CHAPTER 4

DYNAMIC MODELING OF TWO DOF MANIPULATOR

4.1 INTRODUCTION

In this chapter, an introduction to the dynamic model is given, followed by a brief survey on the different issues of the authors for formulation of the mathematical model. Furthermore, the some assumptions have also been made and the mathematical model for the dynamic behaviour of the manipulator has been developed. The mathematical equations, often referred to as manipulator dynamics, are a set of equations of motion (EOM) that describe the dynamic response of the manipulator to input actuator torques. The proposed research uses this approximated dynamic model to position the robot arm in the pre-defined path.

4.2 A BRIEF SURVEY OF THE STATE-OF-ART IN THE FIELD

During the work cycle, a manipulator must accelerate, move at constant speed, and decelerate. This time-varying position and orientation of the manipulator is termed as its dynamic behaviour. Time varying torque is applied at the joints to balance the internal and external forces. The internal forces are caused by motion (velocity and acceleration) of links. Inertial, coriolis and frictional forces are some of the internal forces; the external forces are those forces exerted by the environment and include the load and gravitational forces.
Factories of future need the presence of robots in its operations and the performance of production systems depends largely on such robots. It is essential to study the performance of a robot through widely used techniques in mathematical modeling. Literature is available on the analysis of performance of robots using differential equations. These equations provide simple analytical solutions without non-linear differential equations leading to limited constraints (Mc Clamroch, 1986; Ata and Johar, 2004). They stated that in addition to the effects of differential angular velocity and the joint torque the robotic system performance can also be affected due to the interaction between the joint motion and the angular motion of the constrained surface. The performance of a robot arm with respect to control accuracy and mechanical efficiency is based on the effects of manipulator gravity (Yamawaki and Yashima, 2007).

The following table summarises some of the studies related to operation and performance of robots, as deducted from literature:

**Table 4.1 Summary of Literature on Model Formulated**

<table>
<thead>
<tr>
<th>Author references</th>
<th>Focus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raibert and Craig, 1981</td>
<td>Attempted dealing with the problem of rigid manipulator by applying hybrid position control scheme.</td>
</tr>
<tr>
<td>Qu, 1992</td>
<td>Analyzed robust control of a class of non-linear uncertain systems based on the-linear and coupled characteristics nature; found that the dynamic of robot becomes much more complex</td>
</tr>
<tr>
<td>Matsuno and Yamamoto, 1994</td>
<td>Attempted to control the two link flexible manipulator by applying the dynamic hybrid position/force control</td>
</tr>
<tr>
<td>Krishnan and Mcclamroch, 1994</td>
<td>Developed the Non-Linear differential-algebraic control systems for constrained robot systems</td>
</tr>
<tr>
<td>Steinbach, 1997</td>
<td>Focused on optimizing errors by boundary value problem approach.</td>
</tr>
<tr>
<td>Author references</td>
<td>Focus</td>
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<tr>
<td>Lim and Seraji, 1997</td>
<td>Discussed the configuration of control system.</td>
</tr>
<tr>
<td>Bona and Indri, 1998</td>
<td>Analyzed the robot control problems using observers for improving the performance in linear and non linear system.</td>
</tr>
<tr>
<td>Shi et al., 1999</td>
<td>Derived the mathematical model of a constrained rigid-flexible manipulator based on Hamilton’s principle</td>
</tr>
<tr>
<td>Oucheriah, 1999</td>
<td>Discussed the error minimization in time delayed system.</td>
</tr>
<tr>
<td>Ho et al., 2000</td>
<td>Concentrated on the uncertainties in constraint functions for reducing the inaccuracy levels.</td>
</tr>
<tr>
<td>Featherstone and Orin, 2000</td>
<td>Concentrated on developing an algorithm for dynamic computations</td>
</tr>
<tr>
<td>Ata and Ghazy, 2001</td>
<td>Focused on dynamic modeling and simulation of constrained motion of rigid-flexible manipulator in contact with a compliant surface.</td>
</tr>
<tr>
<td>Bianco and Piazza, 2002</td>
<td>Applied the global optimization approach to obtain the minimum time by considering the joint torque and derivatives for nonlinear manipulator dynamics.</td>
</tr>
<tr>
<td>Ata et al., 2003</td>
<td>Attempted to find an optimal motion trajectory for the constrained motion based on minimum energy consumption</td>
</tr>
<tr>
<td>Hopler et al., 2004</td>
<td>Addressed the equations of motion and sensitivity for optimal control problem in legged robot</td>
</tr>
<tr>
<td>Kalyoncu and Botsal, 2004</td>
<td>Analyzed the elastic manipulator under time-varying cases</td>
</tr>
<tr>
<td>Chitta et al., 2005</td>
<td>Addressed the dynamic equations of motion</td>
</tr>
<tr>
<td>Spong et al., 2005</td>
<td>Discussed the robot modeling and controls</td>
</tr>
<tr>
<td>Stocco and Yedlin, 2007</td>
<td>Concentrated on minimizing constraints by mass/pulley model</td>
</tr>
<tr>
<td>Mist et al., 2008</td>
<td>Discussed the performance of the robot on the basis of finding the positions of base and joint angle of 6 degrees of freedom manipulator.</td>
</tr>
</tbody>
</table>
4.3 ASSUMPTIONS

This model does not take into account actuators dynamics (motors and Gear boxes) as well as friction forces. Besides, the robot is considered rigid, i.e., without flexibility in links and in joints. The model is simple and adequately represent the manipulator for the purpose of this current work.

4.4 FORMULATION OF THE DYNAMIC MODEL

The dynamic model of a manipulator is useful for computation of torque and force required for the execution of a typical work cycle and gives vital information for the design of links, joints, drives, and actuators. The dynamic behaviour of the manipulator provides relationship between joint actuator torques and motion of links for simulation and design of control algorithms. The manipulator control maintains the dynamic response of the manipulator to obtain the desired performance, which directly depends on the accuracy of the dynamic model and efficiency of the control algorithm. The control problem requires specifying the control strategies to achieve the desired response and performance. Simulations of manipulator motion permits testing of control strategies, motion planning and performance studies without a physical prototype of the manipulator. (McClamroch, 1986.)

The serial link manipulator represents a complex dynamic system, which can be modeled systematically using known physical laws of Lagrangian mechanics or Newtonian mechanics. Approaches such as Lagrange – Euler (LE) which are energy based and Newton – Euler (NE) which are force-balance based can be systematically applied to develop the manipulator EOM. The resulting EOM are a set of second order, coupled, non-linear differential equations, consisting of inertia loading and coupling reaction forces between joints. Both Lagrange – Euler and Newton-Euler formulation gives the relationship between the joint and end effector forces.
torques and the manipulator joint-link position, velocities, and accelerations. These formulations are compared on the basis of their computational requirements. The LE formulation has the following characteristics

- The LE is systematic and describes motion in real physical terms.
- The equations of motion obtained are analytical and compact.
- The matrix vector form of equations (inertia matrix, centrifugal, and Coriolis force matrix, and gravitational force vector) are appearing for calculation and control system design.
- The control problem can be simplified by designing the structure of the manipulator with minimal joint coupling that is coefficients may be reduced or eliminated.

Based on the above characteristics, the proposed research uses this LE method for formulating the second order differential equations.

Lagrangian mechanics is based on the differentiation of the energy terms with respect to the system’s variable and time. As the complexity of the system increases, the Lagrangian method becomes relatively simpler to use. (Mittal et al., 2003). The Lagrangian mechanics is based on two generalized equations. One for linear motions and the other for rotational motions.

Summation of all external forces for a linear motion $F_i$ is given by,

\[
F_i = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \left( \frac{\partial L}{\partial x_i} \right) \quad (4.1)
\]
Summation of all external torques in a rotational motion $T_i$ is given by,

$$T_i = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_i} \right) - \left( \frac{\partial L}{\partial \theta_i} \right)$$  \hspace{1cm}(4.2)

To obtain the equations of motion, it is important to derive energy equations for the system and then differentiate the Lagrangian according to Equations (4.1) and (4.2).

![Figure 4.1 Robot Manipulator with Two Degrees of Freedom](image)

The dynamics of a simple manipulator is worked out to illustrate the Lagrange-Euler formulation and to clarify the problems involved in dynamic modeling. For the manipulator links 1 and 2, joint variables are $\theta_1$ and $\theta_2$, link lengths are $l_1$ and $l_2$ and mass of links are $m_1$ and $m_2$ and $r_1$ and $r_2$ are the distance between the joint and centre of mass of the link 1 and 2 respectively which is shown in the Figure 4.1. The linear and angular velocities are $v_1$, $v_2$, $\dot{\theta}_1$, $\dot{\theta}_2$ respectively. In this case, the datum (zero potential energy) is chosen at the axis of rotation “o”.
Lagrangian $L = K - P$ \hspace{1cm} (4.3)

First, the kinetic and potential energies of the system are calculated.

$$K = K_1 + K_2$$ \hspace{1cm} (4.4)

$$K_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$$ \hspace{1cm} (4.5)

To calculate $K_2$ first write the position equation for $m_2$ and then differentiate it for the velocity of $m_2$.

$$x_2 = l_1 \sin \theta_1 + l_2 \sin (\theta_1 + \theta_2) = l_1 S_1 + l_2 S_{12}$$

$$y_2 = - [ l_1 \cos \theta_1 + l_2 \cos (\theta_1 + \theta_2) ] = - l_1 C_1 - l_2 C_{12}$$ \hspace{1cm} (4.6)

$$x'_2 = l_1 C_1 \theta_1' + l_2 C_{12} ( \theta_1' + \theta_2')$$

**Since $V_2 = \dot{x}^2 + \dot{y}^2$,**

$$V_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 ( \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \dot{\theta}_1 \dot{\theta}_2 )$$

$$+ 2 l_1 l_2 ( \dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2 ) (C_1 C_{12} + S_1 S_{12})$$

$$= l_1^2 \dot{\theta}_1^2 + l_2^2 ( \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \dot{\theta}_1 \dot{\theta}_2 ) + 2 l_1 l_2 C_2 ( \dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2 )$$ \hspace{1cm} (4.7)

Then the kinetic energy for the second mass is

$$K_2 = \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 ( \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \dot{\theta}_1 \dot{\theta}_2 ) + m_2 l_1 l_2 C_2 ( \dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2 )$$ \hspace{1cm} (4.8)

and the total kinetic energy is
The derivatives of the Lagrangian are,

\[
K = \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \dot{\theta}_1 \dot{\theta}_2)
\]

\[+ m_2 l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \tag{4.9}\]

The potential energy of the system can be written as

\[
P_1 = -m_1 g l_1 \cos \theta_1 = -m_1 g l_1 C_1 \tag{4.10}\]

\[
P_2 = -m_2 g l_1 C_1 - m_2 g l_2 C_{12} \tag{4.11}\]

\[
P = P_1 + P_2 = -(m_1 + m_2) g l_1 C_1 - m_2 g l_2 C_{12} \tag{4.12}\]

The Lagrangian for the system is

\[
L = K - P
\]

\[
= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \dot{\theta}_1 \dot{\theta}_2)
\]

\[+ m_2 l_1 l_2 C_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) + (m_1 + m_2) g l_1 C_1 + m_2 g l_2 C_{12} \tag{4.13}\]

The derivatives of the Lagrangian are,

\[
\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_2^2 (\dot{\theta}_1' + \theta_2')
\]

\[+ 2 m_2 l_1 l_2 C_2 \theta_1 + m_2 l_1 l_2 C_2 \theta_2 \tag{4.14}\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) = [ (m_1 + m_2) l_1^2 + m_2 l_2^2
\]

\[+ 2 m_2 l_1 l_2 C_2 ] \theta_1'' + [ m_2 l_2^2 + m_2 l_1 l_2 C_2 ] \dot{\theta}_2
\]

\[- 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \tag{4.15}\]
\[ \frac{\partial L}{\partial \dot{\theta}_1} = -(m_1 + m_2) g l_1 S_1 - m_2 g l_2 S_{12} \]  

(4.17)

From Equation (4.2), the first equation of motion is

\[ T_1 = ((m_1 + m_2)) l_1^2 + m_2 l_2^2 \]

\[ + 2 m_2 l_1 l_2 C_2 \dot{\theta}_1 + (m_2 l_2^2 + m_2 l_1 l_2 C_2) \ddot{\theta}_2 \]

\[ - 2 m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 l_1 l_2 S_2 \dot{\theta}_2^2 + (m_1 + m_2) g l_1 S_1 + m_2 g l_2 S_{12} \]  

(4.18)

Similarly, \[ \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2) + m_2 l_1 l_2 C_2 \dot{\theta}_1 \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) = m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 C_2 \ddot{\theta}_1 - m_2 l_1 l_2 S_2 \dot{\theta}_2 \dot{\theta}_1 \dot{\theta}_2 \]

\[ \frac{\partial L}{\partial \theta_2} = -m_2 l_1 l_2 S_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) - m_2 g l_2 S_{12} \]

\[ T_2 = (m_2 l_2^2 + m_2 l_1 l_2 C_2) \ddot{\theta}_1 + m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 S_2 \dot{\theta}_2^2 + m_2 g l_2 S_{12} \]  

(4.19)

Writing these two equations in a matrix form,

\[
\begin{bmatrix}
T_1 \\
T_2
\end{bmatrix} =
\begin{bmatrix}
(m_1 + m_2) l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2 & \{m_2 l_2^2 + m_2 l_1 l_2 C_2\} \\
(m_2 l_2^2 + m_2 l_1 l_2 C_2) & m_2 l_2^2
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta}_1 \\
\ddot{\theta}_2
\end{bmatrix}
\]

\[ + \begin{bmatrix}
0 & -m_2 l_1 l_2 S_2 \\
m_2 l_1 l_2 S_2 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1^2 \\
\dot{\theta}_2^2
\end{bmatrix}
+ \begin{bmatrix}
0 & -m_2 l_1 l_2 S_2 \\
-m_2 l_1 l_2 S_2 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \dot{\theta}_2 \\
\dot{\theta}_2 \dot{\theta}_1
\end{bmatrix}
\]

\[ + \begin{bmatrix}
(m_1 + m_2) g l_1 S_1 + m_2 g l_2 S_{12} \\
m_2 g l_2 S_{12}
\end{bmatrix} \]

(4.20)

Which is of the form,

\[ T = A(Q) \ddot{Q} + B(Q, \dot{Q}) + C(Q) \]  

(4.21)
where, $A(\mathbf{Q})$ is coupled inertia matrix, $B(\mathbf{Q}, \dot{\mathbf{Q}})$ is the matrix of coriolis and centrifugal forces and $C(\mathbf{Q})$ is the gravity matrix. $T$ is the Input torques applied at various joints.

4.5 CONCLUDING REMARKS

The mathematical model for two DOF manipulator with rotary joints has been developed. The mathematical equations, often referred to as manipulator dynamics, are a set of equations of motion (EOM) that describe the dynamic response of the manipulator to input actuator torques. Based on the assumptions made in this section, the approximated mathematical model has been derived and it has been used for simulation and validation purposes in the subsequent chapters. Moreover, this model has been considered in different forms to check the efficiency of the proposed mathematical tool.