Chapter 6
Stability of Ishikawa Iteration for Nonexpansive Type Condition

This Chapter contains some results on stability of Ishikawa iteration procedures for pair of multi-valued maps satisfying nonexpansive type condition. The nonexpansive type condition used is a generalization of many well known contractive conditions. Our result generalizes many well established results on stability of ishikawa iterates available in the literature.

Preliminaries

Let \((X, d)\) be a complete metric space, \(T\) a selfmap of \(X\). Let \(x_0 \in X, x_{n+1} = f(T, x_n)\) denote an iteration procedure which yields a sequence of points \(\{x_n\}\). Suppose that \(\{x_n\}\) converges to a fixed point \(p\) of \(T\). Let \(\{y_n\} \subset X, \varepsilon_n = d(y_{n+1}, f(T, y_n))\). If \(\lim \varepsilon_n = 0\), implies that \(y_n = p\), then the iteration procedure is said to be \(T\) stable.

The study of iteration procedures was initiated by Urbe [144], however Harder and Hicks [46] gave a formal definition of stability of iterative procedures. From literature it appears that Ostrowski [91] was the first to discuss stability of iteration procedures in metric space. Because of increasing use of computational mathematics and revolution in computer programming convergency and stability results for certain classes of mappings have been extensively studied by many authors (see [14], [23], [24], [46], [47], [52], [57], [80], [81], [82], [89] and the references there in). Harder and Hicks [46], [47] pointed out that the study of stability is theoretically as well as numerically interesting.
Mann in 1953 introduced iterative schemes and employed this iterative scheme to obtain the convergence of functions to fixed point. Rhoades [116] established results which showed that continuous self maps of a closed and bounded interval converges to a fixed point of the function but Mann iteration process sometime fails to converge for Lipschitzian pseudo-contractive maps. Ishikawa [57] introduced a new iterative process for the convergence of such maps. In last decade an extensive work has been done by researcher around convergence of Mann and Ishikawa iteration process for single-valued and multivalued mappings under various contractive conditions (see [14], [23], [24], [46], [47], [52], [57], [80], [81], [82], [89] and the references there in).

B.E. Rhoades [116] established the following result for the convergence of Mann iteration procedure.

**Theorem 6.1.** Let $T$ be a self-map of a closed convex subset $K$ of a real Banach space $(X, d)$. Let $\{x_n\}_{n=1}^{\infty}$ be a general Mann iteration of $T$ that converges to a point $p \in X$. If there exists the constants $\alpha, \beta, \gamma \geq 0, \delta < 1$ such that

$$
\|Tx_n - Tp\| \leq \alpha \|x_n - p\| + \beta \|x_n - Tx_n\| + \gamma \|p - Tx_n\|
+ \delta \max\{\|p - Tp\|, \|x_n - p\|\}
$$

Then $p$ is fixed point of $T$.

It is to be noted that, in above theorem, if $T$ is continuous then Mann iteration process converges to fixed point of $T$. But if it is not then there is no guarantee that Mann iterative process converges and even if it converges, it will not necessarily converge to a fixed point of $T$. Hu et.al [52] solved this problem by extending above theorem to the Ishikawa iteration process.

**Theorem 6.2.**[52]. Let $K$ be a compact convex subset of a Hilbert space, $T: K \rightarrow K$ a Lipschitzian pseudo-contractive map and $x_1 \in X$. Then the Ishikawa iteration $\{x_n\}$ defined as:
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\[ x_{n+1} = \alpha_n T[\beta_n TX_n + (1 - \beta_n) x_n] + (1 - \alpha_n)x_n, \]

where \( \{\alpha_n\}^\infty_{n=1} \) and \( \{\beta_n\}^\infty_{n=1} \) are sequence of positive number that satisfy following three conditions.

I. \( 0 \leq \alpha_n \leq \beta_n \leq 1 \), for all positive integer n.
II. \( \lim_{n \to \infty} \beta_n = 0 \),
III. \( s\sum_{n=1}^\infty \alpha_n \beta_n = \infty \),

converges strongly to a fixed point of \( T \).

We need the following Lemma due to Nadler [88] for our main result.

**Lemma6.1.[88]** If \( A, B \in CB(X) \) and \( a \in A \), then for \( \epsilon > 0 \) there exists \( b \in B \) such that

\[ d(a, b) \leq H(a, B) + \epsilon. \]

The Ishikawa iteration scheme for two multivalued mappings is defined as follows:

**Definition6.1.** Let \( K \) is a nonempty subset of \( X \). and \( S, T : K \to CB(X) \). Ishikawa iteration is defined as:

\[
\begin{align*}
\{x_0 \in K \\
y_n &= (1 - \beta_n)x_n + \beta_n a_n, a_n \in TX_n \\
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n b_n, \quad b_n \in Sy_n
\end{align*}
\]

(6.1)

Where \( 0 \leq \alpha_n, \beta_n \leq 1 \) for all n.

Recently for Ishikawa iterates for multivalued mappings on a Banach space Rhoades [111] established a generic theorem with number of corollaries.

**Theorem6.3.** Let \( X \) be a Banach space, \( K \) a closed convex subset of \( X \). \( S \) and \( T \) are multivalued mappings from \( K \) to \( CB(X) \). Suppose that the Ishikawa iteration scheme (6.1), with \( \{\alpha_n\} \) satisfying:
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(i) \( 0 \leq \alpha_n, \beta_n \leq 1 \) for all \( n \),

(ii) \( \lim inf \alpha_n = \delta > 0 \) and \( \{a_n\}, \{b_n\} \), satisfying

\[
\|a_n - b_n\| \leq H(Tx_n, Sy_n) + \epsilon_n \quad \text{with } \lim \epsilon_n = 0
\]  

(6.2)

Converges to a point \( p \). If there exists non-negative number \( \alpha, \beta, \gamma, \delta \) with \( \beta \leq 1 \) such that for sufficiently large \( n \), \( S \) and \( T \) satisfying

\[
H(Tx_n, Sy_n) \leq \alpha \|x_n - b_n\| + \beta \|x_n - a_n\|
\]  

(6.3)

and

\[
H(Sp, Tx_n) \leq \alpha \|x_n - p\| + \gamma d(x_n, Tx_n) + \delta d(p, Tx_n)
\]

\[
+ \beta \max \{d(p, Sp), d(x_n, Sp)\}
\]

(6.4)

then \( p \) is a fixed point of \( S \). If also

\[
H(Sp, Tp) \leq \beta [d(p, Tp) + d(p, sp)].
\]

(6.5)

then \( p \) is a common fixed point of \( S \) and \( T \).

Recently Singh and Dimri [129] extended and generalized the results of Hu et.al [52] and Rhoades [111], [116] and established a common fixed point theorem for Ishikawa iterates for a pair of multivalued mappings in Banach space satisfying nonexpansive type condition. Now we generalize the result of Singh and Dimri [133] for more general nonexpansive type condition. Our result serves as a generalization of results due to Hu et.al [52] Rhoades [111], [116] and Singh and Dimri [129], as well as several well known results.

Main results

Theorem 6.4. Let \( K \) be a nonempty, closed, convex subset of Banach space \( X \) and \( T, S: \rightarrow CB(X) \) satisfying
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\[ H(Tx, Ty) \leq a \|x - y\| + b\{d(x, Tx) + d(y, Sy)\} \]
\[ + c \{d(x, Sy) + d(y, Tx)\}, \]  \hspace{1cm} (6.6)

Where \( a > 0, 0 \leq b < 1, 0 \leq c < 1 \) are such that
\[ a + b + c = 1. \]

for all \( x, y \in K \). If there exists \( x_0 \in K \) such that the \( \{x_n\} \) satisfying (6.1), (6.2) with

- (i) \( 0 \leq \alpha_n, \beta_n \leq 1 \) for all \( n \),
- (ii) \( \lim inf \alpha_n = \delta > 0 \) and \( \{\alpha_n\}, \{\beta_n\} \), satisfying
  \[ \|a_n - b_n\| \leq H(Tx_n, Sy_n) + \epsilon_n \]
- (iii) \( \beta_n = 0 \)

Converges to a point \( p \), then \( p \) is a common fixed point of \( S \) and \( T \).

**Proof.** Since in Theorem 6.3 Rhaodes [111] showed that the conditions (6.3), (6.4) and (6.5) are enough for the convergence of Ishikawa iteration. So for the proof of our theorem it is sufficient to show that \( S \) and \( T \) satisfy the above mention conditions,

From (6.6) we have
\[ H(Tx_n, Sy_n) \leq a \|x_n - y_n\| + b\{d(x_n, Tx_n) + d(y_n, Sy_n)\} \]
\[ + c \{d(x_n, Sy_n) + d(y_n, Tx_n)\}, \]  \hspace{1cm} (6.7)

From (6.1) we have
\[
\begin{align*}
|x_n - y_n| &= \beta_n |x_n - a_n| \\
d(x_n, Tx_n) &\leq |x_n - a_n| \\
d(y_n, Sy_n) &\leq |y_n - b_n| \leq |y_n - x_n| + |x_n - b_n| \\
&\leq \beta_n |x_n - a_n| + |x_n - b_n|, \\
d(x_n, Sy_n) &\leq |x_n - b_n| \\
d(y_n, Tx_n) &\leq |y_n - a_n| = |y_n - x_n| + |x_n - a_n| \\
&\leq (1 + \beta_n) |x_n - a_n|
\end{align*}
\] (6.8)

Now using (6.8) we get
\[
d(x_n, Tx_n) + d(y_n, Sy_n) \leq |x_n - a_n| + \beta_n |x_n - a_n| + |x_n - b_n| \\
\leq (1 + \beta_n) |x_n - a_n| + |x_n - b_n|
\] (6.9)

Also
\[
d(x_n, Sy_n) + d(y_n, Tx_n) \leq |x_n - b_n| + (1 + \beta_n) |x_n - a_n|
\] (6.10)

using (6.10), (6.9) and (6.8) in (6.7) we get
\[
H(Tx_n, Sy_n) \leq a |x_n - a_n| + b \{(1 + \beta_n) |x_n - a_n| + |x_n - b_n|\} \\
\quad + c \{ |x_n - b_n| + (1 + \beta_n) |x_n - a_n| \}
\leq [a + b + c] |x_n - a_n| + [b + c] |x_n - b_n|
\]

and since \(a+b+c=1\) implies \(b+c<1\), we can say that (6.3) is satisfied. Again using (6.6) we get
\[
H(Tx_n, Sp) \leq a |x_n - p| + b \{d(x_n, Tx_n) + d(p, Sp)\}
\]
\[+ c \{d(x_n, Sp) + d(p, Tx_n)\},
\]
\[
\leq a |x_n - p| + b \{|x_n - a_n| + d(p, Sp)\}
\]
\[+ c \{d(x_n, Sp) + d(p, a_n)\}
\] (6.11)

Since (6.3) is satisfied, therefore using (6.2), we get
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\[
\|x_n - a_n\| \leq \|x_n - b_n\| + \|b_n - a_n\|
\]

\[
\leq \|x_n - b_n\| + H(Tx_n, Sy_n) + \epsilon_n
\]

\[
\leq \|x_n - b_n\| + \|x_n - b_n\|
\]

\[
+ (b + c) \|x_n - a_n\| + \epsilon_n
\]

Since \(\lim \|x_n - b_n\| = 0\), we obtain

\[
\lim sup \|x_n - a_n\| \leq (b + c) \lim sup \|x_n - a_n\|
\]

and since \(0 \leq (b + c) \leq 1\), which implies,

\[
\lim \|x_n - a_n\| = 0.
\]  \(\text{(6.12)}\)

Also

\[
\|p - a_n\| \leq \|p - x_n\| + \|x_n - a_n\|
\]  \(\text{(6.13)}\)

Using (6.11), (6.12) and (6.13) we get

\[
H(Tx_n, Sp) \leq a \|x_n - p\| + b d(p, Sp) + c d(x_n, Sp) \leq \|p - x_n\| + \|x_n - a_n\|
\]

\[
\leq (a + c) \|x_n - p\| + b d(p, Sp) + c d(x_n, Sp).
\]  \(\text{(6.14)}\)

The conditions \((b + c) > 0\) and \(0 \leq b < 1, 0 < c < 1\) implies that condition (6.4) is a special case of (6.14). Hence (6.14) implies (6.4). Since (6.3) and (6.4) is satisfied hence by Theorem 6.1 \(p\) is a fixed point of \(S\). Also we can see that condition (6.5) is also satisfied hence \(p\) is a common fixed point of \(S, T\).
Remark 1. It is remarkable that the nonexpansive type condition used in Theorem 6.3 contains the condition used in Singh and Dimri [129], and many other contractive conditions.

Remark 2. The result established by Singh and Dimri [129] and Rhoades [111][116] are special cases of our result and can be obtained as corollaries.
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