CHAPTER 5

INDUCED TOPOLOGY ON INTUITIONISTIC FUZZY SINGLETONS

5.1 INTRODUCTION

The notion of fuzzy sets was introduced by Zadeh (1965) and it was extended to the concept of intuitionistic fuzzy sets by Atanassov (1986). The notion of fuzzy topological spaces was introduced studied by Chang (1968). The concepts of fuzzy Hausdorffness, fuzzy connectedness, fuzzy compactness were studied by Gantner (1978), Wong (1974 b). Pu and Liu (1980a) introduced the concept of fuzzy singletons. The notion of intuitionistic fuzzy topological spaces was introduced and studied by Coker (1997). The concepts of intuitionistic fuzzy Hausdorffness, intuitionistic fuzzy connectedness, intuitionistic fuzzy compactness were studied by Coker (1997), Coker (2000), Gallego (2003), Hanafy (2003). Gallego (2003) introduced the concept of intuitionistic fuzzy points. In this chapter, a new notion of induced topologies on the collection of fuzzy singletons and intuitionistic fuzzy singletons are introduced and studied. The relations of the existing notions in fuzzy topological spaces and intuitionistic fuzzy topological spaces with their analogous notions in this topology have been studied.

Here we give some existing concepts in the literature for the completion of self content of this chapter.

Definition 5.1.1. (Coker (1997)) An intuitionistic fuzzy topology on any nonempty set \( X \) is a collection \( \delta \) of intuitionistic fuzzy subsets of \( X \) satisfying
1. \( \tilde{0} = (0, 1) \in \delta, \tilde{1} = (1, 0) \in \delta \)

2. Intersection of members of any finite subcollection \( \sigma \subseteq \delta \) is also a member of \( \delta \).

3. Union of the members of any subcollection \( \sigma_0 \subseteq \delta \) is again a member of \( \delta \).

The pair \((X, \delta)\) is called an intuitionistic fuzzy topological space. Elements of \( \delta \) are called intuitionistic fuzzy open sets. An intuitionistic fuzzy set \( A = (\mu_A, \gamma_A) \) is said to be intuitionistic fuzzy closed if \( A^c = (\gamma_A, \mu_A) \) is intuitionistic fuzzy open.

As in general topology, the indiscrete intuitionistic fuzzy topology contains only \( \tilde{0} \) and \( \tilde{1} \) while the discrete topology contains all intuitionistic fuzzy subsets of \( X \).

**Definition 5.1.2.** (Coker (1997)) An intuitionistic fuzzy topological space \((X, \delta)\) is said to be intuitionistic fuzzy Hausdorff if for every pair of points \( x, y \in X \) such that \( x \neq y \), there exist intuitionistic fuzzy open sets \( A = (\mu_A, \gamma_A), B = (\mu_B, \gamma_B) \in \delta \) such that \( \mu_A(x) = 1, \mu_B(y) = 1 \) and \( A \cap B = \tilde{0} \).

**Definition 5.1.3.** (Coker (1997)) An intuitionistic fuzzy topological space \((X, \delta)\) is said to be intuitionistic fuzzy compact if for every collection \( \sigma \) of intuitionistic fuzzy open sets such that \( \tilde{1} = \bigcup_{A \in \sigma} A \), there exist a finite subcollection \( A_1, A_2, ..., A_n \) of \( \sigma \) such that \( \tilde{1} = \bigcup_{i=1}^{n} A_i \).

**Definition 5.1.4.** (Coker (2000)) An intuitionistic fuzzy topological space \((X, \delta)\) is said to be intuitionistic fuzzy connected if \( \tilde{1} \) cannot be expressed as the union of two intuitionistic fuzzy open sets \( A = (\mu_A, \gamma_A), B = (\mu_B, \gamma_B) \in \delta \) such that \( A \cap B = \tilde{0} \).
5.2 INDUCED TOPOLOGY ON FUZZY SINGLETONS

In this section, a new notion of induced topology on fuzzy singletons is introduced and some results are proved. Some results are given whose proofs can be easily obtained in similar way to the proof of the analogous results in the next section.

Definition 5.2.1. Let $(G, \delta)$ be a fuzzy topological space. An induced topology $\tau_\delta$ on the collection $\varphi(G)$ of all fuzzy singletons of $G$ is defined as the topology generated by $\sigma = \{V_\mu \mid \mu \in \delta\}$, where $V_\mu = \{p \in \varphi(G) \mid p \in \mu\}$ and hence $(\varphi(G), \tau_\delta)$ is called an induced topological space.

The proof of the following note is immediate from the definition.

Note 5.2.1. Clearly the collection $\sigma = \{V_\mu \mid \mu \in \delta\}$ is a basis for a topology on $\varphi(G)$.

Note 5.2.2. The topology $\tau_\delta$ on $\varphi(G)$ need not be discrete as is seen from the following example.

Example 5.2.1. Let $G = \{e, x, y, xy\}$ be Klein’s four group and $a \in [0, 1/2]$. Let $\delta$ be the collection of all fuzzy sets $\mu$ whose fuzzy value $\mu(z) \in [0, a] \cup \{1\}$, for every $z$ in $G$. Clearly $(G, \delta)$ is a fuzzy topological space.

The topology $\tau_\delta$ on $\varphi(G)$ is not discrete. For, Consider a fuzzy singleton $p$ defined on $x$ with value $b > a$. Any fuzzy open set containing $p$ has the value 1 at $x$. Hence every fuzzy singleton at $x$ with any value lies in $V_1$ and hence $\{p\}$ is not a member of $\tau_\delta$. 
**Theorem 5.2.1.** If \((G, \delta)\) is fuzzy compact, then \((\wp(G), \tau_{\delta})\) is compact. But the converse need not be true.

**Proof:** Let \((G, \delta)\) be fuzzy compact and \(\wp(G) = \bigcup_{\sigma_0 \subseteq \sigma} V_{\sigma_0}\) for some \(\sigma_0 \subseteq \sigma\). Now \(\forall x \in G, 1_x \in \wp(G)\), where \(1_x\) is a fuzzy singleton on \(x\) with fuzzy value 1 and hence \(1_x \in V_{\mu_x}\), for some \(V_{\mu_x} \in \sigma_0\). So \(1_x \in \mu_x\) and hence \(1 = \bigvee_{x \in G} 1_x\). By fuzzy compactness we have \(1 = \bigvee_{i=1}^{n} 1_{\mu_i}\) for some \(i = 1, 2, \ldots, n\) and hence \(\wp(G) = \bigvee_{i=1}^{n} V_{\mu_i}\). Therefore \((\wp(G), \tau_{\delta})\) is compact.

The converse need not be true as can be seen from the following example.

**Example 5.2.2.** Let \(X\) be any set and \(\delta = \{0, 1, 1 - 1/n\}\). Obviously \((X, \delta)\) is a fuzzy topological space. Any open covering of \(\wp(G)\) contains \(V_1\) and hence \((\wp(G), \tau_{\delta})\) is compact. But clearly \((X, \delta)\) is not compact.

**Note 5.2.3.** Two fuzzy singletons of same support but with different values are distinct in \(\wp(G)\). We can’t separate them by two open sets. So we can’t think about Hausdorffness in \(\wp(G)\). But any two fuzzy singletons of different supports can be separated by two open sets. Hence we can speak about pseudo Hausdorff space on \((\wp(G), \tau_{\delta})\) as given below.

The pair \((\wp(G), \tau_{\delta})\) is said to be pseudo Hausdorff if for any two fuzzy singletons \(p, q\) of different supports, there exists two disjoint open sets \(V_{\mu}, V_{\nu}\) such that \(p \in V_{\mu}, q \in V_{\nu}\).

The following theorems can be proved in a similar way as in the proof of theorems 5.3.2 and 5.3.3.

**Theorem 5.2.2.** A fuzzy topological space \((G, \delta)\) is fuzzy Hausdorff if and
only if \((\varphi(G), \tau_\delta)\) is pseudo Hausdorff.

**Theorem 5.2.3.** \((G, \delta)\) is fuzzy connected if and only if \((\varphi(G), \tau_\delta)\) is connected.

**Theorem 5.2.4.** Let \((G, \delta)\) be a fuzzy topological space and \(F \subseteq G\). The subspace on \(\varphi(F)\) inherited from \((\varphi(G), \tau_\delta)\) is equal to the topological space on \(\varphi(F)\) induced by the fuzzy subspace on \(F\) inherited from \((G, \tau_\delta)\).

**Proof:** Let \((G, \delta)\) be a fuzzy topological space and \(F \subseteq G\). First we prove the following claim.

**Claim:** \(V_\mu \cap \varphi(F) = V_{\mu|F}, \) for any fuzzy open set \(\mu \in \delta\).

**Proof of the claim:** Let \(p \in V_\mu \cap \varphi(F)\). Since \(p \in V_\mu, p(x) \leq \mu(x)\). Since \(p\) in \(\varphi(F)\), \(\text{supp } p = x \in F\). Hence \(p(x) \leq (\mu|_F)(x)\). So \(p \in V_{\mu|_F}\). Hence \(V_\mu \cap \varphi(F) \subseteq V_{\mu|_F}\). Now let \(p \in V_{\mu|_F}\). So \(p(x) \leq (\mu|_F)(x)\). If \(\text{supp } p = x \notin F\), \((\mu|_F)(x) = 0\), \(p(x) = 0\), which is a contradiction to the choice of \(p\). So \(\text{supp } p = x \in F\). Therefore \(p \in V_\mu \cap \varphi(F)\). Hence the claim.

Let \(U \cap \varphi(F)\) be an arbitrary open set in \((\varphi(F), \tau_\delta \cap \varphi(F))\), where \(U\) is open in \(\tau_\delta\). By definition \(U = \bigcup_{\mu \in \sigma} V_\mu\), for some \(\sigma \subseteq \delta\). Hence \(U \cap \varphi(F) = \bigcup_{\mu \in \sigma} V_\mu \cap \varphi(F) = \bigcup_{\mu \in \sigma} \{V_\mu \cap \varphi(F)\}\). So by the claim \(U \cap \varphi(F) = \bigcup_{\mu \in \sigma} \{V_{\mu|_F}\} \in (\varphi(F), \tau_{\delta|_F})\).

Similarly any basic open set \(V_{\mu|_F}\) in \((\varphi(F), \tau_{\delta|_F})\) is a basic open set in \((\varphi(F), \tau_\delta \cap \varphi(F))\).

**Theorem 5.2.5.** Let \((G_1, \delta_1), (G_2, \delta_2)\) be fuzzy topological spaces. Then the induced topological space on \(\varphi(G_1 \times G_2)\) by the product fuzzy topological
space \((G_1 \times G_2, \delta_1 \times \delta_2)\) is embedded in the product topological space \((\varphi(G_1) \times \varphi(G_2), \tau_{\delta_1} \times \tau_{\delta_2})\).

**Proof :** It is enough to prove that there exists a function \(f : (\varphi(G_1 \times G_2), \tau_{\delta_1 \times \delta_2}) \rightarrow (\varphi(G_1) \times \varphi(G_2), \tau_{\delta_1} \times \tau_{\delta_2})\) such that \(f\) is \(1-1\) and both \(f\) and \(f^{-1}\) are continuous.

Let \(f : (\varphi(G_1 \times G_2), \tau_{\delta_1 \times \delta_2}) \rightarrow (\varphi(G_1) \times \varphi(G_2), \tau_{\delta_1} \times \tau_{\delta_2})\) be defined by \(f(p) = (p_1, p_2)\), where \(p\) is defined on \((x_1, x_2) \in G_1 \times G_2\) and \(p_1, p_2\) are defined on \(x_1, x_2\) of \(G_1, G_2\) respectively with value \(p(x_1, x_2)\).

First we prove that \(f\) is \(1-1\). Let \(f(p) = f(q)\), where \(p\) and \(q\) are fuzzy singletons defined on \((x_1, y_1)\) and \((x_2, y_2)\) respectively with values \(p(x_1, x_2)\) and \(q(y_1, y_2)\) respectively. By definition, \(f(p) = (p_1, p_2)\) and \(f(q) = (q_1, q_2)\), where \(p_1, p_2, q_1, q_2\) are fuzzy singletons on \(x_1, x_2, y_1\) and \(y_2\) respectively. So by hypothesis, \((p_1, p_2) = (q_1, q_2)\). Hence \(p_1 = q_1\) and \(p_2 = q_2\). So \(x_1 = y_1, x_2 = y_2\) and \(p(x_1, x_2) = p_1(x_1) = q_1(y_1) = q(y_1, y_2)\). Hence \(p = q\).

Let \(F = \{(p_1, p_2) \in \varphi(G_1 \times G_2) \mid p_1(x_1) = p_2(x_2)\} \subseteq \varphi(G_1) \times \varphi(G_2)\). Clearly if \((p_1, p_2) \in F\), then there exists \(p \in \varphi(G_1 \times G_2)\) with \(p(x_1, x_2) = p_1(x_1) = p_2(x_2)\). So \(f : \varphi(G_1 \times G_2) \rightarrow F\) is a bijection.

Now we prove that \(f\) and \(f^{-1}\) are continuous. For this we prove the following claim.

**Claim :** If \(\mu \in \delta_1\) and \(\nu \in \delta_2\), then \(f^{-1}(V_{\mu} \times V_{\nu}) = f^{-1}((V_{\mu} \times V_{\nu}) \cap F) = V_{\mu \times V_{\nu}}\) and \(f(V_{\mu \times V_{\nu}}) = (V_{\mu} \times V_{\nu}) \cap F\).

**Proof of the Claim :** Clearly \(p \in f^{-1}((V_{\mu} \times V_{\nu}) \cap F) \Leftrightarrow f(p) \in (V_{\mu} \times V_{\nu}) \cap F\) \(\Leftrightarrow p_1 \in V_{\mu}\) and \(p_2 \in V_{\nu}\), by definition of \(f\).
\[ p_1(x_1) \leq \mu(x_1), p_2(x_2) \leq \nu(x_2) \text{ and } p_1(x_1) = p_2(x_2) = p(x_1, x_2) \]
\[ p(x_1, x_2) \leq \mu(x_1) \text{ and } p(x_1, x_2) \leq \nu(x_2) \]
\[ p(x_1, x_2) \leq \min\{\mu(x_1), \nu(x_2)\} \]
\[ p(x_1, x_2) \leq (\mu \times \nu)(x_1, x_2) \]
\[ p \in V_{\mu \times \nu} \]

Hence by the claim, the inverse image \( f^{-1}(V_\mu \times V_\nu) \) of a basic open set \( V_\mu \times V_\nu \) of \( \varphi(G_1) \times \varphi(G_2), \tau_{\delta_1} \times \tau_{\delta_2} \) is a basic open set \( V_{\mu \times \nu} \) in \( \varphi(G_1 \times G_2), \tau_{\delta_1} \times \tau_{\delta_2} \) and hence \( f \) is continuous. Similarly the inverse image \( f(V_{\mu \times \nu}) \) of the basic open set \( V_{\mu \times \nu} \) under \( f^{-1} \) is again a basic open set \( (V_\mu \times V_\nu) \cap F \) and hence \( f^{-1} \) is continuous. Hence induced topological space on \( \varphi(G_1 \times G_2) \) by the product fuzzy topological space \( (G_1 \times G_2, \delta_1 \times \delta_2) \) is fuzzy homeomorphic to the subspace \( F \) of the product fuzzy topological space \( (\varphi(G_1) \times \varphi(G_2), \tau_{\delta_1} \times \tau_{\delta_2}) \).

5.3 INDUCED TOPOLOGY ON INTUITIONISTIC FUZZY SINGLETONS

In this section, the notion of induced topology on the collection of intuitionistic fuzzy singletons with respect to an intuitionistic fuzzy topological space is introduced and studied. The notion of intuitionistic fuzzy compact topological space was introduced by Coker (1997). The relation between intuitionistic fuzzy compactness and compactness of induced topology has been studied and the weaker condition is identified in a theorem. The intuitionistic fuzzy connected space and intuitionistic fuzzy Hausdorffness were introduced by Coker (1997) and Coker (2000). The equivalence of intuitionistic fuzzy connected space and connectedness of induced topology and the equivalence of intuitionistic fuzzy Hausdorffness and pseudohausdorffness of induced topology have been proved. The relation between fuzzy subspace and subspace of induced topology has been studied in a theorem. As a main th...
rem, it has also been proved that the induced topological space on \( \varphi(G_1 \times G_2) \) by the product intuitionistic fuzzy topological space \((G_1 \times G_2, \delta_1 \times \delta_2)\) is imbedded in the product topological space \((\varphi(G_1) \times \varphi(G_2), \tau_{\delta_1} \times \tau_{\delta_2})\).

**Definition 5.3.1.** Let \((G, \delta)\) be an intuitionistic fuzzy topological space. An induced topology \(\tau_{\delta}\) on the collection \(\varphi(G)\) of all intuitionistic fuzzy singletons of \(G\) is defined as the topology generated by \(\mathcal{B} = \{V_A \mid A \in \delta\}\), where \(V_A = \{p \in \varphi(G) \mid p \in A\}\) and hence \((\varphi(G), \tau_{\delta})\) is called an induced topological space.

**Note 5.3.1.** Clearly \(\mathcal{B}\) is a basis for a topology in \(\varphi(G)\).

1. Since \(\tilde{1} \in \delta, V_{\tilde{1}} = \varphi(G) \in \mathcal{B}\) and hence for every intuitionistic fuzzy singleton \(p \in \varphi(G)\), we have \(\varphi(G) \in \mathcal{B}\) with \(p \in \varphi(G)\).

2. Let \(V_A, V_B \in \mathcal{B}\). Clearly \(V_A \cap V_B = V_{A \cap B}\) and hence \(V_A \cap V_B \in \mathcal{B}\).

Hence \(\mathcal{B}\) is a basis for a topology on \(\varphi(G)\).

**Example 5.3.1.** Let \(X = \{a, b, c\}\) and \(\delta = \{(\emptyset, \tilde{1}), \{(x, y, 0), (r, s, 1)\}, \{(0, y, z), (1, s, t)\}, \{(0, y, 0), (1, s, 1)\}\}\), where \(y > x > z\) and \(x + r \leq 1, y + s \leq 1, z + t \leq 1\). Here \(\{(t_1, t_2, t_3), (s_1, s_2, s_3)\} \in I^X \times I^X\) denotes an intuitionistic fuzzy subset of \(X\) which has membership values \(t_1, t_2, t_3\) and non-membership values \(s_1, s_2, s_3\) at \(a, b, c\) respectively. Clearly \((X, \delta)\) is an intuitionistic fuzzy topological space.

Now \(\tau_{\delta} = \{V_0, V_1, V_{\{(x, y, 0), (r, s, 1)\}}, V_{\{(0, y, z), (1, s, t)\}}, V_{\{(0, y, 0), (1, s, 1)\}}\}\)

\[
\{\emptyset, \chi(X), \{(\mu_p, \gamma_p), (\mu_q, \gamma_q) \mid \text{supp } \mu_p = \{a\}, \mu_p(a) \leq x \text{ and } \gamma_p(a) \geq r, \text{supp } \mu_q = \{b\}, \mu_q(b) \leq y, \gamma_q(b) \geq s\}, \{(\mu_p, \gamma_p), (\mu_q, \gamma_q) \mid \text{supp } \mu_p = \{b\}, \mu_p(b) \leq y \text{ and } \gamma_p(b) \geq s, \text{supp } \mu_q = \{c\}, \mu_q(c) \leq z, \gamma_q(c) \geq t\}, \{(\mu_p, \gamma_p) \mid \text{supp } \mu_p = \{b\}, \mu_p(b) \leq y \text{ and } \gamma_p(b) \geq s\}\}.\]
**Note 5.3.2.** The topology \( \tau_\delta \) on \( \wp(G) \) need not be discrete as is seen from the above example.

**Definition 5.3.2.** Let \((X, \delta), (Y, \sigma)\) be intuitionistic fuzzy topological spaces and \(f : (X, \delta) \to (Y, \sigma)\) be any map. The function \(i_f : (\wp(X), \tau_\delta) \to (\wp(Y), \tau_\sigma)\) defined by \(i_f(p) = q\) is called induced map of \(f\), where \(p = (\mu_p, \gamma_p)\) is any intuitionistic fuzzy singleton defined on \(x \in X\) and \(q = (\mu_q, \gamma_q)\) is the intuitionistic fuzzy singleton defined on \(f(x) \in Y\) with \(\mu_q(f(x)) = \mu_p(x), \gamma_q(f(x)) = \gamma_p(x)\).

**Theorem 5.3.1.** Let \((X, \delta), (Y, \sigma)\) be intuitionistic fuzzy topological spaces. A function \(f : (X, \delta) \to (Y, \sigma)\) is a intuitionistic fuzzy continuous map if and only if the induced function \(i_f : (\wp(X), \tau_\delta) \to (\wp(Y), \tau_\sigma)\) is continuous.

**Theorem 5.3.2.** If \((G, \delta)\) is intuitionistic fuzzy compact, then \((\wp(G), \tau_\delta)\) is compact. But the converse need not be true.

**Proof:** Let \((G, \delta)\) be intuitionistic fuzzy compact and \(\wp(G) = \cup_{V_A \in \mathcal{B}_0} V_A\) for some \(\mathcal{B}_0 \subseteq \mathcal{B}\), where \(\mathcal{B}\) is the basis for \(\tau_\delta\). Now \(\forall x \in G, (1_x, 0_x) \in \wp(G)\), where \(1_x\) is the characteristic function on \(\{x\}\) and \(0_x\) is the characteristic function on \(A - \{x\}\). Hence \((1_x, 0_x) \in V_{A_x}\), for some \(V_{A_x} \in \mathcal{B}_0\). Hence \(1_x \leq \mu_{A_x}, 0_x \geq \gamma_{A_x}\) and hence \(\mu_{A_x}(x) = 1, \gamma_{A_x}(x) = 0\). So \(\tilde{1} = \vee_{x \in G} A_x\). By intuitionistic fuzzy compactness we have \(\tilde{1} = \vee_{i=1}^{\infty} A_{x_i}\) for some \(i = 1, 2, ..., n\) and hence \(\wp(G) = \vee_{i=1}^{\infty} V_{A_{x_i}}\) and hence \((\wp(G), \tau_\delta)\) is compact.

The converse need not be true as is seen from the following example.

**Example 5.3.2.** Let \(X\) be any set and \(\delta = \{0, \tilde{1}, (1 - 1/n, 1/n)\}\). Obviously \((X, \delta)\) is an intuitionistic fuzzy topological space. Clearly any open covering of \(\wp(G)\) contains \(V_1\) and hence \((\wp(G), \tau_\delta)\) is compact. But clearly \((X, \delta)\) is
not compact.

**Note 5.3.3.** Since two intuitionistic fuzzy singletons defined on same point with different values are distinct in $\varphi(G)$, we can’t separate them by two open sets. So we can’t speak about Hausdorffness in $\varphi(G)$. But any two intuitionistic fuzzy singletons defined on distinct points can be separated by open sets. Hence we can discuss about pseudo Hausdorff space on $(\varphi(G), \tau_3)$ as below.

An induced fuzzy topological space $(\varphi(G), \tau_3)$ is said to be pseudo Hausdorff if for any two intuitionistic fuzzy singletons $p = (\mu_p, \gamma_p), q = (\mu_q, \gamma_q)$ defined on distinct points, there exist two disjoint open sets $V_A, V_B$ such that $p \in V_A, q \in V_B$.

**Theorem 5.3.3.** An intuitionistic fuzzy topological space $(G, \delta)$ is intuitionistic fuzzy Hausdorff if and only if $(\varphi(G), \tau_3)$ is pseudo Hausdorff.

**Proof:** Let $(G, \delta)$ be intuitionistic fuzzy Hausdorff. To prove that $(\varphi(G), \tau_3)$ is pseudo Hausdorff, consider two intuitionistic fuzzy singletons $p = (\mu_p, \gamma_p)$ and $q = (\mu_q, \gamma_q)$ with $\text{supp} \mu_p = \{x\} \neq \text{supp} \mu_q = \{y\}$. By intuitionistic fuzzy Hausdorffness of $(G, \delta)$, there exist $A, B \in \delta$ such that $\mu_A(x) = 1, \mu_B(y) = 1$ and $A \cap B = \emptyset$. Since $A$ and $B$ are intuitionistic fuzzy subsets, $\gamma_A(x) = 0, \gamma_B(y) = 0$. Hence $p \in V_A, q \in V_B$. Since $A \cap B = \emptyset, V_A \cap V_B = \emptyset$. So $V_A \cap V_B = V_A \cap V_B = \emptyset$. Hence there exists disjoint open sets $V_A, V_B \in \tau_3$ such that $p \in V_A, q \in V_B$.

Conversely, if $(\varphi(G), \tau_3)$ is pseudo Hausdorff, then we have to prove that $(G, \delta)$ is intuitionistic fuzzy Hausdorff. Let $x, y \in G$ such that $x \neq y$. Define intuitionistic fuzzy singletons $p, q$ on $x, y$ respectively with $\mu_p(x) = \mu_q(y) = 1$. By pseudo Hausdorffness of $(\varphi(G), \tau_3)$, there exist $V_A, V_B \in \tau_3$
such that \( p \in V_A, q \in V_B \) and \( V_A \cap V_B = \emptyset \), where \( A, B \in \delta \). By definition 
\( \mu_p(x) = 1 \leq \mu_A(x) \), we have \( \mu_A(x) = 1 \) and similarly \( \mu_B(y) = 1 \). Since 
\( V_{A\cap B} = V_A \cap V_B = \emptyset \), \( A \cap B = \emptyset \). Hence \((G, \delta)\) is an intuitionistic fuzzy Hausdorff space.

**Theorem 5.3.4.** An intuitionistic fuzzy topological space \((G, \delta)\) is intuitionistic fuzzy connected if and only if the induced topological space \((\varphi(G), \tau_\delta)\) is connected.

**Proof:** Let \((G, \delta)\) be intuitionistic fuzzy connected. If \((\varphi(G), \tau_\delta)\) is not connected, then \( \varphi(G) = V_A \cup V_B \), for some \( A, B \in \delta \) such that \( V_A \cap V_B = \emptyset \). Obviously \( A \cap B = \emptyset \). For every \( x \in G \), \((1_x, 0_x) \in \varphi(G)\) and so \((1_x, 0_x) \in V_A\) or \((1_x, 0_x) \in V_B\). Hence \( \mu_A(x) = 1 \) and \( \gamma_A(x) = 0 \) or \( \mu_B(x) = 1 \) and \( \gamma_B(x) = 0 \). Therefore \( \mu_A \vee \mu_B = 1 \) and \( \gamma_A \wedge \gamma_B = 0 \). Hence \( A \cup B = \emptyset \), which is a contradiction.

Conversely, suppose that \((\varphi(G), \tau_\delta)\) is connected and \((G, \delta)\) is not intuitionistic fuzzy connected, we have \( \emptyset = A \cup B \) such that \( A \cap B = \emptyset \), for some \( A, B \in \delta \). Hence \( 1 = \mu_A \vee \mu_B, 0 = \gamma_A \wedge \gamma_B \) and \( 0 = \mu_A \wedge \mu_B, 1 = \gamma_A \vee \gamma_B \). Hence \( \forall x \in G, \mu_A(x) = 1, \mu_B(x) = 0, \gamma_A(x) = 0, \gamma_B(x) = 1 \) or \( \mu_B(x) = 1, \mu_A(x) = 0, \gamma_A(x) = 1, \gamma_B(x) = 0 \). So, for any intuitionistic fuzzy singleton \( p = (\mu_p, \gamma_p) \), \( \mu_p(x) \leq \mu_A(x) \) and \( \gamma_p(x) \geq \gamma_A(x) \) or \( \mu_p(x) \leq \mu_B(x) \) and \( \gamma_p(x) \geq \gamma_B(x) \). Hence \( p \in V_A \) or \( V_B \). Clearly \( V_A \cap V_B = V_{A\cap B} = \emptyset \), a contradiction to the fact that \((\varphi(G), \tau_\delta)\) is connected.

**Theorem 5.3.5.** Let \((G, \delta)\) be an intuitionistic fuzzy topological space and \( F \subseteq G \). The subspace on \((\varphi(F))\) inherited from \((\varphi(G), \tau_\delta)\) equals the topological space on \( \varphi(F) \) induced by the intuitionistic fuzzy subspace on \( F \) inherited from \((G, \tau_\delta)\).
**Proof:** Let \((G, \delta)\) be an intuitionistic fuzzy topological space and \(F \subseteq G\). First we prove the following claim.

**Claim:** For any intuitionistic fuzzy open set \(A \in \delta, V_A \cap \varphi(F) = V_{A|F}\).

**Proof of the claim:** Let \(p = (\mu_p, \gamma_p) \in V_A \cap \varphi(F)\). Since \(p \in V_A\), \(\mu_p(x) \leq \mu_A(x)\) and \(\gamma_p(x) \geq \gamma_A(x)\). Since \(p \in \varphi(F)\), \(\text{supp } \mu_p = \{x\} \subseteq F\). Hence \(\mu_p(x) \leq (\mu_A|F)(x)\) and \(\gamma_p(x) \geq (\gamma_A|F)(x)\). So \(p \in V_{A|F}\). Hence \(V_A \cap \varphi(F) \subseteq V_{A|F}\). Now let \(p \in V_{A|F}\). So \(\mu_p(x) \leq (\mu_A|F)(x)\) and \(\gamma_p(x) \geq (\gamma_A|F)(x)\). If \(\text{supp } \mu_p = x \notin F\), \((\mu_A|F)(x) = 0\) and hence \(\mu_p(x) = 0\), a contradiction to the choice of \(p\). So \(\text{supp } \mu_p = x \in F\). Therefore \(\mu_p(x) \leq \mu_A(x)\) and \(\gamma_p(x) \geq \gamma_A(x)\). So \(p \in V_A \cap \varphi(F)\). Hence the claim.

Let \(U \cap \varphi(F)\) be an arbitrary open set in \((\varphi(F), \tau_\delta \cap \varphi(F))\), where \(U\) is open in \(\tau_\delta\). By definition \(U = \bigcup_{A \in B} V_A\), for some \(B \subseteq \delta\). Hence \(U \cap \varphi(F) = \bigcup_{A \in B} V_A \cap \varphi(F) = \bigcup_{A \in B} \{V_A \cap \varphi(F)\}\). So, by the claim, \(U \cap \varphi(F) = \bigcup_{A \in B} \{V_{A|F}\} \in (\varphi(F), \tau_{\delta|F})\).

Similarly any basic open set \(V_{A|F}\) in \((\varphi(F), \tau_{\delta|F})\) is a basic open set in \((\varphi(F), \tau_\delta \cap \varphi(F))\).

**Theorem 5.3.6.** Let \((G_1, \delta_1), (G_2, \delta_2)\) be intuitionistic fuzzy topological spaces. Then the induced topological space on \(\varphi(G_1 \times G_2)\) by the product intuitionistic fuzzy topological space \((G_1 \times G_2, \delta_1 \times \delta_2)\) is imbedded in the product topological space \((\varphi(G_1) \times \varphi(G_2), \tau_{\delta_1} \times \tau_{\delta_2})\).

**Proof:** It is enough to prove that there exists a function \(f : (\varphi(G_1 \times G_2, \tau_{\delta_1} \times \delta_2) \to (\varphi(G_1) \times \varphi(G_2), \tau_{\delta_1} \times \tau_{\delta_2})\) such that \(f\) is 1–1 and both \(f\) and \(f^{-1}\) are continuous.
Let \( f : (\varphi(G_1 \times G_2), \tau_{\delta_1 \times \delta_2}) \to (\varphi(G_1) \times \varphi(G_2), \tau_{\delta_1} \times \tau_{\delta_2}) \) be defined by \( f(p) = f(\mu_p, \gamma_p) = (p_1, p_2) = ((\mu_{p_1}, \gamma_{p_1}), (\mu_{p_2}, \gamma_{p_2})) \), where \( p \) is defined on \((x_1, x_2) \in G_1 \times G_2\) and \( p_1, p_2 \) are defined on \(x_1, x_2\) of \( G_1, G_2\) respectively and \( \mu_{p_1}(x_1) = \mu_{p_2}(x_2) = \mu_p(x_1, x_2), \gamma_{p_1}(x_1) = \gamma_{p_2}(x_2) = \gamma_p(x_1, x_2) \).

First we prove that \( f \) is \( 1-1 \). Let \( f(p) = f(q) \), where \( p = (\mu_p, \gamma_p) \) and \( q = (\mu_q, \gamma_q) \) are intuitionistic fuzzy singletons defined on \((x_1, y_1)\) and \((x_2, y_2)\) respectively. By definition, \( f(p) = (p_1, p_2) \) and \( f(q) = (q_1, q_2) \), where \( p_1, p_2, q_1, q_2 \) are intuitionistic fuzzy singletons on \( x_1, x_2, y_1 \) and \( y_2 \) respectively. So, by hypothesis, \((p_1, p_2) = (q_1, q_2)\) and hence \( p_1 = q_1 \) and \( p_2 = q_2 \). So \( x_1 = y_1, x_2 = y_2 \) and \( \mu_p(x_1, x_2) = \mu_{p_1}(x_1) = \mu_{q_1}(y_1) = \mu_q(y_1, y_2) \) and \( \gamma_p(x_1, x_2) = \gamma_{p_1}(y_1) = \gamma_q(y_1, y_2) \). Hence \( p = q \).

Let \( F = \{(p_1, p_2) \in \varphi(G_1) \times \varphi(G_2) \mid \mu_{p_1}(x_1) = \mu_{p_2}(x_2), \gamma_{p_1}(x_1) = \gamma_{p_2}(x_2)\} \subseteq \varphi(G_1) \times \varphi(G_2) \). Clearly, if \((p_1, p_2) \in F\), then there exists \( p \in \varphi(G_1 \times G_2) \) with \( \mu_p(x_1, x_2) = \mu_{p_1}(x_1) = \mu_{p_2}(x_2) \) and \( \gamma_p(x_1, x_2) = \gamma_{p_1}(x_1) = \gamma_{p_2}(x_2) \). So \( f : \varphi(G_1 \times G_2) \to F \) is a bijection.

To prove that \( f \) and \( f^{-1} \) are continuous, we need this claim.

**Claim:** If \( A \in \delta_1 \) and \( B \in \delta_2 \), then \( f^{-1}(V_A \times V_B) = f^{-1}((V_A \times V_B) \cap F) = V_{A \times B} \) and \( f(V_{A \times B}) = (V_A \times V_B) \cap F \).

**Proof of the Claim:** Clearly \( p \in f^{-1}((V_A \times V_B) \cap F) \Leftrightarrow f(p) \in V_A \times V_B \cap F \)

\[ \Leftrightarrow p_1 \in V_A \text{ and } p_2 \in V_B, \text{ by definition of } f \]

\[ \Leftrightarrow \mu_{p_1}(x_1) \leq \mu_A(x_1), \gamma_{p_1}(x_1) \geq \gamma_A(x_1), \mu_{p_2}(x_2) \leq \mu_B(x_2), \gamma_{p_2}(x_2) \geq \gamma_B(x_2) \text{ and } \mu_{p_1}(x_1) = \mu_{p_2}(x_2) = \mu_p(x_1, x_2), \gamma_{p_1}(x_1) = \gamma_{p_2}(x_2) = \gamma_p(x_1, x_2) \]

\[ \Leftrightarrow \mu_p(x_1, x_2) \leq \mu_A(x_1), \gamma_p(x_1, x_2) \geq \gamma_A(x_1) \text{ and } \]
\[ \mu_p(x_1, x_2) \leq \mu_B(x_2), \gamma_p(x_1, x_2) \geq \gamma_B(x_2) \]

\[ \Leftrightarrow \mu_p(x_1, x_2) \leq \min\{\mu_A(x_1), \mu_B(x_2)\} \text{ and } \gamma_p(x_1, x_2) \geq \max\{\gamma_A(x_1), \gamma_B(x_2)\} \]

\[ \Leftrightarrow \mu_p(x_1, x_2) \leq (\mu_{A \times B})(x_1, x_2) \text{ and } \gamma_p(x_1, x_2) \geq (\gamma_{A \times B})(x_1, x_2), \text{ by definition of } A \times B \]

\[ \Leftrightarrow p \in V_{A \times B} \]

Hence, by the claim, the inverse image \( f^{-1}(V_A \times V_B) \) of a basic open set \( V_A \times V_B \) of \( (\wp(G_1) \times \wp(G_2), \tau_{\delta_1} \times \tau_{\delta_2}) \) is a basic open set \( V_{A \times B} \) in \( (\wp(G_1 \times G_2), \tau_{\delta_1 \times \delta_2}) \) and hence \( f \) is continuous. Similarly the inverse image \( f(V_{A \times B}) \) of the basic open set \( V_{A \times B} \) under \( f^{-1} \) is again a basic open set \( (V_A \times V_B) \cap F \) and hence \( f^{-1} \) is continuous. So induced topological space on \( \wp(G_1 \times G_2) \) by the product intuitionistic fuzzy topological space \( (G_1 \times G_2, \delta_1 \times \delta_2) \) is homeomorphic to the subspace \( F \) of the product topological space \( (\wp(G_1) \times \wp(G_2), \tau_{\delta_1} \times \tau_{\delta_2}) \).

**Note 5.3.4.** Let \((G, \delta)\) be an intuitionistic fuzzy topological space on \( G \) and \( f : G \to G' \) be a surjective map. The induced topology on \( \wp(G') \) induced by the quotient intuitionistic fuzzy topology \( Q(\delta) \) on \( G' \) equals the quotient topology on \( \wp(G') \) by the map \( f' : \wp(G) \to \wp(G') \) defined by \( f'(p) = p' \), where \( p' \) is an intuitionistic fuzzy singleton of \( G' \) defined on \( f(x) \) with \( \mu_{p'}(f(x)) = \mu_p(x) \) and \( \gamma_{p'}(f(x)) = \gamma_p(x) \).

**5.4 CONCLUSION**

In this chapter, a new notion of induced topology on intuitionistic fuzzy singletons has been defined and its properties have been studied. Using this induced topology, one can study the topological algebraic structures like quotient intuitionistic fuzzy topological group.