy numbers for linguistic terms. The two dimensional canonical fuzzy number was introduced by Ramik and Nakamura (1993) and it was extended to n-dimensional fuzzy number by Wang and Wu (2002), Guixiang Wang et al (2007). In fact, when we collect data from the expert, he always shows hesitation or confidence. So in this chapter, a new method of fuzzy decision making using two dimensional fuzzy numbers by including hesitation has been introduced and is illustrated by an example in health care.

First we give a brief review of definitions.
Note 6.1.1. (Chen and Hwang (1992)) Let \((l, m, n)\) be a triangular fuzzy number. Then the total score of \(M\) is given by

\[ T(M) = \left[ 1 + R(M) - L(M) \right] / 2, \]

where left score \(L\) and right score \(R\) of this fuzzy number are given by

\[ L = \frac{1-l}{1+m-n}, \quad R = \frac{n}{1+n-m}. \]

Definition 6.1.1. (Wang and Wu (2002)) A two dimensional triangular fuzzy number is an ordered pair of fuzzy numbers \((\mu_1, \mu_2)\), where \(\mu_1, \mu_2\) are triangular fuzzy numbers.

Definition 6.1.2. Let \(A = \{(l_1, m_1, n_1), (a_1, b_1, c_1)\}\) and \(B = \{(l_2, m_2, n_2), (a_2, b_2, c_2)\}\) be two triangular two dimensional fuzzy numbers. Then \(\alpha A, A+B\) and \(AB\) are given by

\(\alpha A = \{(\alpha l_1, \alpha m_1, \alpha n_1), (\alpha a_1, \alpha b_1, \alpha c_1)\}\), where \(\alpha\) is a real number,

\(A + B = \{(l_1 + l_2, m_1 + m_2, n_1 + n_2), (a_1 + a_2, b_1 + b_2, c_1 + c_2)\}\)

\(AB = \{(l_1l_2, m_1m_2, n_1n_2), (a_1a_2, b_1b_2, c_1c_2)\}\) and

if \(l_2, m_2, n_2, a_2, b_2, c_2\) are nonzero real numbers, then \(\frac{A}{B}\) is given by

\(\frac{A}{B} = \{(\frac{l_1}{l_2}, \frac{m_1}{m_2}, \frac{n_1}{n_2}), (\frac{a_1}{b_1}, \frac{b_1}{b_2}, \frac{c_1}{c_2})\}\)

6.2 TWO DIMENSIONAL FUZZY LINGUISTIC TERMS

In this section, the notion of two dimensional fuzzy linguistic terms are introduced and studied.

6.2.1. Two dimensional linguistic terms

Whenever we collect data, the expert expresses his opinion with hesitation. For example, a doctor says that a particular medicine is 0.6 membership grade of preferred medicine. If we ask him does it mean that the
medicine is 0.4 membership grade of non-preferred medicine, then definitely he shows his confidence or hesitation. So in real life problems, one can model an expert’s opinion more precisely by this two dimensional linguistic terms.

**Definition 6.2.1.** A two dimensional linguistic term is given by an ordered pair \((l_1, l_2)\) of linguistic terms.

**Note 6.2.1.** In this paper two dimensional linguistic terms \((l_1, l_2)\) are used to represent the expert’s opinion with their hesitation. Hence \(l_1\) denotes the expert’s opinion and \(l_2\) denotes the expert’s hesitation on his opinion. This way of approach generalises ordinary fuzzy linguistic terms by including no hesitation with it.

In the above example, if the doctor’s opinion is ‘preferred with little hesitation’ or ‘preferred with strong hesitation’ or ‘preferred with no hesitation’, then according to his hesitation or confidence, we can model these as two dimensional linguistic terms (preferred, little), (preferred, strong), (preferred, no hesitation).

Some of the two dimensional linguistic terms are (preferred, little), (preferred, moderate), (excellent, very little). If the expert’s opinion is excellent, then he will not have very strong hesitation. Hence (excellent, very strong) is not possible. In this paper, the linguistic terms for the level of hesitation are very little, little, moderate, strong, very strong.

**6.2.2. Conversion of two dimensional linguistic terms**

A two dimensional linguistic term can be converted into two dimensional fuzzy number using triangular fuzzy numbers. The conversion of opinion into triangular fuzzy numbers are given in the conversion table.
The conversion of the levels of hesitation into triangular fuzzy numbers are also given in the conversion table.

6.2.3. Scores of two dimensional fuzzy numbers

The score of two dimensional fuzzy number can be found using the crisp scores of fuzzy numbers defined as in the Note 6.1.1.

Let \((M, H)\) be a two dimensional fuzzy number. Then the score of the two dimensional fuzzy number is

\[
T = \left( \frac{1 + R(M) - L(M)}{2}, \frac{1 + R(H) - L(H)}{2} \right),
\]

where \(L(M), R(M)\) and \(L(H), R(H)\) are left and right scores of opinion and hesitancy fuzzy numbers respectively. Let \(A = \{(l, m, n), (a, b, c)\}\) be two dimensional fuzzy number. Hence \(T_A\) is given by

\[
T_A = \left( \frac{1 + \frac{n}{1 + n - m} - \frac{1 - l}{1 + m - l}}{2}, \frac{1 + \frac{c}{1 + c - b} - \frac{1 - a}{1 + b - a}}{2} \right).
\]

6.3 TWO DIMENSIONAL FUZZY DECISION MAKING

The two dimensional fuzzy decision making is the problem of selecting the best alternative from \(M_1, M_2, \cdots, M_m\) based on the criteria \(C_1, C_2, \cdots, C_n\) in which \(A_{ij}\), the performance of the \(i^{th}\) alternative \(M_i\) with respect to the \(j^{th}\) criterion (i.e., \(C_j\)), is two dimensional fuzzy in nature.

6.3.1. Priorities of criteria

In decision making problems, the decision differs with different priorities of criteria. The pairwise comparison matrix between criteria is formed using Saaty’s scale and the relative priority vector of each criterion is found.
6.3.2. Priorities of Alternatives

The pairwise comparison of alternatives with respect to each criterion is formed carefully using the following two dimensional linguistic comparative words like (equally preferred, no hesitaion) and (moderately preferred, little), where equally preferred, moderately preferred are expert’s opinion and no hesitaion, little are their levels of hesitation. The verbal judgment of preferences are converted into two dimensional fuzzy numbers using 6.2.2 and the conversion table.

Now each entry of the above pairwise two dimensional comparison matrix with respect to each criterion is divided by its column sum using definition 6.1.2. Now the row average of the resulting matrix is the two dimensional priority vector with respect to each criterion.

6.3.3. Two Dimensional fuzzy score of Alternatives

Finally the two dimensional fuzzy score of each alternative is the sum of the products of weights of criteria with their corresponding two dimensional fuzzy numbers in two dimensional fuzzy priority vectors. Let $w_j$ be the weight of the criterion $C_j$, $j = 1, 2, \ldots n$ and $(P_{A_i})_j, i = 1, 2, \ldots m$ be the two dimensional fuzzy priority vector of $A_i$ corresponding to $C_j$.

The fuzzy score $F_{A_i}$ of $A_i$ is given by

$$ F_{A_i} = \sum w_j(P_{A_i})_j $$

6.3.4. Crisp scores of Alternatives

The two dimensional crisp score $\{M(A_i), H(A_i)\}$ for each alternative $A_i$ can be found by 6.2.3.
Now the score of hesitancy variable \((a, b, c)\) are found by

\[
\text{Hesitancy score} = 10 \left( \frac{1 + R - L}{2} \right),
\]

where \(R = \frac{c}{10 + c - b}\) and \(L = \frac{10 - a}{10 + b - a}\).

The crisp score \((T(A_i))\) of each alternative = Membership score \(M(A_i)\) - \(k\) times the hesitancy score \(H(A_i)\), where

\[
k = \frac{\text{The minimum difference of scores in hesitancy variable}}{\text{number of hesitancy variable}}.
\]

From the crisp score, the best alternative which has maximum score can be chosen.

6.4 ILLUSTRATION

The structure of the typical two dimensional decision problem considered in this chapter consists of five antibiotics \(A_1, A_2, A_3, A_4, A_5\) which are all very close to one another based on the following four criteria \(C_1, C_2, C_3\) and \(C_4\). Let \(w_j\) be the weight of the criterion \(C_j\).

6.4.1 Priorities of criteria

According to the experts, the pair wise comparisons between criteria in linguistic terms are given by,

1. Each criterion is equally preferred to itself
2. \(C_2\) is equally to moderately preferred to \(C_1\)
3. \(C_3\) is strongly preferred to \(C_1\)
4. \(C_1\) is equally to moderately preferred to \(C_4\)
5. \(C_2\) is moderately preferred to \(C_4\)
6. $C_3$ is moderately preferred to $C_2$
7. $C_3$ is strongly preferred to $C_4$

Using 6.3.1, the weights of criteria $C_1$, $C_2$, $C_3$ and $C_4$ can be found. They are given by $w_1 = .1296, w_2 = .2268, w_3 = .5601$ and $w_4 = .0835$.

6.4.2. Priorities of Medicines

In this paper several doctors are consulted and the data of this problem have been collected. The pairwise comparison of medicines with respect to each criterion is formed carefully using two dimensional linguistic comparative words.

The pairwise comparison of medicines with respect to each criterion are given below.

Each alternative is equally preferred to itself with no hesitation for each criterion.

**Pairwise Comparison of Medicines based on $C_1$**

1. $A_2$ is (equally to moderately preferred, strong) to $A_1$
2. $A_4$ is (moderately to strongly preferred, very little) to $A_1$
3. $A_5$ is (moderately preferred, little) to $A_1$
4. $A_4$ is (moderately preferred, little ) to $A_2$
5. $A_5$ is (equally to moderately preferred, moderate ) to $A_2$
6. $A_1$ is (equally to moderately preferred, moderate ) to $A_3$
7. $A_2$ is (moderately preferred, no hesitation ) to $A_3$
8. $A_4$ is (strongly preferred, no hesitation ) to $A_3$
9. $A_5$ is (moderately to strongly preferred, very little) to $A_3$
10. $A_5$ is (equally to moderately preferred, strong) to $A_4$

**Pairwise Comparison of Medicines based on $C_2$**

1. $A_5$ is (equally to moderately preferred, strong) to $A_1$
2. $A_1$ is (equally to moderately preferred, moderate) to $A_2$
3. $A_5$ is (strongly preferred, very little) to $A_2$
4. $A_1$ is (strongly preferred, no hesitation) to $A_3$
5. $A_2$ is (equally to moderately preferred, moderate) to $A_3$
6. $A_5$ is (strongly to very strongly preferred, no hesitation) to $A_3$
7. $A_1$ is (very strong to extremely preferred, no hesitation) to $A_4$
8. $A_2$ is (strongly to very strongly preferred, no hesitation) to $A_4$
9. $A_3$ is (strongly preferred, little) to $A_4$
10. $A_5$ is (extremely preferred, no hesitation) to $A_4$

**Pairwise Comparison of Medicines based on $C_3$**

1. $A_5$ is (equally to moderately preferred, strong) to $A_1$
2. $A_1$ is (moderately preferred, little) to $A_2$
3. $A_5$ is (moderately to strongly preferred, moderate) to $A_2$
4. $A_1$ is (moderately to strongly preferred, very little) to $A_3$
5. $A_2$ is (moderately preferred, strong) to $A_3$
6. $A_5$ is (strongly preferred, little) to $A_3$
7. $A_1$ is (strongly preferred, no hesitation) to $A_4$
8. $A_2$ is (moderately to strongly preferred, very little) to $A_4$
9. $A_3$ is (moderately preferred, moderate) to $A_4$
10. $A_5$ is (very strongly preferred, no hesitation) to $A_4$
**Pairwise Comparison of Medicines based on $C_4$**

1. $A_3$ is (moderately to strongly preferred, very little) to $A_1$
2. $A_4$ is (moderately preferred, very little) to $A_1$
3. $A_1$ is (moderately preferred, very little) to $A_2$
4. $A_3$ is (strongly to very strongly preferred, no hesitation) to $A_2$
5. $A_4$ is (strongly preferred, no hesitation) to $A_2$
6. $A_3$ is (equally to moderately preferred, very little) to $A_4$
7. $A_1$ is (strongly preferred, no hesitation) to $A_5$
8. $A_2$ is (moderately preferred, very little) to $A_5$
9. $A_3$ is (extremely preferred, no hesitation) to $A_5$
10. $A_4$ is (very strong to extremely preferred, no hesitation) to $A_5$

These two dimensional linguistic terms have been converted into two dimensional fuzzy numbers using 6.2.2.

Using 6.3.2, the two dimensional fuzzy priority vectors (TF priority vectors) of the medicines based on each criteria are given in the table.

Using 6.3.3 and 6.3.4, the two dimensional crisp scores of $A_1$, $A_2$, $A_3$, $A_4$, and $A_5$ with their corresponding hesitancy degree are

\{(0.298524, 0.232199)\}, \{(0.16476, 0.32445)\}, \{(0.144375, 0.31476)\},
\{(0.14015, 0.08143)\} and \{(0.42255, 0.330105)\} respectively.

Finally, the crisp score of each alternative for this problem can be found as follows.

The crisp score of each alternative = Membership score - $\frac{2}{11}$ hesitancy score.
The crisp score of $A_1$, $A_2$, $A_3$, $A_4$, and $A_5$ are given by 0.256306, 0.10577, 0.08714, 0.125344 and 0.36253 respectively.

6.5 CONCLUSION

From the problem under consideration and data collected from experts, the total score of each medicine can be found and finally we conclude that $A_5$ is the most appropriate medicine. Alternatively $A_1$ can also be preferably used. Even if the opinion score of $A_2$ is higher than that of $A_4$, the final score of $A_4$ is higher than that of $A_2$ due to hesitation. Hence finally $A_4$ is preferred to $A_2$. So two dimensional fuzzy decision making is an effective tool for the decision making problem where hesitation plays a vital role. Since the problems of computer aided drug prescription and drug interaction are of great importance to practice, this is a good starting point in this field.
Conversion Table

<table>
<thead>
<tr>
<th>Levels of hesitation</th>
<th>Fuzzy Numbers</th>
<th>Crisp scores</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very little</td>
<td>(1, 1, 2)</td>
<td>$\frac{10}{11}$</td>
</tr>
<tr>
<td>little</td>
<td>(1, 2, 3)</td>
<td>$\frac{25}{11}$</td>
</tr>
<tr>
<td>moderate</td>
<td>(2, 3, 4)</td>
<td>$\frac{35}{11}$</td>
</tr>
<tr>
<td>strong</td>
<td>(3, 4, 5)</td>
<td>$\frac{45}{11}$</td>
</tr>
<tr>
<td>Very strong</td>
<td>(4, 5, 6)</td>
<td>$\frac{55}{11}$</td>
</tr>
</tbody>
</table>

Verbal judgment of preference  Fuzzy Number

| Equally preferred                | (1, 1, 1) |
| Equally to moderately preferred  | (1, 2, 3) |
| Moderately preferred             | (2, 3, 4) |
| Moderately to strongly preferred  | (3, 4, 5) |
| Strongly preferred               | (4, 5, 6) |
| Strongly to very strongly preferred | (5, 6, 7) |
| Very strongly preferred          | (6, 7, 8) |
| Very strong to extremely preferred | (7, 8, 9) |
| Extremely preferred              | (8, 9, 10) |

TF Priority Vectors Table

A. TF priority vectors for Medicines based on $C_1$

$A_1 \quad \{(0.040401, 0.098573, 0.195571), (0.11845, 0.2459, 0.49)\}$
$A_2 \quad \{(0.084036, 0.161051, 0.321320), (0.8167, 0.1523, 0.2967)\}$
$A_3 \quad \{(0.038220, 0.062376, 0.116866), (0.0423, 0.1091, 0.1812)\}$
$A_4 \quad \{(0.226218, 0.416212, 0.738133), (0.0965, 0.1285, 0.2121)\}$
$A_5 \quad \{(0.137110, 0.261788, 0.505329), (0.2631, 0.3643, 0.6176)\}$
B. TF priority vectors for Medicines based on $C_2$

\[ A_1 \quad \{(0.202463, 0.3071, 0.492912), (0.09153, 0.17846, 0.34139)\} \]
\[ A_2 \quad \{(0.078457, 0.122509, 0.188921), (0.61409, 0.34681, 0.65934)\} \]
\[ A_3 \quad \{(0.059036, 0.085985, 0.132636), (0.07436, 0.21538, 0.63077)\} \]
\[ A_4 \quad \{(0.022627, 0.031068, 0.043072), (0.01333, 0.02857, 0.08571)\} \]
\[ A_5 \quad \{(0.287049, 0.453338, 0.703853), (0.10909, 0.23077, 0.43077)\} \]

C. TF priority vectors for Medicines based on $C_3$

\[ A_1 \quad \{(0.170924, 0.284081, 0.491579), (0.04582, 0.13743, 0.32346)\} \]
\[ A_2 \quad \{(0.079932, 0.136802, 0.232723), (0.09394, 0.23881, 0.50364)\} \]
\[ A_3 \quad \{(0.055442, 0.089973, 0.155109), (0.12212, 0.28667, 0.62821)\} \]
\[ A_4 \quad \{(0.030737, 0.045344, 0.070888), (0.01676, 0.04109, 0.0731)\} \]
\[ A_5 \quad \{(0.266626, 0.443800, 0.721679), (0.15276, 0.296, 0.59137)\} \]

D. TF priority vectors for Medicines based on $C_4$

\[ A_1 \quad \{(0.092159, 0.143910, 0.228114), (0.15, 0.3, 0.5)\} \]
\[ A_2 \quad \{(0.046356, 0.071305, 0.110575), (0.12, 0.2667, 0.48)\} \]
\[ A_3 \quad \{(0.279312, 0.449211, 0.711002), (0.1067, 0.1667, 0.4267)\} \]
\[ A_4 \quad \{(0.193073, 0.301053, 0.496174), (0.09, 0.1667, 0.26)\} \]
\[ A_5 \quad \{(0.024536, 0.034521, 0.050424), (0.0333, 0.1, 0.1333)\} \]