Chapter 1

Introduction

Nonlinear dynamics is one of the branches of science in which extensive research activities have been done in the last few decades [1]. Basically, this is an interdisciplinary area of science, which deals with the study of the systems described by nonlinear mathematical equations. Since most of the natural and engineering systems are nonlinear, the study of nonlinear dynamics has fundamental importance in science and technology. Nonlinear systems show a rich variety of phenomena such as self sustained oscillations [2], multistability [2, 4, 5], quasiperiodicity [6], pattern formation [7] and chaos [8]. A proper understanding of such effects is helpful in their control in technology, where such effects may cause unwanted results. On the other hand, the nonlinearities have been found sometimes to be applicable in technology and they can be applied for practical purposes such as chaotic cryptography [15] and soliton-based optical communications [10].

1.1 Chaos: The fundamental concepts

Chaos is one of the most widely discussed and fascinating phenomena in nonlinear dynamics. Literally, the word ‘chaos’ means the total disorder or utter confusion. However, in nonlinear dynamics, it has a different meaning. The unpredictable and complex evolution of deterministic systems is commonly referred to as chaos. The randomness associated with a chaotic system comes from the intrinsic dynamics of the system and it is entirely different from the randomness one encounters in systems with the stochastic external forces. The fundamental characteristic of a chaotic system is its extreme sensitivity to the initial conditions, i.e., the phase space trajectories started with slightly different initial conditions will diverge exponentially and they will become totally uncorrelated after a finite time. Thus a very small variation in the initial condition produces an infinitely large effect on the long term behavior of the system. For the same reason, the long term prediction of a chaotic
system is practically impossible.

The first numerical evidence of chaos was given by Edward Lorenz in 1963 [11]. He observed certain non-repeating solutions while simulating a truncated version of atmospheric convection. However, the history of chaos theory starts from the time of the renowned French mathematician Henry Poincaré. There was a belief that the complete evolution of a physical system can be predicted if its dynamical equations and the corresponding initial conditions are given. However, no physical quantity can be measured with infinite precision and there will be certain amount of error associated with each dynamical variable. The predictability of the dynamical system definitely depends on the evolution of this error in computations. Poincaré pointed out that, the errors will grow exponentially in certain nonlinear systems and hence the long term prediction of such system becomes practically impossible. This issue is currently known as the sensitive dependence on initial conditions and it is the fundamental property of chaotic systems. Even though Poincaré had predicted the possibility of chaotic solutions, it has taken a long time for the discovery of chaos in many of the physical systems. One of the reasons for this delay was the lack of sophisticated computational systems. After Lorenz’ discovery of chaos in the convection model, chaos has been observed in many nonlinear systems such as lasers [31], population models [13] and electronic circuits [14]. The realm of chaos has now been extended to the diverse branches of science such as chemistry [15] and biology [16]. The field of applications of chaos theory include the study of turbulence [17], pattern formation [18], secure communication [19], EEG data analysis [20, 21] and the study of ECG signals of arrhythmic heart [22].

A remarkable development of chaos theory is the universality of chaotic systems established by Feigenbaum [21]. He showed that there are certain universal constants associated with the transitions of the systems to chaos irrespective of the details of the systems. Further, there are certain universal categories of chaotic systems and most of the fundamental characteristics of the chaotic systems belonging to each category does not vary within them.

Chaos is common even in very simple deterministic systems. Very simple dynamical models of nonlinear systems have been found to behave chaotically. For a continuous dynamical system, the necessary number of degree of freedom for observing chaos is 3 or more. If it is a non-invertible discrete mapping (a dynamical system represented by discrete-time difference equations), there is no such restriction. For example, one dimensional maps such as logistic map and tent map are known to exhibit chaotic evolutions. The mechanism behind the complex nature of chaotic trajectories is the so-called stretching and folding of the trajectories in the phase space. The trajectories may experience stretching and folding in different directions and the exponential sensitivity is an outcome of these effects. The
sensitive dependence on initial conditions is quantitatively described by the logarithmic average of divergence of the trajectories which are commonly known as Lyapunov exponents. There are $N$ Lyapunov exponents associated with an $N$-dimensional deterministic system. A system is said to be chaotic if at least one of these exponents is positive.

Both the conservative and dissipative physical systems show chaos under various conditions. Hamiltonian systems are certain conservative systems, the dynamical behavior of which can be completely described by the so-called Hamiltonian function [24]. Kolmogorov Arnold Moser (KAM) theorem is known to be a paradigm for Hamiltonian chaos [25]. Plasma physics [26], mixing of fluids [27] and celestial mechanics [28] are some of the fields wherein the concepts of Hamiltonian Chaos can be applied. The fundamental issues in physics such as ergodicity have been discussed within the framework of Hamiltonian chaos [29]. Most of the natural and engineering systems are dissipative. The phase space trajectories of the dissipative systems asymptotically approaches some limit sets called attractors. The attractors associated with chaotic systems have non-integer dimensions and they are known as strange attractors. It should be noted that the Hamiltonian chaotic systems do not have attractors because of the phase-volume preservation.

1.2 Necessary computational tools

Dynamical aspects of the nonlinear systems are usually investigated with a number of numerical techniques. In this section, we give a brief account of the computational techniques which are used for characterizing different dynamical states of semiconductor lasers and the nonlinear oscillators.

1.2.1 Poincaré section

Poincaré section is a method used to construct a discrete mapping of a deterministic dynamical system that is originally described by a system of nonlinear differential equations [24]. For example consider a three dimensional flow described by a system of autonomous differential equations. We consider a two dimensional surface in the three dimensional phase space and mark every crossing of the trajectories in the same direction. The points obtained by this method constitute the Poincaré section. Since the system is deterministic, there will be certain definite relation (mapping) between the points obtained by two successive crossings i.e, $P_{n+1} = f(P_n)$, where $f$ is a nonlinear function. Hence the three dimensional flow is reduced to a two dimensional map. Fig.1.1 shows the construction of Poincaré section for a three dimensional flow. Similarly, for obtaining the Poincaré section of an $N$ dimensional continuous dynamical system we should take the crossings of the trajectories on an $N - 1$
dimensional hyperplane of the $N$-dimensional phase space. The dimensionality reduction is an additional advantage since it is very difficult to visualize the attractors in higher dimensions. The dimension of the attractor can further be reduced by taking first return map $(X_{n+1} \text{ vs } X_n)$ of just one variable of the section obtained. This type of maps can be used in bifurcation diagrams.

1.2.2 Bifurcation diagrams

Bifurcation is the event in which the qualitative properties of attractor of a dynamical system is changed as a control parameter of the system is varied. In the bifurcation phenomena, attractor may appear, disappear or an attractor is replaced by another one. Bifurcation diagrams helps us to visualize these transitions. They are the plot of the attractor points versus the control parameter. Period doubling in logistic map is a good example for bifurcations [8]. In each period doubling of the map, a period $n$ orbit becomes unstable and a stable period $n + 1$ orbit appears. After an infinite number of period doublings, the map becomes chaotic. Actually a number of continuous and discrete systems follow the period doubling route to chaos. For plotting the bifurcation of continuous dynamical systems, a set of values of a single variable representing the attractor must be obtained. This is usually done by the return map obtained from the Poincaré section [21]. There is another method for obtaining discrete mappings from the flows. Lorenz has constructed a one dimensional
map from the three dimensional flow \((X, Y, Z)\) by taking consecutive maxima of the single variable \(Z\) \[1\]. Such methods also can be used for plotting bifurcation diagrams.

### 1.2.3 Power spectrum

Fourier techniques are commonly used in nonlinear dynamics for characterizing periodic, quasiperiodic and chaotic states of a system. Fourier spectra give the power distribution of the observed signal as a function of frequencies. Consider a signal \(y(t)\) which has \(N\) discrete values sampled at intervals of \(\Delta t\). The discrete fourier transform of this signal is

\[
Y(f_k) = \sum_{l=0}^{N-1} y(t_l) e^{-2\pi ilk/N}, \quad k = 0, 1, \ldots, N - 1, \quad i = \sqrt{-1}, \tag{1.2.1}
\]

where the discrete frequency components are given by

\[
f_k = k/(N \Delta t) \tag{1.2.2}
\]

and the coefficients

\[
t_l = l \Delta t \tag{1.2.3}
\]

The separation \(\Delta t\) of sampled points determines the maximum frequency component of \(y(t)\) and the total time span \(N \Delta t\) determines the minimum frequency. The squares of the Fourier coefficients given in the Eq. (1.2.1) is referred to as power spectrum.

Fast Fourier Transform (FFT) is a convenient algorithm for obtaining the power spectrum of a time series\[33\]. In this method, the number of numerical calculation are considerably reduced using the recurrence properties of the fourier series.

The expression given in Eq.(1.2.1) can be written as

\[
Y_k = \sum_{l=0}^{N-1} y_l W^{kl}, \tag{1.2.4}
\]

where, \(W = e^{-2\pi ilk/N}\)

There exist certain simple relations between the terms appearing in the series represented by Eq.(1.2.4) and hence the total number of steps needed for calculating power spectra reduces significantly. (For instance, suppose we have a time series of 8 samples, we require all the terms from \(W^0\) to \(W^{49}\). However the number of terms can be conveniently reduced to eight \(W\)'s from \(W^0\) to \(W^7\). Further more, \(W^7 = -W^3\), \(W^6 = -W^2\), \(W^5 = -W^1\) and \(W^4 = -W^0\).) The FFT algorithm utilize this opportunity to complete the computation of power spectra with in a relatively short time.
Chapter 1

The power spectrum of a time series gives us the information about its periodicity. If the system is periodic, the spectrum is peaked at a single point and at its higher harmonics. If it is quasi periodic, the peaks will be at all linear combinations of two (or three) fundamental incommensurate frequencies. If the system is chaotic, we will get broadband power spectra.

1.2.4 Lyapunov exponents

Lyapunov exponents are the widely accepted tools for characterizing chaotic and periodic states of a system. They quantify the exponential divergence of phase space trajectories of the systems which is the fundamental property of chaotic system by definition. Lyapunov exponents can also be used for determining the stability of periodic orbits of a dynamical system. In this section, we discuss briefly the definition and computational aspects of the Lyapunov exponents of the continuous and discrete dynamical systems.

Consider a continuous system represented by the following nonlinear differential equation

\[
\frac{dX(t)}{dt} = F(X(t)), \quad X \in \mathbb{R}^N,
\]

where \(F(X(t))\) is a nonlinear function of the vector \(X(t)\) representing the state variables, given by

\[
X(t) = \begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_N
\end{pmatrix}
\]

Assume that a solution \(X(t)\) of the equations exists for a particular initial condition \(X(0)\). This solution can be obtained using numerical integration of the above system of differential equations. Consider another trajectory starting from a slightly different point \(X(0) + \delta X(0)\). Let \(\delta X(t)\) be the vector representing the separation between the trajectories after a time \(t\).

Lyapunov exponent of the system (corresponding to these initial conditions) is defined as [8]

\[
\lambda(X(0), \delta X(0)) = \lim_{t \to \infty} \frac{1}{t} \log \frac{\|\delta X(t)\|}{\|\delta X(0)\|}
\]

The Lyapunov exponent defined by Eq.1.2.7 depends on the initial values \(X(0)\) and \(\delta X(0)\) and hence it is no longer an invariant measure of the attractor of the system. Totally,
there are $N$ Lyapunov exponents for an $N$ dimensional system and they are independent of the initial conditions chosen for calculations. Secondly, the long term integration of the trajectories does not assure the smallness of separation vectors. In order to overcome these issues, the calculation of Lyapunov exponents are usually done using a different technique where a reference trajectory is obtained by the integration of nonlinear equations and the evolution of small deviations from this trajectory (separation vectors) is determined by integrating a set of linearized equations of the corresponding nonlinear equations. The vectors are normalized frequently using the Gram Schmidt Orthogonolization (GSR) procedure. The eigen values obtained by this process are averaged throughout the attractor in order to find the Lyapunov exponents. The Wolf’s algorithm is one of the commonly used scheme for calculating the complete set of Lyapunov exponents of a system [34]. To find the Lyapunov exponents of a system described by the delay differential equations (Eg.,Laser with delayed feedback), the Farmer’s algorithm can be used [35]. In this algorithm, the infinite dimensional delay differential equations are approximated to finite dimensional maps for the convenience of computation.

1.3 Simple examples of chaotic systems

We illustrate the general characteristics of chaotic systems using two well known dynamical systems. The former is the logistic map, which is a discrete system derived from a model of insect population. The latter is the Lorenz system, which is a continuous model obtained by the Raleigh Benard thermal convection model. Even though these models are represented by very simple and deterministic mathematical equations, their solutions are very complex and unpredictable for certain regimes of parameters.

1.3.1 Logistic map

Logistic map is a simple mathematical model proposed by R. M. May for studying the yearly variations in population of an insect species living with limited resources [13]. The map is given by

$$X_{n+1} = f(X_n),$$

where $X_n$ represents the normalized population of insects in the $n^{th}$ generation and the nonlinear function $f(X) = \lambda X (1 - X)$. $\lambda$ is the control parameter depends on the reproduction rate of insects. The above function can be represented graphically by a parabola as shown in Fig.1.2 ($\lambda = 4$). The domain of the map is the interval $[0,1]$.

The dynamical behavior of the map depends on the control parameter $\lambda$ which can be varied from 0 to 4. For $\lambda > 4$ the values in the interval $[0,1]$ are not mapped into itself. Let
us consider the fixed points of the map and their stability. A point $X^*$ is said to be a fixed point of the map if it satisfy the condition

$$f(X^*) = X^*$$

(1.3.2)

Since the function is quadratic, there will be two fixed points for the map.

The first fixed point is

$$X^* = 0$$

(1.3.3)

and the second one is,

$$X^* = 1 - 1/\lambda$$

(1.3.4)

The map shows a sequence of period doubling bifurcations when the control parameter $\lambda$ is varied from 0 to 4. For all the value of $\lambda < 1$, the fixed point at 0 is stable and attracting the trajectories originating from the domain of the map. If $\lambda$ is increased beyond 1, the attractor at 0 becomes unstable and the other fixed pint becomes stable. This fixed point becomes unstable at the value $\lambda = 3$ and a stable period 2 cycle is formed. This is the first period doubling bifurcation of the map. On increasing $\lambda$ further, a sequence of period doubling bifurcations take place and finally the map becomes chaotic for a limiting value $\lambda = \lambda_\infty = 3.57...$. The bifurcation diagram of the logistic map is shown in Fig.1.3. The parameter regime ranging from $\lambda_\infty$ to 4 is mainly chaotic domain where certain periodic windows are also present.
Feigenbaum has shown that there exist certain universal constants associated with the bifurcation phenomena [21]. It has been found that these constants apply not only to the logistic map, but to many of the chaotic systems following a period doubling route [8].

1.4 Lorenz system: A typical chaotic flow

Lorenz model is a well known example for continuous dissipative dynamical system showing chaos. The model was derived by E. N. Lorenz in 1963 [11]. It is a highly simplified model of Raleigh - Benard convection [36]. The original model considers the convection of a fluid contained between two rigid plates placed horizontally and kept with a temperature difference in between them. The importance of the Lorenz model was not in quantitatively describing the convection, but it illustrates how a simple deterministic model can show a rich variety of complex phenomena depending on the values of parameters. The model is described by a system of three nonlinear differential equations, which is a truncated version of Navier-Stokes equations

\[
\begin{align*}
\frac{dX}{dt} &= -\sigma(X - Y) \\
\frac{dY}{dt} &= -XZ + rX - Y \\
\frac{dZ}{dt} &= XY - bZ,
\end{align*}
\]

(1.4.1)
where $\sigma$, $r$ and $b$ are dimensionless parameters. The variable $X$ is a quantity proportional to the circulatory fluid flow velocity, $Y$ represents the temperature difference and $Z$ is the deviation of the vertical temperature profile from linearity. Lorenz numerically studied the case for $\sigma = 10$, $b = 8/3$ and $r = 28$. The system has two fixed points where $\frac{dX}{dt} = \frac{dY}{dt} = \frac{dZ}{dt} = 0$. For the parameters specified above, these fixed points are unstable. The trajectories near to a fixed point spirals outward then switches to spiraling outward from the other fixed point. The projection of Lorenz attractor in the $Y - Z$ plane is shown in Fig.1.4(a). The patterns repeat forever and the jump from one wing to other is in an erratic manner. Lorenz obtained a sequence $m_n$ by giving the $n_{th}$ maxima of the variable $Z(t)$ and plotted $m_{n+1}$ versus $m_n$. The plot is shown in Fig.1.4(b). It is an approximate one-dimensional map and resembles the tent map. The magnitude of the slope of the plot is always greater than unity indicating that all the points are unstable similar to the case of tent map.

![Figure 1.4: Lorenz attractor: (a) projection in the $X - Z$ plane (b) return map obtained from the maxima of $Z(t)$](image)

### 1.5 Attractors and dimensions

Attractors are the limit sets of points to which the phase space trajectories of different dynamical systems converge asymptotically. These attractors are usually characterized by their dimensions. The important dimensions are the capacity dimension, information dimension and the correlation dimension. The capacity dimension or the fractal dimension is related to the scaling properties of the attractor [37]. The information dimension quantify the extra information required to specify an initial condition on the attractor [38]. The
correlation dimension[39] is useful for describing the local inhomogeneity of the attractor i.e., the small scale variations of density of fractal objects over small scales.

A fixed point is considered as a zero dimensional attractor. The dimension of the periodic limit cycles is 1. A bi-periodic torus is the attractor of a two-frequency quasiperiodic system and it is having a dimension 2. The attractors having non-integer dimensions are called strange attractors and they are often associated with the chaotic states.

1.6 Different routes to chaos

In section 1.3, we have discussed about the period doubling route to chaos. Besides the logistic map many dissipative systems follow the period doubling scenario[14, 31]. For a continuous dissipative system, usually there will be a stable fixed point (dimension 0) which bifurcates in to a limit cycle (dimension 1) when varying the control parameter. This process is called a Hopf bifurcation [40]. This period 1 cycle then bifurcates into a period 2 cycle. This period doubling process continues as the control parameter is again varied and the system finally reaches at a chaotic state. The other important route is the quasi periodic route to chaos [41]. The Hopf bifurcation is followed by a transition of singly periodic limit cycle into a doubly periodic torus (dimension 2) and this torus bifurcates into a chaotic attractor having fractal dimension. In certain systems, the three frequency quasi periodicity route has also been reported [42]. The third route is the intermittency route which has no relevance to our work and it is not discussed here. A schematic description of period doubling and quasi periodic routes are given in the Fig.1.5.

Figure 1.5: Two main routes to chaos: (a) period doubling route (b) quasiperiodicity route
Chapter 1

1.7 Chaos in laser systems

Lasers are good examples of nonlinear systems which show many complex phenomena. Actually, lasers are the earlier experimental systems wherein chaos has been observed. Haken [43] formulated Lorenz like model of lasers by applying the Rotating Wave Approximation (RWA) and the Slowly Varying Envelope Approximation (SVEPA) to Laser equations. However the first experimental observation of chaos in a quantum - optical molecular system was reported by Arecchi et. al. in 1982 [31]. They observed subharmonic bifurcations, chaos and multistability in a Q - Switched CO$_2$ laser modulated by an electro optical modulator.

Later, chaos has been reported in laser systems such as single mode inhomogeneously broadened xenon laser [44], NdP$_5$O$_{14}$ tunable laser with modulated pump [45], NH$_3$ laser [47], Nd:YAG [48] and semiconductor lasers. The chaotic behavior observed in the far infrared NH$_3$ lasers are good example of Lorenz type chaos in lasers [49].

Chaos and other instabilities in the semiconductor lasers are particularly important since such lasers are widely used in optical communications and optical data processing. Semiconductor lasers show chaos under various physical conditions such as high speed modulation [50, 51, 20, 53], external optical injection [54] and optical or optoelectronic feedback [55, 56]. The feedback systems are usually delay systems and hence infinite dimensional. The high dimensional chaotic attractors shown by the external cavity laser diode have been widely studied [57, 58, 59].

In addition to the fundamental importance in nonlinear dynamics, study of chaotic laser systems have good practical importance. Study of chaotic dynamics in lasers help us to control chaos in laser systems and ensure their stable operation. On the other hand, chaos in lasers has an important application in technology. Synchronization of various types of chaotic laser systems have been successfully used for the optical data encryption. Hence, the study of nonlinear dynamics of lasers is helpful in the generation of chaos also.

1.8 Control of chaos

Since most of the engineering systems are expected to be operated in a steady or stable periodic state, chaos is an unwanted phenomenon in these systems. Hence, a number of methods have been developed for controlling chaos [14]. Most of these schemes are based on the fact that an infinite number of unstable periodic orbits (UPO) are embedded with in the chaotic attractors. These unstable orbits can be stabilized by applying small perturbations to an accessible system-parameter or a state variable. In this section some important
methods those are employed for controlling chaos are briefly discussed.

1.8.1 Unstable periodic orbits

It is well known that an infinite number of unstable periodic orbits (UPO) are embedded within the chaotic attractor. The presence of the UPOs can be easily understood by the bifurcation sequence through which the chaos is developed in the systems [61]. For example, on every period doubling bifurcation, one period $n$ orbit becomes unstable and a stable period $2n$ orbit is formed in the place of them, where $n$ is any integer. The system becomes chaotic through an infinite number of such bifurcations and that much UPOs are formed on the attractor. The presence of UPOs can be determined by time delay reconstruction of a chaotic system [14]. Many characteristic features of the attractors such as fractal dimensions and entropy can be extracted from the UPOs [62].

1.8.2 Ott, Grebogy and Yorke (OGY) method

The first attempt to control chaos by stabilising UPOs was done by Ott, Grebogy and Yorke [6] in 1990. They proposed that the UPOs of a chaotic system can be stabilized by applying small discrete perturbations to an accessible and suitable parameter. The UPOs can be located by time delay reconstruction of the considered system. This is possible even if we don’t know the model equations of the system. The second step is to choose the specific periodic orbits to be stabilized. The eigen values of the unstable fixed points in the Poincaré section is determined from the reconstructed time series of the system. The perturbation required to stabilize the particular UPO can be obtained from these eigen values. The major advantage of this control scheme is that no detailed information of the dynamical model of the system is required to implement it. Ditto et. al. have successfully applied the OGY method for a magneto elastic ribbon working in chaotic domain [7]. Roy and coworkers used a modified version of OGY method for controlling a chaotic Q-switched Nd YAG laser [15]. Even though it was successful in controlling chaos, it had two main disadvantages; 1) The implementation of the control scheme requires time delay reconstruction of the measured time series and hence computer assistance is needed for OGY method. 2) The controlled systems do not have good tolerance to the presence of noise. The effect of noise causes intermittent bursts in the output. In spite of these difficulties, OGY method attracted a wide attention and was implemented in a number of practical systems.
1.8.3 Targeting

The first targeting algorithm was proposed by Huberman and Lumer [8]. The method is based on an adaptive feedback aimed at bringing the chaotic systems to a desired state. The parameter would be adjusted in such a way that the system gives the required output. The desired state may be one of the periodic orbits or another chaotic state. Ramaswamy, Sinha and Rao have extended the targeting algorithm to multi parameter and higher dimensional systems [9]. They found that if a sudden perturbation is applied to the controlled system, the system will recover the controlled state after a short while and the recovery time is always proportional to the inverse of control stiffness.

1.8.4 Periodic parametric perturbation

It was theoretically shown by Lima and Pettini that a small, periodic perturbation to a parameter of a chaotic system may suppress chaos [10]. They applied a periodic perturbation to the amplitude of the cubic term in the Duffing-Holmes equation describing the chaotic oscillator and found that the regular periodic behavior was achieved by perturbations of small strength. Colet and Braiman have shown that chaos in a multimode solid state laser could be controlled by using periodic parametric perturbations [11].

1.8.5 Delay feedback control

K. Pyragas introduced a control scheme for chaotic systems [26, 27]. It was based on the synchronization of unstable periodic orbits to their past states. For controlling a particular UPO, a continuous feedback is applied to an accessible state variable of the system with a time delay equal to the period of the specific periodic orbit. Consider a chaotic system described by the ordinary differential equations,

\[ \dot{X} = F(X) \]  \hspace{1cm} (1.8.1)

Where \( X = (X_1, X_2, \ldots X_N)^T \) represents the state variables of the \( N \) dimensional phase space. Let us assume that a state variable \( X_k \) can be measured. For controlling a UPO of period \( \tau \), a feedback proportional to the difference between the value of \( X_k \) delayed by \( \tau \) units of time and the current value of \( X_k \) is applied to the system.

The modified dynamical equation of the system is given by,

\[ \dot{X} = F(X) + C[X_k(t - \tau) - X_k(t)] \] \hspace{1cm} (1.8.2)

where \( C \) is the feedback strength. If the control is achieved the feedback term will vanish and the system becomes periodic. The delay feedback control has many advantages
over earlier methods. It does not require any computation since no delay reconstruction is needed for implementing the control. No external signal is required to apply the perturbation. In contrast to the OGY method, the Pyragas method is very much robust to noise. The delay feedback control scheme is currently known as the Time Delay Auto Synchronization (TDAS). It has been successfully applied for controlling chaos in several nonlinear systems such as electronic circuits [72], glow discharge [24], magneto-elastic ribbon [74], and periodically driven yttrium iron garnet film [75]. Recently, Arecchi et al. have employed this technique for stabilizing high period orbits in a $CO_2$ laser [76]. The TDAS scheme also has some limitations. It is difficult to control high period orbits and the period orbits of the attractors with large positive Lyapunov exponents. Multistability of controlled orbits is another problem [26]. In spite of these limitations, delay feedback is known to be one of the most efficient and simple methods for controlling chaos.

1.9 Synchronization of chaos

Synchronization of chaos is a novel area of research in nonlinear dynamics which has emerged in 1980s. Chaotic systems are known to show extreme sensitivity to the initial conditions. The phase space trajectories of two identical chaotic systems diverge exponentially and they will become totally uncorrelated after a finite time. Hence it is impossible to construct two independent chaotic systems with the same temporal evolution. However, certain techniques have been developed for synchronizing chaotic systems. Yamada and Fujisaka have shown that two identical chaotic systems are synchronized when they are coupled together by sending information between them [1]. Afraimovich, Verichev and Rabinovich have studied the features of synchronized chaos in detail [2]. In 1990, Pecora and Caroll introduced a new synchronization scheme based on the complete replacement a variable of one of the two identical subsystems (response) by the corresponding variable of the other subsystem (drive) for synchronizing chaotic systems [3]. This method has been shown to be efficient in synchronizing many types of analogue electronic circuits [80]. However, coupling is commonly used for synchronizing other types of chaotic systems including the chaotic lasers operating in very high frequency regime.

1.9.1 Coupling of chaotic systems

Consider two identical chaotic systems described by the following differential equations
\[
\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}) \quad (1.9.1)
\]
\[
\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}) \quad (1.9.2)
\]

where, \( \mathbf{X} \) and \( \mathbf{Y} \) represent the vectors representing state variables of two systems given as

\[
\mathbf{X}(t) = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{pmatrix} \quad (1.9.3)
\]

and

\[
\mathbf{Y}(t) = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{pmatrix} \quad (1.9.4)
\]

and

\[
\mathbf{F}(\mathbf{X}) = \begin{pmatrix} f_1(X_1, X_2, \ldots, X_N) \\ f_2(X_1, X_2, \ldots, X_N) \\ \vdots \\ f_N(X_1, X_2, \ldots, X_N) \end{pmatrix} \quad (1.9.5)
\]

where, \( f_1, f_2, \ldots, f_N \) are some nonlinear functions.

Coupling of these systems can be done by sending the variables of each individual system to the other and applying feedbacks proportional to the difference of similar variables of the systems. A general expression for the dynamical equations for a coupled system is

\[
\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}) + \mathbf{C}_X(\mathbf{Y} - \mathbf{X})
\]
\[
\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}) + \mathbf{C}_Y(\mathbf{X} - \mathbf{Y}), \quad (1.9.6)
\]
where

\[ C_X = [C_{X1}, C_{X2}, ... C_{XN}]^T \]  \hspace{1cm} (1.9.7) 
\[ C_Y = [C_{Y1}, C_{Y2}, ... C_{YN}]^T \]  \hspace{1cm} (1.9.8)

are matrices representing the strength of perturbations applied to the first and second systems respectively. The coupling described by Eq.1.9.6 involves the measurement of all the variables of individual systems and the feedback is assumed to be given to all the variables. This is difficult to implement in practice. Usually, a single variable is measured and the feedback is applied to anyone of the accessible variables of the systems.

Figure 1.6: Schematic diagram of the coupling schemes (a) bidirectional and (b) unidirectional coupling.

The coupling scheme may be bidirectional or unidirectional depending on the specific situations. In bidirectional coupling, the measured signal corresponding to one of the variables are sent mutually in between the system. The schematic diagram of such a coupling scheme is given in Fig.1.6 (a). In the unidirectional method, the signal is sent in one direction only and all the coefficients of one of the matrices (say \( C_X \)) are assumed to be zero. The schematic diagram of the unidirectional scheme is given in the Fig. 1.6(b). This
coupling scheme is widely used for developing the secure communication schemes using synchronization of chaos.

### 1.9.2 Pecora and Caroll method: The replacement synchronization

This method was proposed by Pecora and Caroll in 1990 [3]. The basic criteria of this scheme is to construct two identical subsystems (drive and response) of a chaotic system and replacing one of the variables of the response completely with the corresponding variable of the drive.

Consider a chaotic system described by an $m$ dimensional state vector

$$ \mathbf{W}(t) = \begin{pmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{pmatrix} $$

(1.9.9)

where $\mathbf{X}$ and $\mathbf{Y}$ are the subsystems of $\mathbf{W}$ and they are having dimensions $m_1$ and $m_2$ ($m_1 + m_2 = m$). The evolution of the system $\mathbf{W}$ can be described by

$$ \frac{d\mathbf{W}}{dt} = \mathbf{F}(\mathbf{W}) $$

(1.9.10)

where the nonlinear function,

$$ \mathbf{F}(\mathbf{W}) = \begin{pmatrix} \mathbf{G}(\mathbf{X}, \mathbf{Y}) \\ \mathbf{H}(\mathbf{X}, \mathbf{Y}) \end{pmatrix} $$

(1.9.11)

Thus the $m$ dimensional system can be decomposed into two subsystems

$$ \frac{d\mathbf{X}}{dt} = \mathbf{G}(\mathbf{X}, \mathbf{Y}) $$

(1.9.12)

$$ \frac{d\mathbf{Y}}{dt} = \mathbf{H}(\mathbf{X}, \mathbf{Y}) $$

(1.9.13)

The next step is to construct another subsystem in such a way that its dynamics can be described as

$$ \frac{d\hat{\mathbf{Y}}}{dt} = \mathbf{H}(\mathbf{X}, \hat{\mathbf{Y}}) $$

(1.9.14)

This system is identical to the subsystems $\mathbf{Y}$ and it is called the driven replica subsystem. The schematic diagram of the replacement synchronization is given in Fig.1.7. One of the variable of the response sub system is completely replaced by the time series $\hat{\mathbf{Y}}$ received from the drive system. The subsystems $\mathbf{Y}$ and $\hat{\mathbf{Y}}$ are said to be synchronized if

$$ \lim_{t \to \infty} |\mathbf{Y}(t) - \hat{\mathbf{Y}}| = 0 $$

(1.9.15)

The synchronization of subsystems described by the above equations can be realized by analogue electronic circuits and it is one of the practical ways of chaotic data encryption.
1.9.3 Stability of the Synchronized State

A coupled system can be considered as a higher dimensional dynamical system. When the systems are synchronized, the phase space trajectories of the combined system are confined to a low dimensional hyperplane called synchronization manifold. The synchronization manifold of the coupled system described by Eq.1.9.6 is the hyperplane \( X = Y \). The stability of synchronized state can be determined by calculating the so-called Transverse Lyapunov Exponents (TLE) introduced by Pecora and Caroll [3]. They are the Lyapunov Exponents in the direction normal to the synchronization manifold. The synchronization is said to be stable if all the TLEs are negative.

1.9.4 Perfect synchronization and partial synchronization

Two chaotic systems are said to be perfectly synchronized if the variables of one system exactly coincide with the corresponding variables of the other [80]. There are many factors such as parameter mismatches, noise and improper coupling which may lead to the loss of perfect synchronization. Since these factors are common in real physical systems, perfect synchronization is only an ideal case and we can achieve only a practical or almost synchronization in coupled chaotic systems. If there is strong mismatch in parameters or asymmetry in coupling, different types of partial synchronization is obtained. The study of such phenomena is significant as much as the study of perfect synchronization. In the next sections, we discuss briefly the concept of well known phase and lag synchronization.
1.9.5 Phase synchronization

Phase synchronization is a weak entrainment between the weekly coupled chaotic systems [33]. We can define a phase of chaotic oscillations which is analogous to the phase angle of the periodic oscillations by various methods such as Hilbert transform. Two chaotic systems are said to be phase synchronized if there exists a constant relationship between the phases of the oscillators while their amplitudes are varying chaotically. Phase synchronization is analogous to the phase locking in coupled periodic systems. It is generally a natural phenomena while perfect synchronization is possible only in the laboratory.

1.9.6 Lag synchronization

This type of synchronization is observed in the mutually coupled chaotic oscillators when the coupling strength is increased to relatively higher values. In this case the individual systems would have almost the same chaotic evolution. However one of the system lags from the other by constant amount of time. This phenomenon was first reported by Rosenblum, Pikovski and Kurths [33]. They have characterized the extent of synchronization using statistical measures such as the similarity function.

1.9.7 Generalized synchronization

It is the entrainment between two coupled non-identical chaotic oscillators [83]. Generalized synchronization is possible between two similar chaotic systems with large parameter mismatches or even between two chaotic systems belonging to entirely different classes. As the result of this entrainment, a new chaotic attractor is formed in which the evolution of the response system is uniquely determined by that of the drive system.

1.9.8 Secure communication using synchronization of chaos

Some novel secure communications schemes based on chaotic synchronization have been developed recently. The basic criteria behind these methods are given as follows. Two identical chaotic systems are synchronized by unidirectional coupling or replacement. The secure communication is possible between these systems. In one of the encryption schemes, the encrypting message is added with the output of the drive and this message can be recovered by taking the difference of the outputs of the drive and response [19] In the other method called Chaos Shift Keying (CSK), one of the parameter is slightly varied in proportion to the message and this variation produce the loss of synchronization. The encrypted signal can be recovered in the terms of synchronization errors [12].
1.10 Present work

Direct current modulation of semiconductor lasers has enormous applications in photonics technology. The modulation is usually done in the GHz frequency domain where the nonlinear effects such as subharmonic generation, quasiperiodicity and chaos are produced as a result of the nonlinear interaction of charge carriers and photons in the laser cavity. The main objective of our work is to theoretically study the nonlinear dynamics of high speed modulated lasers with a particular emphasis to the control and synchronization of chaos. We expect that the numerical investigations on the possible methods for controlling chaos in directly modulated lasers would be helpful for the practical implementation of such schemes. The delay produced by the external transit of the optical signal, phase mismatches and frequency detuning of the modulating signal etc. are some of the practical issues in synchronizing modulated laser diodes. We address these issues while considering the application of these laser systems in chaotic secure communications.

1.11 Conclusion

A general introduction to the chaos theory and its applications is presented in this chapter. Fundamental concepts of chaos are explained and illustrated using two well known dynamical systems- the logistic map and the Lorenz model. The numerical techniques necessary for the study of chaotic semiconductor lasers are presented. The different methods used for controlling chaos are described. The concept of synchronization and its application in secure communications are explained in brief. The motivation behind the present work is also discussed.
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