CHAPTER 6

INTUITIONISTIC FUZZY IDEALS OF TM-ALGEBRAS

6.1 INTRODUCTION

Generalization of fuzzy sets leads to intuitionistic approach, that can further utilized to relate the membership and non-membership functions, introduced by Atanassov (1986). Every intuitionistic approach is explained in terms of upper and lower level sets. In this chapter, Intuitionistic fuzzy ideals and intuitionistic fuzzy closed ideals are discussed. Properties of the homomorphic image and inverse image of intuitionistic fuzzy ideals of TM-algebra are also highlighted in this chapter.

6.2 INTUITIONISTIC FUZZY IDEALS

Definition 6.2.1 (Atanassov 1986)

An Intuitionistic Fuzzy Set (IFS) $A$ in a non-empty set $X$ is an object having the form $A = \{(X, \mu_A(x), \lambda_A(x))/x \in X\}$, where the function $\mu_A : X \rightarrow [0, 1]$ and $\lambda_A : X \rightarrow [0, 1]$ denotes the degree of membership (namely $\mu_A(x)$) and the
degree of non-membership (namely $\lambda_A(x)$) of each element $x \in X$ to the set $A$ respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$, for all $x \in X$.

For the sake of simplicity, the symbol $A = (X, \mu_A, \lambda_A)$ is used for the intuitionistic fuzzy set $A = \{(X, \mu_A(x), \lambda_A(x)/x \in X\}$.

**Definition 6.2.2 (Atanassov 1986)**

Let $A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy set in $X$. Then

(i) $\square A = (X, \mu_A, \overline{\mu}_A)$ and

(ii) $\diamond A = (X, \overline{\lambda}_A, \lambda_A)$.

**Definition 6.2.3 (Jun and Kim 2000)**

An intuitionistic fuzzy set $A = (X, \mu_A, \lambda_A)$ in a TM-algebra $X$ is said to be an intuitionistic fuzzy ideal of $X$ if,

(i) $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$

(ii) $\mu_A(x) \geq \min\{\mu_A((x * y), \mu_A(y)\}$

(iii) $\lambda_A(x) \leq \max\{\lambda_A((x * y), \lambda_A(y)\}$,

for all $x, y \in X$.

**Definition 6.2.4**

An intuitionistic fuzzy set $A = (X, \mu_A, \lambda_A)$ in a TM-algebra $X$ is called an intuitionistic fuzzy $T$-ideal of $X$ if,
\((i)\) \(\mu_A(0) \geq \mu_A(x)\) and \(\lambda_A(0) \leq \lambda_A(x)\)

\((ii)\) \(\mu_A(x \ast z) \geq \min\{\mu_A((x \ast y) \ast z), \mu_A(y)\}\)

\((iii)\) \(\lambda_A(x \ast z) \leq \max\{\lambda_A((x \ast y) \ast z), \lambda_A(y)\}\),

for all \(x, y, z \in X\).

**Definition 6.2.5**

An intuitionistic fuzzy set \(A = (X, \mu_A, \lambda_A)\) in a TM-algebra \(X\) is called an intuitionistic fuzzy closed \(T\)-ideal of \(X\), if it satisfies

\[(i)\] \(\mu_A(0 \ast x) \geq \mu_A(x)\) and \(\lambda_A(0 \ast x) \leq \lambda_A(x)\)

\[(ii)\] \(\mu_A(x \ast z) \geq \min\{\mu_A((x \ast y) \ast z), \mu_A(y)\}\)

\[(iii)\] \(\lambda_A(x \ast z) \leq \max\{\lambda_A((x \ast y) \ast z), \lambda_A(y)\}\),

for all \(x, y, z \in X\).

**Example 6.2.6**

Consider the set \(X = \{0, a, b, c\}\) with the following Cayley Table 6.1

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
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<td>b</td>
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<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Define an intuitionistic fuzzy set \(A = (X, \mu_A, \lambda_A)\) in \(X\) as follows.

\(\mu_A(0) = \mu_A(a) = 1\) and \(\mu_A(b) = \mu_A(c) = t\), \(\lambda_A(0) = \lambda_A(a) = 0\) and \(\lambda_A(b) = \lambda_A(c) = s\) where \(t, s \in (0, 1)\) and \(t + s \leq 1\).
The routine calculation shows that it is an intuitionistic fuzzy ideal of $X$.

**Definition 6.2.7 (Jun and Kim 2000)**

Let $A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy set in a TM-algebra $X$. The set $U(\mu_A; s) = \{x \in X/\mu_A(x) \geq s\}$ is called an upper $s$-level of $\mu_A$ and the set $L(\lambda_A; t) = \{x \in X/\lambda_A(x) \leq t\}$ is called lower $t$-level of $\lambda_A$.

### 6.3 INTUITIONISTIC Q-FUZZY IDEALS IN TM-ALGEBRAS

The notion of intuitionistic $Q$-Fuzzy ideals in TM-algebra is introduced and its properties are investigated.

**Definition 6.3.1 (Eun Hwan Roh et al 2006)**

A fuzzy subset $\mu_Q : X \times Q \to [0, 1]$, where $Q$ is any non-empty set, in a TM-algebra $X$ is called a $Q$-fuzzy ideal of $X$, if

(i) $\mu_Q(0, q) \geq \mu_Q(x, q)$

(ii) $\mu_Q(x, q) \geq \min\{\mu_Q(x \ast y), q), \mu_Q(y, q)\}$,

for all $x, y \in X$ and $q \in Q$.

**Definition 6.3.2 (Eun Hwan Roh et al 2006)**

An Intuitionistic $Q$-fuzzy set (IQFS) $A_Q$ in a non-empty set $X$ is an object having the form $A_Q = \{(X \times Q, \mu_{AQ}(x, q), \lambda_{AQ}(x, q)/x \in X, q \in Q\}$,
where the functions $\mu_{AQ} : X \times Q \to [0, 1]$ and $\lambda_{AQ} : X \times Q \to [0, 1]$ denote
the degree of membership (namely $\mu_{AQ}(x, q)$) and the degree of non-membership
(namely $\lambda_{AQ}(x, q)$) of each element $(x, q) \in X \times Q$ to the set $A_Q$ respectively, and
$0 \leq \mu_{AQ}(x, q) + \lambda_{AQ}(x, q) \leq 1$, for all $x \in X$ and $q \in Q$.

For the sake of simplicity, the symbol $A_Q = (X, \mu_{AQ}, \lambda_{AQ})$ is used for the
intuitionistic $Q$-fuzzy set

$$A_Q = \{(X, \mu_{AQ}(x, q), \lambda_{AQ}(x, q)/x \in X, q \in Q\}.$$

**Definition 6.3.3**

An intuitionistic $Q$-fuzzy set $A_Q = (X, \mu_{AQ}, \lambda_{AQ})$ in a TM-algebra $X$ is
called an intuitionistic $Q$-fuzzy ideal of $X$, if it satisfies the following axioms.

(i) $\mu_{AQ}(0, q) \geq \mu_{AQ}(x, q)$ and $\lambda_{AQ}(0, q) \leq \lambda_{AQ}(x, q)$

(ii) $\mu_{AQ}(x, q) \geq \min\{\mu_{AQ}((x * y), q), \mu_{AQ}(y, q)\}$

(iii) $\lambda_{AQ}(x, q) \leq \max\{\lambda_{AQ}((x * y), q), \lambda_{AQ}(y, q)\}$,

for all $x, y \in X$ and $q \in Q$.

**Definition 6.3.4**

An intuitionistic $Q$-fuzzy set $A_Q = (X, \mu_{AQ}, \lambda_{AQ})$ in a TM-algebra $X$ is
called an intuitionistic $Q$-fuzzy $T$-ideal of $X$, if it satisfies the following axioms:

(i) $\mu_{AQ}(0, q) \geq \mu_{AQ}(x, q)$ and $\lambda_{AQ}(0, q) \leq \lambda_{AQ}(x, q)$

(ii) $\mu_{AQ}(x * z, q) \geq \min\{\mu_{AQ}((x * y) * z, q), \mu_{AQ}(y, q)\}$
(iii) $\lambda_{AQ}(x \ast z, q) \leq \max \{\lambda_{AQ}((x \ast y) \ast z, q), \lambda_{AQ}(y, q)\}$,

for all $x, y, z \in X$ and $q \in Q$.

**Proposition 6.3.5**

Let $A_Q = (X, \mu_{AQ}, \lambda_{AQ})$ be an intuitionistic $Q$-fuzzy ideal of a TM-algebra $X$. If $x \leq y$ in $X$ then $\mu_{AQ}(x, q) \geq \mu_{AQ}(y, q)$ and $\lambda_{AQ}(x, q) \leq \lambda_{AQ}(y, q)$.

**Proof**

$x \leq y$ implies $x \ast y = 0$.

\[
\mu_{AQ}(x, q) = \mu_{AQ}(x, q) \\
\geq \min \{\mu_{AQ}((x \ast y), q), \mu_{AQ}(y, q)\} \\
= \min \{\mu_{AQ}(0, q), \mu_{AQ}(y, q)\} \\
= \mu_{AQ}(y, q)
\]

Therefore $\mu_{AQ}(x, q) \geq \mu_{AQ}(y, q)$.

Similarly, $\lambda_{AQ}(x, q) \leq \lambda_{AQ}(y, q)$.

**Definition 6.3.6**

Let $A_Q = (X, \mu_{AQ}, \lambda_{AQ})$ be an intuitionistic $Q$-fuzzy set in $X$. Then (i) $\Box A_Q = (X, \mu_{AQ}, \mu_{AQ})$ and (ii) $\Diamond A_Q = (X, \lambda_{AQ}, \lambda_{AQ})$.

**Theorem 6.3.7**

Let $A_Q = (X, \mu_{AQ}, \lambda_{AQ})$ be an intuitionistic $Q$-fuzzy ideal (IQF ideal) of a TM-algebra $X$. Then so is $\Box A_Q = (X, \mu_{AQ}, \mu_{AQ})$. 
Proof

Since $A_Q$ is an intuitionistic $Q$-fuzzy ideal of $X$,

\[ \mu_{AQ}(0, q) \geq \mu_{AQ}(x, q) \]

and

\[ \mu_{AQ}(x, q) \geq \min\{\mu_{AQ}(x * y, q), \mu_{AQ}(y, q)\} \].

Now, $\mu_{AQ}(0, q) \geq \mu_{AQ}(x, q)$

\[ \Rightarrow 1 - \overline{\mu}_{AQ}(0, q) \geq 1 - \overline{\mu}_{AQ}(x, q) \]

\[ \Rightarrow \overline{\mu}_{AQ}(0, q) \leq \overline{\mu}_{AQ}(x, q), \text{ for any } x \in X \text{ and } q \in Q, \text{ and} \]

\[ \mu_{AQ}(x, q) \geq \min\{\mu_{AQ}(x * y, q), \mu_{AQ}(y, q)\} \].

\[ \Rightarrow 1 - \overline{\mu}_{AQ}(x, q) \geq \min\{1 - \overline{\mu}_{AQ}(x * y, q), 1 - \overline{\mu}_{AQ}(y, q)\} \]

\[ \Rightarrow \overline{\mu}_{AQ}(x, q) \leq 1 - \min\{1 - \overline{\mu}_{AQ}(x * y, q), 1 - \overline{\mu}_{AQ}(y, q)\} \]

\[ \leq \max\{\overline{\mu}_{AQ}(x * y, q), \overline{\mu}_{AQ}(y, q)\} \].

Hence $\square A_Q = (X, \mu_{AQ}, \overline{\mu}_{AQ})$ is an intuitionistic $Q$-fuzzy ideal of $X$.

Theorem 6.3.8

Let $A_Q = (X, \mu_{AQ}, \lambda_{AQ})$ be an intuitionistic $Q$-fuzzy ideal of a TM-algebra $X$. Then so is $\diamond A_Q = (X, \overline{\lambda}_{AQ}, \lambda_{AQ})$.

Proof

\[ \lambda_{AQ}(0, q) \leq \lambda_{AQ}(x, q) \Rightarrow 1 - \overline{\lambda}_{AQ}(0, q) \leq 1 - \overline{\lambda}_{AQ}(x, q) \]

\[ \Rightarrow \overline{\lambda}_{AQ}(0, q) \geq \overline{\lambda}_{AQ}(x, q), \text{ for any } x \in X \text{ and for any } q \in Q. \]

Now, for any $x, y \in X$ and for $q \in Q$

\[ \lambda_{AQ}(x, q) \leq \max\{\lambda_{AQ}((x, q) * (y, q)), \lambda_{AQ}(y, q)\} \]

\[ \Rightarrow 1 - \overline{\lambda}_{AQ}(x, q) \leq \max\{1 - \overline{\lambda}_{AQ}((x, q) * (y, q)), 1 - \overline{\lambda}_{AQ}(y, q)\} \]

\[ \Rightarrow \overline{\lambda}_{AQ}(x, q) \geq 1 - \max\{1 - \overline{\lambda}_{AQ}((x, q) * (y, q)), 1 - \overline{\lambda}_{AQ}(y, q)\} \]
\[= \min\{\overline{\lambda}_{AQ}((x, q) \ast (y, q)), \overline{\lambda}_{AQ}(y, q)\}\]

\[\Rightarrow \overline{\lambda}_{AQ}(x, q) \geq \min\{\overline{\lambda}_{AQ}((x, q) \ast (y, q)), \overline{\lambda}_{AQ}(y, q)\}.\]

Hence, \(\Diamond A = (X, \overline{\lambda}_{AQ}, \lambda_{AQ})\) is an intuitionistic fuzzy ideal of \(X\).

**Theorem 6.3.9**

\(A_Q = (X, \mu_{AQ}, \lambda_{AQ})\) is an intuitionistic \(Q\)-fuzzy ideal of a TM-algebra \(X\) if and only if \(\Diamond A_Q, \Diamond A_Q\) are intuitionistic \(Q\)-fuzzy ideals of a TM-algebra \(X\).

**Definition 6.3.10**

An intuitionistic \(Q\)-fuzzy set \(A_Q = (X, \mu_{AQ}, \lambda_{AQ})\) in a TM-algebra \(X\) is called an intuitionistic \(Q\)-fuzzy closed ideal of \(X\), if it satisfies

(i) \(\mu_{AQ}(0 \ast x, q) \geq \mu_{AQ}(x, q)\) and \(\lambda_{AQ}(0 \ast x, q) \leq \lambda_{AQ}(x, q)\)

(ii) \(\mu_{AQ}(x \ast z, q) \geq \min\{\mu_{AQ}((x \ast y) \ast z, q), \mu_{AQ}(y, q)\}\)

(iii) \(\lambda_{AQ}(x \ast z, q) \leq \max\{\lambda_{AQ}((x \ast y) \ast z, q), \lambda_{AQ}(y, q)\}\),

for all \(x, y, z \in X\) and \(q \in Q\).

**Theorem 6.3.11**

If \(A_Q = (X, \mu_{AQ}, \lambda_{AQ})\) is an intuitionistic \(Q\)-fuzzy closed ideal of a TM-algebra \(X\), then so is \(\Box A_Q = (X, \mu_{AQ}, \overline{\mu}_{AQ})\).

**Proof**

For any \(x \in X\) and \(q \in Q\), \(\mu_{AQ}(0 \ast x, q) \geq \mu_{AQ}(x, q)\).

\[\Rightarrow 1 - \overline{\mu}_{AQ}(0 \ast x, q) \geq 1 - \overline{\mu}_{AQ}(x, q)\]
⇒ \( \mu_{A_Q}(0 \ast x, q) \leq \mu_{A_Q}(x, q) \), for any \( x \in X \) and \( q \in Q \).

Hence \( \Box A_Q = (X, \mu_{A_Q}, \mu_{A_Q}) \) is an intuitionistic \( Q \)-fuzzy closed ideal of \( X \).

**Theorem 6.3.12**

If \( A_Q = (X, \mu_{A_Q}, \lambda_{A_Q}) \) is an intuitionistic \( Q \)-fuzzy closed ideal of TM-algebra \( X \), then so is \( \Diamond A_Q = (X, \lambda_{A_Q}, \lambda_{A_Q}) \).

**Proof**

\[
\lambda_{A_Q}(0 \ast x, q) \leq \lambda_{A_Q}(x, q) \Rightarrow 1 - \lambda_{A_Q}(0 \ast x, q) \leq 1 - \lambda_{A_Q}(x, q)
\]

\( \Rightarrow \lambda_{A_Q}(0 \ast x, q) \geq \lambda_{A_Q}(x, q) \), for any \( x \in X \) and \( q \in Q \).

Hence, \( \Diamond A = (X, \lambda_{A}, \lambda_{A}) \) is an intuitionistic fuzzy closed ideal of \( X \).

**Theorem 6.3.13**

\( A_Q = (X, \mu_{A_Q}, \lambda_{A_Q}) \) is an intuitionistic \( Q \)-fuzzy closed ideal of a TM-algebra \( X \) if and only if \( \Box A_Q \) and \( \Diamond A_Q \) are intuitionistic \( Q \)-fuzzy closed ideals of a TM-algebra \( X \).

**Theorem 6.3.14**

\( A_Q = (X, \mu_{A_Q}, \lambda_{A_Q}) \) is an intuitionistic \( Q \)-fuzzy ideal of a TM-algebra \( X \) if and only if the non-empty upper \( s \)-level cut \( U(\mu_{A_Q}; s) \) and the non-empty lower \( t \)-level cut \( L(\lambda_{A_Q}; t) \) are ideals of \( X \), for any \( s, t \in [0, 1] \).

**Proof**

Suppose \( A_Q = (X, \mu_{A_Q}, \lambda_{A_Q}) \) is an intuitionistic \( Q \)-fuzzy ideal of a TM-algebra \( X \).
Since $U(\mu_{AQ};s) \neq \phi$, for $(x, q) \in U(\mu_{AQ};s) \Rightarrow \mu_{AQ}(x, q) \geq s$

$\Rightarrow \mu_{AQ}(0, q) \geq \mu_{AQ}(x, q) \geq s$

$\Rightarrow \mu_{AQ}(0, q) \geq s$

$\Rightarrow (0, q) \in U(\mu_{AQ};s)$.

Let $(x \ast y, q) \in U(\lambda_{AQ}, s)$ and $(y, q) \in U(\lambda_{AQ}, s)$.

$\Rightarrow \mu_{AQ}(x \ast y, q) \geq s$ and $\mu_{AQ}(y, q) \geq s$.

Since $\mu_{AQ}(x, q) \geq \min\{\mu_{AQ}(x \ast y, q), \mu_{AQ}(y, q)\}$

$\geq \min\{s, s\} = s$.

Thus $\mu_{AQ}(x, q) \geq s$.

$\Rightarrow (x, q) \in U(\lambda_{AQ};s)$.

Hence $U(\mu_{AQ};s)$ is an ideal of $X$.

Similarly it can be proved that $L(\lambda_{AQ};t)$ is an ideal of $X$.

Conversely, suppose that for any $s, t \in [0, 1]$, $U(\mu_{AQ};s)$ and $L(\lambda_{AQ};t)$ are ideals of $X$. If possible, assume $x_0, y_0 \in X$ and $q_0 \in Q$ such that $\mu_{AQ}(0, q_0) < \mu_{AQ}(x_0, q_0)$ and $\lambda_{AQ}(0, q_0) > \lambda_{AQ}(y_0, q_0)$.

Put

$s_0 = \frac{1}{2}[\mu_{AQ}(0, q_0) + \mu_{AQ}(x_0, q_0)]$.

$\Rightarrow s_0 < \mu_{AQ}(x_0, q_0)$ and $0 \leq \mu_{AQ}(0, q_0) < s_0 < 1$

$\Rightarrow (x_0, q_0) \in U(\mu_{AQ};s_0)$ and $(0, q_0) \notin U(\mu_{AQ};s_0)$.

Since $U(\mu_{AQ};s_0)$ is an ideal of $X$, we have $(0, q_0) \in U(\mu_{AQ};s_0)$ and $\mu_{AQ}(0, q_0) \geq s_0$.

Therefore, the assumption is wrong. Hence, $\mu_{AQ}(0, q) \geq \mu_{AQ}(x, q)$, for all $x \in X$ and $q \in Q$. 

Similarly, by taking $t_0 = \frac{1}{2}[\lambda_{AQ}(0, q) + \lambda_{AQ}(y_0, q)]$ it can be shown that $\lambda_{AQ}(0, q) \leq \lambda_{AQ}(y, q)$ for all $y \in X$ and $q \in Q$.

If possible, assume that $x_0, y_0 \in X$, $q_0 \in Q$ such that $\mu_{AQ}(x_0, q_0) < \min\{\mu_{AQ}(x_0 \ast y_0, q_0), \mu_{AQ}(y_0, q_0)\}$.

Take $s_0 = \frac{1}{2}[\mu_A(x_0, q_0) + \min\{\mu_{AQ}(x_0 \ast y_0, q_0), \mu_{AQ}(y_0, q_0)\}]$.

$\Rightarrow s_0 > \mu_{AQ}(x_0, q_0)$ and $s_0 < \min\{\mu_{AQ}(x_0 \ast y_0, q_0), \mu_{AQ}(y_0, q_0)\}$

$\Rightarrow s_0 > \mu_{AQ}(x_0, q_0)$, $s_0 < \mu_{AQ}(x_0 \ast y_0, q_0)$ and $s_0 < \mu_{AQ}(y_0, q_0)$

$\Rightarrow (x_0, q) \notin U(\mu_{AQ}; s_0)$. Since $U(\mu_{AQ}; s_0)$ is an ideal of $X$, $(x_0 \ast y_0, q_0) \in U(\mu_{AQ}; s_0)$ and $y_0 \in U(\mu_{AQ}; s_0)$ imply that $(x_0, q_0) \in U(\mu_{AQ}; s_0)$.

Therefore, the assumption is wrong. Hence $\mu_{AQ}(x, q) \geq \min\{\mu_{AQ}(x \ast y, q), \mu_{AQ}(y, q)\}$, for any $x, y, z \in X$ and $q \in Q$.

Similarly it can be proved that, $\lambda_{AQ}(x, q) \leq \max\{\lambda_{AQ}(x \ast y, q), \lambda_{AQ}(y, q)\}$

for all $x, y, z \in X$ and $q \in Q$.

Hence $A_Q$ is an intuitionistic $Q$-fuzzy ideal of a TM-algebra $X$.

**Theorem 6.3.15**

$A_Q = (X, \mu_{AQ}, \lambda_{AQ})$ is an intuitionistic $Q$-fuzzy closed ideal of a TM-algebra $X$ if and only if the non-empty upper $s$-level cut $U(\mu_{AQ}; s)$ and the non-empty lower $t$-level cut $L(\lambda_{AQ}; t)$ are closed ideals of $X$, for any $s, t \in [0, 1]$.

**Proof**

Suppose $A_Q = (X, \mu_{AQ}, \lambda_{AQ})$ is an intuitionistic fuzzy closed ideal of a
TM-algebra $X$. Then $\mu_{AQ}(0 \ast x, q) \geq \mu_{AQ}(x, q)$ and $\lambda_{AQ}(0 \ast x, q) \leq \lambda_{AQ}(x, q)$ for any $x \in X$ and $q \in Q$. Therefore for any $(x, q) \in U(\mu_{AQ}; s)$ implies that $\mu_{AQ}(0 \ast x, q) \geq s$ which shows that $(0 \ast x, q) \in U(\mu_{AQ}; s)$.

Similarly $(x, q) \in L(\lambda_{AQ}; t)$ implies that $(0 \ast x, q) \in L(\lambda_{AQ}; t)$. Hence $U(\mu_{AQ}; s)$ and $L(\lambda_{AQ}; t)$ are closed ideals of $X$, for any $s, t \in [0, 1]$.

Conversely, $U(\mu_{AQ}; s)$ and $L(\lambda_{AQ}; t)$ are closed ideals of $X$. To show that $A_Q = (X, \mu_{AQ}, \lambda_{AQ})$ is an intuitionistic fuzzy closed ideal of $X$, it is enough to show that $\mu_{AQ}(0 \ast x, q) \geq \mu_{AQ}(x, q)$ and $\lambda_{AQ}(0 \ast x, q) \leq \lambda_{AQ}(x, q)$ for any $x \in X$ and $q \in Q$. If possible assume that there exists some $x_0 \in X$ such that $\mu_{AQ}(0 \ast x_0, q) < \mu_{AQ}(x_0, q)$. Take $s_0 = [\mu_{AQ}(0 \ast x_0, q) + \mu_{AQ}(x_0, q)]$. Then $\mu_{AQ}(0 \ast x_0, q) < s_0 < \mu_{AQ}(x_0, q)$. This shows that $(x_0, q) \in U(\mu_{AQ}; s_0)$, but $(0 \ast x, q) \notin U(\mu_{AQ}; s_0)$, a contradiction to the definition of ideal. Hence $\mu_{AQ}(0 \ast x, q) \geq \mu_{AQ}(x, q)$ for any $x \in X$ and $q \in Q$. Similarly it can be proved that $\lambda_{AQ}(0 \ast x, q) \leq \lambda_{AQ}(x, q)$ for any $x \in X$ and for any $q \in Q$.

6.4 HOMOMORPHISMS OF INTUITIONISTIC Q-FUZZY IDEALS OF TM-ALGEBRAS

Definition 6.4.1

Let $f$ be a mapping on a set $X \times Q$ and $A_Q = (X, \mu_{AQ}, \lambda_{AQ})$ an intuitionistic $Q$-fuzzy set in $X$. Then the fuzzy sets $u$ and $v$ on $f(X \times Q)$ defined
by
\[ u(y, q) = \sup_{(x,q) \in f^{-1}(y,q)} \mu_{AQ}(x, q) \quad \text{and} \quad v(y, q) = \inf_{(x,q) \in f^{-1}(y,q)} \lambda_{AQ}(x, q) \]
for all \((y, q) \in f(X \times Q)\), are called the images of \(A\) under \(f\). If \(u, v\) are fuzzy sets in \(f(X \times Q)\) then the fuzzy sets \(\mu_{AQ} = u \circ f\) and \(\lambda_{AQ} = v \circ f\) are called the pre-images of \(u\) and \(v\) respectively under \(f\).

**Definition 6.4.2**

A function \(f : X \times Q \to X^1 \times Q\) is said to be a homomorphism of TM-algebras if \(f((x, q) * (y, q)) = f(x, q) * f(y, q) = f(x * y, q)\).

**Theorem 6.4.3**

Let \(f : X \times Q \to X^1 \times Q\) be an onto homomorphism of TM-algebras. If \(A^1_Q = (X^1, u, v)\) is an intuitionistic \(Q\)-fuzzy ideal of \(X^1\) then the pre-image of \(A^1_Q\) under \(f\) is an intuitionistic \(Q\)-fuzzy ideal of \(X\).

**Proof**

Let \(A_Q = (X, \mu_{AQ}, \lambda_{AQ})\) where \(\mu_{AQ} = u \circ f\) and \(\lambda_{AQ} = v \circ f\) is the pre-image of \(A^1_Q = (X^1, u, v)\) under \(f\). Since \(A^1_Q = (X^1, u, v)\) is an intuitionistic \(Q\)-fuzzy ideal of \(X^1\),

\[ u(0^1, q) \geq u(f(x, q)) = u \circ f(x, q) = \mu_{AQ}(x, q) \quad \text{and} \]
\[ v(0^1, q) \leq v(f(x, q)) = v \circ f(x, q) = \lambda_{AQ}(x, q). \]

On the other hand,

\[ u(0^1, q) = u(f(0, q)) = u \circ f(0, q) = \mu_{AQ}(0, q) \quad \text{and} \]
\[ v(0^1, q) = v(f(0, q)) = v \circ f(0, q) = \lambda_{AQ}(0, q). \]
Therefore,
\[ \mu_{AQ}(0, q) = u(0, q) \geq \mu_{AQ}(x, q) \]
and
\[ \lambda_{AQ}(0, q) = v(0^1, q) \leq \lambda_{AQ}(x, q) \]
for all \( x, y \in X \) and \( q \in Q \).

Now, it can be shown that, for all \( x, y, z \in X \) and \( q \in Q \),
\[ \mu_{AQ}(x, q) \geq \min\{\mu_{AQ}(x \ast y, q), \mu_{AQ}(y, q)\} \]
and
\[ \lambda_{AQ}(x, q) \leq \max\{\lambda_{AQ}(x \ast y, q), \lambda_{AQ}(y, q)\}. \]

For this,
\[
\mu_{AQ}(x, q) = u \circ f(x, q) \\
= u(f(x, q)) \\
\geq \min\{u((f(x, q) \ast f(y, q)), u(f(y, q))\} \\
= \min\{u(f(x \ast y, q)), u(f(y, q))\} \\
= \min\{u(f((x \ast y), q)), u(f(y, q))\} \\
= u \circ f((x \ast y), q), u \circ f(y, q)\} \\
= \min\{\mu_{AQ}((x \ast y), q), \mu_{AQ}(y, q), \}
\]
for all \( x, y \in X \) and \( q \in Q \).

Hence, the result \( \mu_{AQ}(x, q) \geq \min\{\mu_{AQ}(x \ast y, q), \mu_{AQ}(y, q)\} \) is true for all \( x, y \in X \) and \( q \in Q \).

Similarly, it can be proved that \( \lambda_{AQ}(x, q) \leq \max\{\lambda_{AQ}(x \ast y, q), \lambda_{AQ}(y, q)\}, \)
for all \( x, y \in X \) and \( q \in Q \).

Hence the pre image \( A_Q = (X, \mu_{AQ}, \lambda_{AQ}) \) of \( A^1_Q = (X^1, \mu_{AQ}, \lambda_{AQ}) \) is an
intuitionistic \( Q \)-fuzzy ideal of \( X \).
Note 6.4.4

The results proved for intuitionistic $Q$-fuzzy ideals of TM-algebras can also be proved for intuitionistic fuzzy ideals of TM-algebras by removing $Q$ in intuitionistic $Q$-fuzzy ideals.

6.5 CONCLUSION

The concepts of intuitionistic fuzzy ideals and intuitionistic $Q$-fuzzy ideals in TM-algebras have been introduced in this chapter with investigation of some of their properties. It can be further extended to intuitionistic fuzzy points on TM-algebras and on hyper TM-algebras.