CHAPTER I
INTRODUCTION

Gabour Szegö's (1895-1985) most important work was in the area of extremal problems and Toeplitz matrices. He proved a number of limit theorems, now known as Szegö's limit theorem, the strong Szegö's limit theorem [2] and Szegö's orthogonal polynomials.

In early twenties G. Szegö studied in detail the distribution of eigenvalues of the section of Toeplitz forms associated with a function defined in \([-\pi,\pi]\).

The basic idea used by Szegö is the so called equidistribution of sequences introduced by H. Weyl.

**Equidistribution of Sequence** [13]

Let \((u_k)\) \((k \geq 1)\) be a sequence of real numbers contained in an interval \(I\) of length \(|I|\). For any subinterval \(J\) of \(I\), of length \(|J|\), let \(J(n)\) denote the number of points among \(u_1,u_2,\ldots,u_n\) that lie in \(J\). The sequence is said to be equidistributed or uniformly distributed on \(I\) if for each \(J\) contained in \(I\),

\[
\lim_{n \to \infty} \frac{J(n)}{n} = \frac{|J|}{|I|}.
\]

(Interals may be open or half-open.)

The following measure theoretic version can also found in [13].

**Theorem**

The sequence \((u_k)\) contained in \([0,2\pi)\) is uniformly distributed on that interval if and only if

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} f(u_k)}{n} = \frac{2\pi}{\int f \, d\sigma}
\]

For every function \(f\) that is continuous and periodic with period \(2\pi\).
Toeplitz has studied the distribution of eigenvalues of an infinite matrix \( (C_{u-m}) \) where the indices \( u \) and \( m \) range from \(-\infty\) to \(\infty\). The asymptotic distribution of the eigenvalues of Toeplitz forms can be expressed in the terminology of theory of equal distribution due to H. Weyl. The well known Szegò’s theorem throws light into the asymptotic distribution of eigenvalues of truncations.

Szegö’s Theorem [12]

The Szegö’s theorem on Toeplitz matrices states that if \( \lambda_1(A)_N, \lambda_2(A)_N, \ldots, \lambda_N(A)_N \) are the eigenvalues of the \( N \times N \) truncations \( (A)_N \) of the matrix \( A = (a_{i,j}) \), where

\[
a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx
\]

is the \( k^{th} \) Fourier coefficient of the multiplier \( f \) in \( L^\infty(-\pi, \pi) \), and \( F \) is any continuous function on \( \mathbb{R} \), then

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} F(\lambda_k(A)_N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(f(x)) dx \quad \ldots \quad (1)
\]

The above theorem is well known for its applications to trigonometric moment problems, stochastic process [12] and to problems in edge detection [14].

The classical Szegö’s theorem is based on Fourier system \( \{ e_n : n \in \mathbb{Z} \} \) where \( e_n(x) = e^{inx} \). In this thesis we study similar results in the context of Haar System.

1.1 Summary of the Thesis

The problem considered is the validity of conclusion of Szegö under the following changes in the hypothesis.
(i) The Fourier basis is reordered
(ii) The Fourier system is replaced by other systems like Haar wavelet system with various ordering.

The thesis is divided into five chapters including introductory Chapter I.

In Chapter II we look into the effect of change in the ordering of the Fourier system on Szegö's classical observations of asymptotic distribution of eigenvalues of finite Toeplitz forms. This is done by checking proofs and Szegö's properties in the new set up. It is observed that there is no change in the conclusion of Szegö. The first section deals with minimum property of Toeplitz forms and its limits in the changed system. The second one deals with asymptotic distribution of eigenvalues of finite Toeplitz forms in the new system. This is an imitation of the method adopted by Szegö in the original case.

In Chapter III we consider the multiplication operators under Haar system in $L^2(0,1)$. To be more precise the corner $N \times N$ truncations and the associated asymptotic distribution of eigenvalues are analyzed, analogous to Szegö’s theorem classical version. This chapter is divided into two sections. In section one, $L^2(0,1)$ with Haar system under lexicographic ordering is considered. The main theorem of this chapter [3.1.3] says that the conclusion of classical theorem does not remain valid in the changed setup. It is also observed that when the same operator is considered with respect to another ordering, the distribution of eigenvalues converges. In section two we consider spectral approximations of multiplication operators under Haar system in $L^2(0,1)$. This work is quite similar to the work of Kent E. Morrison.[17].

In chapter IV analogous to classical Szegö's theorem we define Szegö's Type theorem for operators in $L^2(R_+)$ and in $L^2(R)$ and checks its validity for certain multiplication operators with respect to a chosen ordering of the Haar basis. It is observed that for certain multiplication operators $T_f$ with
multiplier \( f = h_i \), \( i \geq 0 \), the distribution of eigenvalues converges but not to the "Szegő limit" and for multiplication operators \( T_f \) with \( f = h_j, i \geq 0, j > 0 \), the distribution of eigenvalues exists and Szegő's Type theorem is valid. This can be considered the main result of this chapter. The theorem 4.11 provides a partial \( L^2(R) \) version of the above result.

In the fifth and final chapter, we discuss classes of orderings of Haar System in \( L^2(R) \) and in \( L^2(R) \) in which Szegő's Type Theorem is valid for certain multiplication operators. This chapter is divided into two sections. In the first section, we give an ordering to Haar system in \( L^2(R) \) and prove that with respect to this ordering, Szegő's Type Theorem holds for general class of multiplication operators \( T_f \) with multiplier \( f \in L^2(R) \), subject to some conditions on \( f \). This is given in 5.1.13, which is the main result of this chapter. Finally in second section more general classes of orderings of Haar system in \( L^2(R) \) and in \( L^2(R) \) are identified in such a way that for certain classes of multiplication operators the asymptotic distribution of eigenvalues exists. Some illustrative examples are also given.

Apart from these five chapters a result on spectral approximation and a proposal for future investigation to higher dimensional \( L^2(R^n) \) is given in the appendix.

1.2 Basic Definitions and Theorems

Some basic definitions and theorems which are quoted in the subsequent chapters are given here.

1.2.1 Toeplitz's Forms [12]

Let \( f(x) \) be a real-valued function of class \( L \) and

\[
f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}\]

its Fourier Series, where

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \quad c_{-n} = c_n^* .
\]
Then the Hermitian form \( T_n = \sum_{\nu=0}^{n-1} c_{\nu,\mu} u_{\mu} \overline{u}_{\nu} \), \( \nu, \mu = 0, 1, \ldots, n \) is called the Toeplitz form associated with the function \( f(x) \) and the matrix \( (c_{\nu,\mu}) \) is called Toeplitz matrix. We have in this case

\[
T_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| u_0 + u_1 e^{ix} + \ldots + u_n e^{inx} \right|^2 f(x) \, dx.
\]

1.2.2. Equal distribution of numbers [12]

For each \( n \) we consider a set of \( n+1 \) real numbers \( a_1^{(n)}, a_2^{(n)}, \ldots, a_{n+1}^{(n)} \) and another set of the same kind \( b_1^{(n)}, b_2^{(n)}, \ldots, b_{n+1}^{(n)} \). We assume that for each \( v \) and \( n \)

\[
|a_v^{(n)}| < K, \quad |b_v^{(n)}| < K
\]

where \( K \) is independent of \( v \) and \( n \). We say that \( \{a_v^{(n)}\} \) and \( \{b_v^{(n)}\} \), \( n \to \infty \), are equally distributed in the interval \([-K, K]\) if the following holds. Let \( F(t) \) be an arbitrary continuous function in the interval \([-K, K]\); we have then

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{v=1}^{n+1} \left| F(a_v^{(n)}) - F(b_v^{(n)}) \right| = 0.
\]

1.2.3 Multiplication Operator [17]

Suppose \( I \subseteq \mathbb{R} \) is an interval and \( f : I \to C \) is a bounded measurable function. Define the multiplication operator

\[
T_f : L^2[I] \to L^2[I] : g \to fg, \ g \in L^2[I].
\]

Let \( \{e_1, e_2, \ldots\} \) be an orthonormal basis of \( L^2[I] \). We define the \( N \times N \) matrix

\[
(T_f)_N = (a_{ij}), \quad 1 \leq i, j \leq N,
\]

where

\[
a_{ij} = \int_{I} f(x) e_j^*(x) e_i(x) \, dx.
\]

The infinite matrix \( (T_f) = (a_{ij}) \) \( 0 \leq i, j \), represents the operator \( T_f \). \( T_f \) is the bounded linear operator and we use the operator norm.
\[ \|T_f\| = \sup_{|k|=1} \|T_f(\xi)\|. \]

Let \( P_N \) denote the orthogonal projection of \( H \) onto the span \( \{e_1, e_2, \ldots, e_n\} \) and put \( T_{fN} = P_N T_f P_N \). As it is done in [1], we freely consider \( T_{fN} \) as \( N \times N \) corner truncation of the matrix \( (T_f) \). We can regard \( (T_f)_N \) as a matrix approximation of \( T_f \).

1.2.4 Hausdorff metric [15,17]

Let \( H(c) \) denote the set of compact subsets of \( C \). Define the Hausdorff metric \( h \) on \( H(C) \) by

\[ h(M,N) = \max[h^*(M,N), h^*(N,M)] \]

(The Hausdorff distance between \( M \) & \( N \)) where

\[ h^*(M,N) = \sup \inf \{m - n\} \]

1.2.5 Essential Range [6]

Let \( E \) be a measurable subset of \( R \) and \( f \in L^\infty(E) \). The set

\[ \{k \in R : m\{t \in E : |f(t) - k| < \varepsilon\} > 0 \text{ for every } \varepsilon > 0\} \]

is called the essential range of \( f \) and is denoted by \( R(f) \).

1.2.6 Haar Wavelet Theory [3,5]

Wavelets are mathematical functions that cut up data into different frequency components and then study each component with a resolution matched to its scale. They have advantages over traditional Fourier methods in analyzing physical situations where the signal contains discontinuities and sharp spikes. A comparison of Fourier transform and Wavelet transform is given in [3].

The first mention of wavelet appeared in an appendix to the thesis of A.Haar. The theory of wavelets lies in the boundaries between (i)Mathematics (ii) Scientific Calculations (iii) Signal Processing (iv) Image
Processing. The main branch of mathematics leading to wavelets began with Joseph Fourier who introduced Fourier Synthesis.

In 1910 Haar constructed an orthonormal basis for $L^2(0,1)$ now known as Haar system which provides a local analysis.

For $m, n \in \mathbb{Z}$, let $I_{mn}$ be the closed interval

$$I_{mn} = \left[ \frac{n}{2^m}, \frac{n+1}{2^m} \right] \subseteq \mathbb{R}. $$

Such intervals are called dyadic intervals. The collection $\{I_{mn} : m, n \in \mathbb{Z}\}$ of all dyadic intervals has the nesting property: if the interiors of $I_{mn}$ and $I_{pq}$ have nonempty intersection, then either $I_{mn} \subseteq I_{pq}$ or $I_{pq} \subseteq I_{mn}$. The Haar function $\{h_{mn} : m, n \in \mathbb{Z}\}$ on $\mathbb{R}$ are defined as

$$h_{mn}(x) = \begin{cases} 
2^{m/2} & \frac{n}{2^m} \leq x < \frac{n+1/2}{2^m} \\
-2^{m/2} & \frac{n+1/2}{2^m} \leq x < \frac{n+1}{2^m} \\
0 & \text{otherwise}
\end{cases}$$

Each $h_{mn}$ is nonzero on $I_{mn}$ and $\{h_{mn} : m, n \in \mathbb{Z}\}$ is an orthonormal set. $\{h_{mn} : m, n \in \mathbb{Z}\}$ is complete in $L^2(\mathbb{R})$, so we have the identity

$$f = \sum_{m,n \in \mathbb{Z}} \langle f, h_{mn} \rangle h_{mn} \quad \text{in} \quad L^2(\mathbb{R}).$$

This expansion is local in the sense that if $f = 0$ on $h_{mn}$, then $\langle f, h_{mn} \rangle = 0$.

Let

$$h = h_{00} = \begin{cases} 
1 & 0 \leq x < 1/2 \\
-1 & 1/2 \leq x < 1 \\
0 & \text{otherwise}
\end{cases}$$

Then for each $m, n \in \mathbb{Z}$

$$h_{mn}(x) = 2^{\frac{m}{2}} h(2^m x - n).$$

Hence all basis elements are obtained by certain translations and dilations of one element. This is the characteristic structure of wavelet basis. That one
element is called the wavelet. Thus Haar system is regarded as a simplest example of a wavelet basis. 'h' is known as Haar Wavelet.

One of the properties of the Haar wavelet is that it has compact support. It is the only known simple symmetrical wavelet with compact support. Also the simplest wavelet basis suitable for edge detection problems is the Haar basis [11,24]. Unfortunately Haar wavelet is not continuously differentiable which somewhat limits its applications like problems in differential equation.

Let \( \phi \) be the characteristic function of the unit interval \([0,1]\) and for each \( j \in \mathbb{Z} \), \( \phi_j(x) = \phi(x - j) \). Then the collection \( \{ \phi_j(x) \} \) is an orthonormal set in \( L^2(\mathbb{R}) \). Let \( V_0 \) be the closed linear span of \( \phi_j(x) \). Let

\[
h_{ij}(x) = 2^{ij} \cdot h(2^i x - j), \quad i, j \in \mathbb{Z}.\]

For each \( i \geq 0 \), let \( W_i \) be the closed linear span of \( \{ h_{ij}(x) \mid j \in \mathbb{Z}, \quad i \geq 0 \} \). Then it is known that

\[
L^2(\mathbb{R}) = V_0 \bigoplus \left\{ \bigoplus_{i=0}^{\infty} W_i \right\}.
\]

Hence the collection \( \{ \phi_j, h_{ij}, j \in \mathbb{Z}, \quad i \geq 0 \} \) is an orthonormal basis in \( L^2(\mathbb{R}) \). The analysis carried in \( L^2(\mathbb{R}) \) is using the above orthonormal basis. In the case of \( L^2(\mathbb{R}, \chi) \) and in \( L^2(0,1) \), the restriction of these functions are considered.

1.2.7 Weyl's Theorem[8]

Let \( A \) and \( B \) be the Hermitian matrices. Then

\[
\max_j \left| \lambda_{ij}(A) - \lambda_{ij}(B) \right| \leq \|A - B\|
\]

where \( \lambda_{ij}(A) \) and \( \lambda_{ij}(B) \) be the eigenvalues of \( A \) and \( B \) arranged in decreasing order.
1.2.8 Iterated Limit Theorem [7]

Let \((a_{mn})\) be the double sequence. Suppose that the single limits
\[y_m = \lim_{n} (a_{mn}), \quad z_n = \lim_{m} (a_{mn})\]
exist for all \(m, n \in \mathbb{N}\), and that the convergence of one of these collections is uniform. Then both iterated limits and the double limit exist and all three are equal.

We conclude this chapter by giving some of Kent E. Morrison’s work on Szegö’s Type theorem based on Walsh system. A brief sketch of Morrison’s work [17] is as follows:

In his paper he considered how well the eigenvalues of the matrices approximate the spectrum of the multiplication operator, which is the essential range of the multiplier. The choice of the orthonormal basis strongly affects the convergence. He considered the spectral convergence of multiplication operators acting on the \(L^2\) functions on an interval with respect to Fourier basis, Legendre basis and Walsh basis in the following sense.

(i) \(\Lambda_{n}(f) \to R(f)\) in \(H(C)\)
(ii) \(\mu_{n}(f) \to \phi \ast(m)\) weakly.

Where

- \(\Lambda_{n}(f)\) - The set of eigenvalues of \(\{T_{f}\}_{n}\)
- \(R(f)\) - Essential range of \(f\)
- \(H(C)\) - The Hausdorff space of \(C\)
- \(\mu_{n}(f)\) - The measure on \(C\) such that \(\mu_{n}(f) = \frac{\sum_{\lambda \in \Lambda_{n}} \delta_{\lambda}}{n}\) where \(\delta_{\lambda}\) is the Dirac delta measure concentrated at \(\lambda\).
- \(\phi \ast(m)\) - The measure defined in \(C\) such that
  \[\phi \ast(m) [F] = \frac{1}{b-a} \int_{a}^{b} F[f(x)] \, dx\]
  for any continuous function \(F\) on \(C\).
In the case of Legendre basis, Szegö proved the following theorem and another version of this is given in Morrison's paper.

1.2.9 Theorem [12]

Let $f$ be a real valued $L^\infty$ function on $[-1, 1]$. Then the sequence of spectral measures $\mu_n(f)$ converges weakly to the measure $\mu$ defined by

$$\mu(a, b) = \frac{1}{\pi} \left[ \phi(\cos^{-1} b) - \phi(\cos^{-1} a) \right]$$

In the case of Walsh basis, Morrison proved the following theorem.

1.2.10 Theorem

Let $f(x) = \sum_{i=0}^{k} c_i \psi_i(x)$ with $k$ less than $2^m$ where $\psi_i$ is the Walsh functions for $i \geq 0$. Then

(i) $\mu_n(f)$ converges weakly to $\phi \ast (dx)$ .

(ii) For $n = 2^m$ and $m$ sufficiently large, $\Lambda_n(f) = R(f)$ .

In this thesis, theorems 3.2.1 and 3.2.2 are analogous to the above mentioned theorem, with Haar system as the underlined basis.

1.3 Notations that are frequently used

- $T_f$ - Multiplication Operator with multiplier $f$.
- $(T)$ - The matrix of a bounded linear operator on a Hilbert space with respect to a chosen base.
- $(T)_N$ - The $N \times N$ corner truncation of $(T)$.
- $P_N$ - Orthogonal projection of $L^2$ space to span of first $n$ basis elements.
- $T_N$ - $P_N TP_N$