7.1. Rough Ideals in a Lattice

7.2. Rough Fuzzy Ideals and Homomorphism

7.3. Rough Intuitionistic Fuzzy Sets in a Lattice
The concept of rough sets proposed by Pawlak [56] is a powerful mathematical method for the study of incomplete or imprecise information. The usefulness of rough set theory has been demonstrated by some successful application in artificial intelligence. Equivalence relation is a key notion in Pawlak’s rough set model. The equivalence classes are employed to construct the lower and upper approximations.

Theories of rough sets and fuzzy sets are related but distinct and complementary theories. Integration of these two theories has been made in recent years. The fuzzy generalization of rough sets is a typical example. Dubois and Prade [30] proposed the notions of rough fuzzy sets and fuzzy rough sets.

Rough set theory has found practical application in many areas such as knowledge discovery, machine learning, data analysis etc. Also there is connection between rough sets and algebraic systems. Some authors for example Biswas and Nanda [21] introduced the notion of rough subgroups. Kuroki [48] introduced rough ideals in a semigroup, Kuroki and Wang [49] studied properties of lower and upper approximation with respect to normal subgroups. B Davaaz [28] introduced rough subring and ideals with respect to an ideal of a ring.

In this chapter we studied rough sets in lattice theory. We give the rough set approximation of a set with respect to an ideal of a lattice and introduced rough sublattice (ideals and prime ideals). Also some properties of the lower and upper approximations of a set in a lattice are studied. Moreover we introduced the concepts of rough fuzzy sublattices, rough fuzzy ideal and prime ideal in a lattice and studied their properties. Also we discussed relationship between upper and lower rough ideals (prime ideals) and their homomorphic images. Lastly we give the rough approximation of an intuitionistic fuzzy set and introduced rough intuitionistic fuzzy sublattices, ideals etc. Also we defined intuitionistic fuzzy rough sets, intuitionistic fuzzy rough sublattices (ideals) and studied their properties.
7.1. Rough Ideals in a Lattice

In this section some properties of the lower and upper approximations of a set in a lattice with respect to ideals are studied. We know that in a distributive lattice $L$, for any ideal $I$ of $L$ we can find a congruence relation $C$ over $L$ defined by

$$\forall a, b \in L \text{ a } C \text{ b if } \exists x \in I \text{ such that } a \lor x = b \lor x .$$

Corresponding to this we have the congruence class $[x]_I = \{ y \in L / x Cy \}$.

So in this case we can use $(L, I)$ instead of approximation space $(U, \emptyset)$ (Refer Definition 2.3.1). From now onwards $L$ denotes a distributive lattice.

**Definition 7.1.1.** Let $X$ be any non-empty subset of $L$. Then the sets

$$\underline{\text{Apr}}_I(X) = \{ x \in L / [x]_I \subseteq X \},$$

$$\overline{\text{Apr}}_I(X) = \{ x \in L / [x]_I \cap X \neq \emptyset \}$$

are called the **lower and upper rough approximations** of the set $X$ with respect to the ideal $I$ of $L$.

The pair $\text{Apr}_I(X) = (\underline{\text{Apr}}_I(X), \overline{\text{Apr}}_I(X))$ is called a **rough set in the approximation space** $(L, I)$.

**Proposition 7.1.2.** For every approximation space $(L, I)$ and for every subsets $A$, $B \subseteq L$, we have

(i) $\underline{\text{Apr}}_I(A) \subseteq A \subseteq \overline{\text{Apr}}_I(A)$,

(ii) $\underline{\text{Apr}}_I(\emptyset) = \emptyset = \overline{\text{Apr}}_I(\emptyset)$,

(iii) $\underline{\text{Apr}}_I(L) = L = \overline{\text{Apr}}_I(L)$,

(iv) If $A \subseteq B$ then $\underline{\text{Apr}}_I(A) \subseteq \underline{\text{Apr}}_I(B)$ and $\overline{\text{Apr}}_I(A) \subseteq \overline{\text{Apr}}_I(B)$,

(v) $\underline{\text{Apr}}_I(\overline{\text{Apr}}_I(A)) = \underline{\text{Apr}}_I(A),$

(vi) $\overline{\text{Apr}}_I(\underline{\text{Apr}}_I(A)) = \overline{\text{Apr}}_I(A),$. 
(vii) \( \overline{\text{Apr}}_1(\text{Apr}_1(A)) = \text{Apr}_1(A) \).

(viii) \( \text{Apr}_1(\overline{\text{Apr}}_1(A)) = \text{Apr}_1(A) \).

(ix) \( \text{Apr}_1(A) = [\text{Apr}_1(A^c)]^c \).

(x) \( \text{Apr}_1(A) = [\text{Apr}_1(A^c)]^c \).

(xi) \( \text{Apr}_1(A \cap B) = \text{Apr}_1(A) \cap \text{Apr}_1(B) \).

(xii) \( \text{Apr}_1(A \cap B) \subseteq \text{Apr}_1(A) \cap \text{Apr}_1(B) \).

(xiii) \( \text{Apr}_1(A \cup B) \supseteq \text{Apr}_1(A) \cup \text{Apr}_1(B) \).

(xiv) \( \text{Apr}_1(A \cup B) = \text{Apr}_1(A) \cup \text{Apr}_1(B) \).

(xv) \( \text{Apr}_1([x]_I) = \text{Apr}_1([x]_I), \forall x \in L. \)

**Proof:** (i), (ii) and (iii) follows from the definition of \( \text{Apr}_1(X) \) and \( \text{Apr}_1(X) \).

(iv) Let \( x \in \text{Apr}_1(A) \). Then \( [x]_1 \subseteq A \). This implies \( [x]_1 \subseteq B \), since \( A \subseteq B \).

Hence \( x \in \text{Apr}_1(B) \). Thus \( \text{Apr}_1(A) \subseteq \text{Apr}_1(B) \). Next, if \( x \in \text{Apr}_1(A) \), then \( [x]_1 \cap A \neq \emptyset \) and hence \( [x]_1 \cap B \neq \emptyset \). Hence \( x \in \text{Apr}_1(B) \). Thus \( \text{Apr}_1(A) \subseteq \text{Apr}_1(B) \).

(v) \( \text{Apr}_1(\overline{\text{Apr}}_1(A)) = \{x \in L/ [x]_1 \subseteq \text{Apr}_1(A)\} \)

\[ = \{x \in L/ [x]_1 \subseteq A\}, \text{since } \text{Apr}_1(A) \subseteq A \]

\[ = \text{Apr}_1(A) \]

(vi) We have \( \text{Apr}_1(A) = \{x \in L/ [x]_1 \cap A \neq \emptyset\} \)

\[ = \{x \in L/ [x]_1 \cap \overline{\text{Apr}}_1(A) \neq \emptyset\}, \text{since } A \subseteq \overline{\text{Apr}}_1(A) \]

\[ = \text{Apr}_1(\overline{\text{Apr}}_1(A)). \]
(vii) Firstly, we show that \( \overline{\text{Apr}}_j (\text{Apr}_j (A)) \subseteq \text{Apr}_j (A). \) Let \( a \in \overline{\text{Apr}}_j [\text{Apr}_j (A)]. \) Then \( [a]_1 \cap \text{Apr}_j (A) \neq \emptyset. \) So that \( \exists x \in [a]_1 \) and \( x \in \text{Apr}_j (A). \) This implies \( [x]_1 \subseteq A \Rightarrow [a]_1 \subseteq A \Rightarrow a \in \text{Apr}_j (A). \) Thus
\[
\overline{\text{Apr}}_j (\text{Apr}_j (A)) \subseteq \text{Apr}_j (A). \] ………………………………….. (1)
Next, we show that \( \text{Apr}_j (A) \subseteq \text{Apr}_j (\text{Apr}_j (A)). \) Suppose, If possible \( \text{Apr}_j (A) \not\subseteq \text{Apr}_j (\text{Apr}_j (A)). \) Then \( \exists a \in \text{Apr}_j (A) \) such that \( a \notin \text{Apr}_j (\text{Apr}_j (A)). \) Now a \( \notin \text{Apr}_j (\text{Apr}_j (A)) \) implies \( [a]_1 \cap \text{Apr}_j (A) = \emptyset \) and hence \( a \notin \text{Apr}_j (A), \) which is a contradiction. Hence
\[
\text{Apr}_j (A) \subseteq \text{Apr}_j (\text{Apr}_j (A)) \] ………………………………….. (2)
From (1) and (2), \( \text{Apr}_j (\text{Apr}_j (A)) = \text{Apr}_j (A). \)

(viii) Firstly, we show that \( \overline{\text{Apr}}_j (\overline{\text{Apr}}_j (A)) \subseteq \text{Apr}_j (A). \) Let \( a \in \text{Apr}_j \overline{\text{Apr}}_j (A). \) Then \( [a]_1 \subseteq \text{Apr}_j (A). \) This implies \( a \in \text{Apr}_j (A). \) Thus
\[
\text{Apr}_j (\overline{\text{Apr}}_j (A)) \subseteq \text{Apr}_j (A) \] ………………………………….. (1)
Conversely, we show that \( \text{Apr}_j (A) \subseteq \text{Apr}_j (\overline{\text{Apr}}_j (A)). \) If possible, let \( \text{Apr}_j (A) \not\subseteq \text{Apr}_j (\overline{\text{Apr}}_j (A)). \) Then \( \exists a \in \text{Apr}_j (A) \) such that \( a \notin \text{Apr}_j (\overline{\text{Apr}}_j (A)). \) Hence there exist \( x \in [a]_1 \) such that \( x \notin \text{Apr}_j (A). \) This implies \( [x]_1 \cap A = \emptyset. \) Thus \( [a]_1 \cap A = \emptyset \) (since \( x \in [a]_1 \Rightarrow [x]_1 = [a]_1 \)) which is a contradiction. Hence
\[
\text{Apr}_j (A) \subseteq \text{Apr}_j (\overline{\text{Apr}}_j (A)) \] ………………………………….. (2)
From (1) and (2), \( \text{Apr}_j (\overline{\text{Apr}}_j (A)) = \text{Apr}_j (A). \)

(ix) We have \( \overline{\text{Apr}}_j (A^c)^c = \{ x \in [x] \cap A \neq \emptyset \}^c = \{ x \in [x] \cap A^c = \emptyset \} = \{ x \in [x] \subseteq A \} = \text{Apr}_j (A). \)
(x) We have $[Apr_j A]^c \subset \{x \in L | x \subseteq A^c \}^c = \{x \in L | x \cap A \neq \emptyset \} = \overline{Apr}_j (A)$

(xi) Let $x \in \overline{Apr}_j (A \cap B)$. Then $[x]_I \subseteq A \cap B$

$\iff [x]_I \subseteq A$ and $[x]_I \subseteq B$
$\iff x \in \overline{Ar}_j (A)$ and $x \in \overline{Ar}_j (B)$
$\iff x \in \overline{Ar}_j (A) \cap \overline{Ar}_j (B)$.

Hence $\overline{Apr}_j (A \cap B) = \overline{Ar}_j (A) \cap \overline{Ar}_j (B)$.

(xii) Let $x \in \overline{Ar}_j (A \cap B)$. Then $[x]_I \cap A \cap B \neq \emptyset$

$\Rightarrow [x]_I \cap A \neq \emptyset$ and $[x]_I \cap B \neq \emptyset$
$\Rightarrow x \in \overline{Ar}_j (A)$ and $x \in \overline{Ar}_j (B)$
$\Rightarrow x \in \overline{Ar}_j (A) \cap \overline{Ar}_j (B)$.

Thus $\overline{Ar}_j (A \cup B) \subseteq \overline{Ar}_j (A) \cup \overline{Ar}_j (B)$.

(xiii) Let $x \in \overline{Ar}_j (A) \cup \overline{Ar}_j (B)$. Then $x \in \overline{Ar}_j (A)$ or $x \in \overline{Ar}_j (B)$

$\Rightarrow [x]_I \subseteq A$ or $[x]_I \subseteq B \Rightarrow [x]_I \subseteq A \cup B$
$\Rightarrow x \in \overline{Ar}_j (A \cup B)$.

Hence $\overline{Ar}_j (A) \cup \overline{Ar}_j (B) \subseteq \overline{Ar}_j (A \cup B)$.

(xiv) Let $x \in \overline{Ar}_j (A \cup B)$. Then $[x]_I \cap (A \cup B) \neq \emptyset$

$\iff [x]_I \cap A \neq \emptyset$ or $[x]_I \cap B \neq \emptyset$
$\iff x \in \overline{Ar}_j (A)$ or $x \in \overline{Ar}_j (B)$
$\iff x \in \overline{Ar}_j (A) \cup \overline{Ar}_j (B)$.

Hence $\overline{Ar}_j (A \cup B) = \overline{Ar}_j (A) \cup \overline{Ar}_j (B)$.

(xv) Obviously, $\overline{Ar}_j [x]_I \subseteq \overline{Ar}_j [x]_I$. 
For the converse, let \( a \in \overline{Ap r}_I[x]_I \). Then \([a]_I \cap [x]_I \neq \phi \Rightarrow [a]_I = [x]_I \)

\[ \Rightarrow a \in \overline{Ap r}_I[x]_I \Rightarrow \overline{Ap r}_I[x]_I \subseteq \overline{Ap r}_I[x]_I \]

Hence \( \overline{Ap r}_I[x]_I = \overline{Ap r}_I[x]_I \)

The converse of proposition (xii) & (xiii) is not true, which is illustrated by the following example

**Example 7.1.3.** Let \( L = \{1, 2, 4, 5, 10, 20, 25, 50, 100\} \), the factors of 100, which forms a lattice under divisibility. Its Hasse Diagram is given below. Let \( I = \{1, 2, 5, 10\} \) be an ideal of \( L \).

![Hasse Diagram](image)

Define \( A = \{2, 10, 20, 100\} \) and \( B = \{1, 5, 10\} \). Then different equivalence classes are

\( [1]_I = \{1, 2, 5, 10\} \), \( [4]_I = \{4, 20\} \), \( [25]_I = \{25, 50\} \), and \( [100]_I = \{100\} \).

Then \( \overline{Ap r}_I(A) = \{100\} \), \( \overline{Ap r}_I(B) = \phi \) and \( \overline{Ap r}_I(A \cup B) = \{1, 2, 5, 10, 100\} \).

So that \( \overline{Ap r}_I(A \cup B) \not\subseteq \overline{Ap r}_I(A) \cup \overline{Ap r}_I(B) \).

Similarly, if we take the ideal \( I = \{1, 5, 25\} \), \( A = \{1, 2, 10, 50\} \) and \( B = \{2, 4, 10, 25, 50\} \). Then different equivalence classes are

\( [1]_I = \{1, 5, 25\} \), \( [2]_I = \{2, 10, 50\} \) and \( [4]_I = \{4, 20, 100\} \). Then
\( \overline{A_{\text{pr}}}(A) = \{1, 2, 5, 10, 25, 50\} \) and \( \overline{A_{\text{pr}}}(B) = \{1, 2, 4, 5, 10, 20, 25, 50, 100\} \).

But \( \overline{A_{\text{pr}}}(A \cap B) = \{2, 10, 50\} \). Therefore \( \overline{A_{\text{pr}}}(A \cap B) \supseteq \overline{A_{\text{pr}}}(A) \cap \overline{A_{\text{pr}}}(B) \).

**Corollary 7.1.4.** For every approximation space \((L, I)\)

(i) \( \overline{A_{\text{pr}}}(A) \) and \( \overline{A_{\text{pr}}}(A) \) are definable sets, \( \forall A \subseteq L \).

(ii) \([x]_I\) is a definable set, \( \forall x \in L \).

**Proof:** By proposition 7.1.2, we have

\[
\overline{A_{\text{pr}}}(\overline{A_{\text{pr}}}(A)) = \overline{A_{\text{pr}}}(A)
\]

\[
\overline{A_{\text{pr}}}(\overline{A_{\text{pr}}}(A)) = \overline{A_{\text{pr}}}(A) \text{ and } \overline{A_{\text{pr}}}[x]_I = \overline{A_{\text{pr}}}[x]_I.
\]

So \( \overline{A_{\text{pr}}}(A) \), \( \overline{A_{\text{pr}}}(A) \) and \([x]_I\) are definable sets.

**Definition 7.1.5.** Let \( A \) and \( B \) are two non-empty subsets of \( L \). Then we define

\( A \lor B = \{a \lor b/a \in A, b \in B\} \) and \( A \land B = \{a \land b/a \in A, b \in B\} \).

**Proposition 7.1.6.** Let \( I \) be an ideal of \( L \) and \( A \) and \( B \) be two non-empty subsets of \( L \).

Then \( \overline{A_{\text{pr}}}(A) \lor \overline{A_{\text{pr}}}(B) = \overline{A_{\text{pr}}}(A \lor B) \).

**Proof:** Let \( x \in \overline{A_{\text{pr}}}(A) \lor \overline{A_{\text{pr}}}(B) \). Then \( x = a \lor b \), where \( a \in \overline{A_{\text{pr}}}(A) \) and \( b \in \overline{A_{\text{pr}}}(B) \). This implies \([a]_I \cap A \neq \emptyset\) and \([b]_I \cap B \neq \emptyset\). So \( \exists c \in [a]_I \cap A \) and \( d \in [b]_I \cap B \). Hence \( c \lor d \in A \lor B \) and \( c \lor d \in [a]_I \lor [b]_I = [a \lor b]_I = [x]_I \).

Thus \( [x]_I \cap (A \lor B) \neq \emptyset \) \( \Rightarrow x \in \overline{A_{\text{pr}}}(A \lor B) \).

Hence \( \overline{A_{\text{pr}}}(A) \lor \overline{A_{\text{pr}}}(B) \subseteq \overline{A_{\text{pr}}}(A \lor B) \).

Conversely, let \( x \in \overline{A_{\text{pr}}}(A \lor B) \). Then \( [x]_I \cap (A \lor B) \neq \emptyset \). Let \( y \in [x]_I \cap (A \lor B) \).

Then \( y \in [x]_I \) and \( y \in (A \lor B) \). Hence \( y = a \lor b \), where \( a \in A, b \in B \). So that \( x \in [y]_I \)
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= [a \lor b]_I = [a]_I \lor [b]_I . This implies \( x = c \land d \), where \( c \in [a]_I \) and \( d \in [b]_I \) and hence \( a \in [c]_I \) and \( b \in [d]_I \).

Thus \( A \cap [c]_I \neq \emptyset \) and \( B \cap [d]_I \neq \emptyset \) \( \Rightarrow c \in \overline{A r} I (A) \) and \( d \in \overline{A r} I (B) \).

Thus \( x = c \land d \in \overline{A r} I (A) \lor \overline{A r} I (B) \). Hence \( \overline{A r} I (A \lor B) \subseteq \overline{A r} I (A) \lor \overline{A r} I (B) \).

Thus \( \overline{A r} I (A) \lor \overline{A r} I (B) = \overline{A r} I (A \lor B) \).

**Proposition 7.1.7.** Let \( I \) be an ideal of \( L \) and \( A \) and \( B \) be two non-empty subsets of \( L \). Then \( \overline{A r} I (A) \land \overline{A r} I (B) = \overline{A r} I (A \land B) \).

**Proof.** Let \( x \in \overline{A r} I (A) \land \overline{A r} I (B) \). Then \( x = a \land b \), where \( a \in \overline{A r} I (A) \) and \( b \in \overline{A r} I (B) \). Hence \( [a]_I \land A \neq \emptyset \) and \( [b]_I \land B \neq \emptyset \). This implies \( \exists c, d \in L \) such that \( c \in [a]_I \cap A \) and \( d \in [b]_I \cap B \). Hence \( c \land d \in [a]_I \land [b]_I = [a \land b]_I = [x]_I \) and \( c \land d \in A \land B \). Hence \( [x]_I \land (A \land B) \neq \emptyset \). This implies \( x \in \overline{A r} I (A \land B) \).

Thus \( \overline{A r} I (A \land B) \subseteq \overline{A r} I (A) \land \overline{A r} I (B) \). Hence \( \overline{A r} I (A) \land \overline{A r} I (B) = \overline{A r} I (A \land B) \).

Conversely let \( x \in \overline{A r} I (A \land B) \). Then \( [x]_I \cap (A \land B) \neq \emptyset \). This implies \( \exists y \in L \) such that \( y \in [x]_I \cap (A \land B) \Rightarrow y \in [x]_I \) and \( y \in A \land B \) and hence \( y = a \land b \), where \( a \in A, b \in B \).

Hence \( x \in [y]_I = [a \land b]_I = [a]_I \land [b]_I \). This implies \( x = c \land d \), where \( c \in [a]_I \) and \( d \in [b]_I \). Thus \( a \in [c]_I, b \in [d]_I \). So that \( [c]_I \cap A \neq \emptyset \) and \( [d]_I \cap B \neq \emptyset \). This implies \( c \in \overline{A r} I (A) \) and \( d \in \overline{A r} I (B) \Rightarrow x = c \land d \in \overline{A r} I (A) \land \overline{A r} I (B) \).

Thus \( \overline{A r} I (A \land B) \subseteq \overline{A r} I (A) \land \overline{A r} I (B) \). Hence \( \overline{A r} I (A) \land \overline{A r} I (B) = \overline{A r} I (A \land B) \).

**Proposition 7.1.8.** Let \( I \) be an ideal of \( L \) and \( A \) and \( B \) be two non-empty subsets of \( L \). Then \( \overline{A r} I (A) \land \overline{A r} I (B) \subseteq \overline{A r} I (A \land B) \).

**Proof.** Let \( x \in \overline{A r} I (A) \land \overline{A r} I (B) \). Then \( x = a \land b \), where \( a \in \overline{A r} I (A) \) and \( b \in \overline{A r} I (B) \). This implies \( x = a \land b \), where \( a \in \overline{A r} I (A) \) and \( b \in \overline{A r} I (B) \). Hence \( x \in \overline{A r} I (A \land B) \). Thus \( \overline{A r} I (A) \land \overline{A r} I (B) \subseteq \overline{A r} I (A \land B) \).
b ∈ \text{Apr}_I(B). Hence \([a]_I \subseteq A\) and \([b]_I \subseteq B\). Thus \([a]_I \land [b]_I \subseteq A \land B \Rightarrow [a \land b]_I \subseteq A \land B\). This implies \([x]_I \subseteq A \land B\) and hence \(x \in \text{Apr}_I(A \land B)\). Thus \(\text{Apr}_I(A) \land \text{Apr}_I(B) \subseteq \text{Apr}_I(A \land B)\).

**Proposition 7.1.9.** Let \(I\) be an ideal of \(L\) and \(A, B\) two non-empty subsets of \(L\). Then \(\text{Apr}_I(A) \lor \text{Apr}_I(B) \subseteq \text{Apr}_I(A \lor B)\).

**Proof:** Let \(x \in \text{Apr}_I(A) \lor \text{Apr}_I(B)\). Then \(x = a \lor b\), where \(a \in \text{Apr}_I(A)\) and \(b \in \text{Apr}_I(B)\). So that \([a]_I \subseteq A\) and \([b]_I \subseteq B\). Thus \([a]_I \lor [b]_I \subseteq A \lor B \Rightarrow [a \lor b]_I \subseteq A \lor B \Rightarrow [x]_I \subseteq A \lor B\) and hence \(x \in \text{Apr}_I(A \lor B)\). Thus \(\text{Apr}_I(A) \lor \text{Apr}_I(B) \subseteq \text{Apr}_I(A \lor B)\).

The following example shows that the reverse inclusion of Propositions 7.1.8 and 7.1.9 are not generally true.

**Example 7.1.10.** Consider the lattice given in Example 7.1.3. Let \(I = \{1, 2, 5, 10\}\) be the ideal of \(L\). Define \(A = \{2, 10, 20, 50, 100\}\) and \(B = \{1, 2, 4, 10, 20\}\). Then \(A \lor B = \{2, 4, 10, 20, 50, 100\}\). So that \(\text{Apr}_I(A) = \{100\}\), \(\text{Apr}_I(B) = \{4, 20\}\).

Then \(\text{Apr}_I(A) \lor \text{Apr}_I(B) = \{100\}\) and \(\text{Apr}_I(A \lor B) = \{4, 20, 100\}\). Therefore \(\text{Apr}_I(A) \lor \text{Apr}_I(B) \not\subseteq \text{Apr}_I(A \lor B)\).

In the same lattice, if we take the ideal \(I = \{1, 5, 25\}\), \(A = \{1, 2, 10, 50\}\) and \(B = \{2, 10, 5, 50\}\). Then \(\text{Apr}_I(A) = \{2, 10, 50\}\) and \(\text{Apr}_I(B) = \{2, 10, 50\}\). So that \(\text{Apr}_I(A) \land \text{Apr}_I(B) = \{2, 10, 50\}\). Also \(A \land B = \{1, 2, 5, 10, 25, 50\}\). So that \(\text{Apr}_I(A \land B) = \{1, 2, 5, 10, 25, 50\}\). Therefore \(\text{Apr}_I(A) \land \text{Apr}_I(B) \not\subseteq \text{Apr}_I(A \land B)\).
Lemma 7.1.11. Let \( I \) and \( J \) be two ideals of \( L \) such that \( I \subseteq J \) and let \( A \) be a non-empty subset of \( L \). Then

(i) \( \text{Apr}_J(A) \subseteq \text{Apr}_I(A) \)  

(ii) \( \overline{\text{Apr}_J(A)} \subseteq \overline{\text{Apr}_I(A)} \)

Proof: (i) Let \( x \in \text{Apr}_J(A) \). Then \([x]_J \subseteq A \Rightarrow [x]_I \subseteq A\), since \([x]_I \subseteq [x]_J\). Thus \( x \in \text{Apr}_I(A) \). Hence \( \text{Apr}_J(A) \subseteq \text{Apr}_I(A) \)

(ii) Let \( x \in \overline{\text{Apr}_I(A)} \). Then \([x]_I \cap A \neq \emptyset \Rightarrow [x]_J \cap A \neq \emptyset\), since \([x]_I \subseteq [x]_J\). Thus \( x \in \text{Apr}_I(A) \). Hence \( \overline{\text{Apr}_I(A)} \subseteq \overline{\text{Apr}_J(A)} \).

The following Corollary follows from Lemma 7.1.11.

Corollary 7.1.12. Let \( I, J \) be two ideals of \( L \) and \( A \) a non-empty subset of \( L \). Then

(i) \( \text{Apr}_J(A) \cap \text{Apr}_I(A) \subseteq \text{Apr}_{I \cap J}(A) \)

(ii) \( \overline{\text{Apr}_{I \cap J}(A)} \subseteq \overline{\text{Apr}_I(A)} \cap \overline{\text{Apr}_J(A)} \).

The following example shows that the reverse inclusion is not generally true in Corollary 7.1.12.

Example 7.1.13. Consider the lattice \( L \) given in Example 3.2. Let \( I = \{1, 2, 5, 10\} \) and \( J = \{1, 5, 25\} \) be two ideals of \( L \). Then \( I \cap J = \{1, 5\} \). Let \( A = \{1, 2, 4, 10, 20\} \).

Then \( \text{Apr}_I(A) = \{4, 20\} \) and \( \text{Apr}_J(A) = \emptyset \). Also \( \overline{\text{Apr}_{I \cap J}(A)} = \{2, 4, 10, 20\} \). So that \( \text{Apr}_I(A) \cap \text{Apr}_J(A) \) \( \not\subseteq \overline{\text{Apr}_{I \cap J}(A)} \).

Now, if we take \( A = \{2, 4, 10, 25\} \). Then \( \overline{\text{Apr}_I(A)} = \{1, 2, 4, 5, 10, 20, 25, 50\} \), \( \overline{\text{Apr}_J(A)} = \{1, 2, 4, 5, 10, 20, 25, 50, 100\} \) and \( \overline{\text{Apr}_{I \cap J}(A)} = \{2, 4, 10, 20, 25\} \). Therefore \( \overline{\text{Apr}_I(A)} \cap \overline{\text{Apr}_J(A)} \not\subseteq \overline{\text{Apr}_{I \cap J}(A)} \).
Proposition 7.1.14. If $I$ is an ideal and $J$ is a sublattice of $L$, then $\overline{\text{Apr}}_I(J)$ and $\overline{\text{Apr}}_J(I)$ are sublattices of $L$.

**Proof:** Let $a, b \in \overline{\text{Apr}}_I(J)$. Then $[a]_I \cap J \neq \emptyset$ and $y \in [b]_I \cap J \neq \emptyset$. This implies $\exists x, y \in L$ such that $x \in [a]_I \cap J$ and $y \in [b]_I \cap J$. Now $x \in J$ implies $x \lor y \in J$ and $x \land y \in J$, as $J$ is a sublattice of $L$. Also $x \in [a]_I$ and $y \in [b]_I$ implies $x \lor y \in [a]_I \lor [b]_I = [a \lor b]_I$, and $x \land y \in [a]_I \land [b]_I = [a \land b]_I$. Thus $[a \lor b]_I \cap J \neq \emptyset$ and $[a \land b]_I \cap J \neq \emptyset$.

Hence $a \lor b \in \overline{\text{Apr}}_I(J)$ and $a \land b \in \overline{\text{Apr}}_J(I)$. Thus $\overline{\text{Apr}}_I(J)$ is a sublattice of $L$.

Now, let $a, b \in \overline{\text{Apr}}_J(I)$. Then $[a]_I \subseteq J$ and $[b]_I \subseteq J \Rightarrow [a]_I \lor [b]_I \subseteq J$ and $[a]_I \land [b]_I \subseteq J$. That is $[a \lor b]_I \subseteq J$ and $[a \land b]_I \subseteq J$. Hence $a \lor b \in \overline{\text{Apr}}_J(I)$ and $a \land b \in \overline{\text{Apr}}_J(I)$. Thus $\overline{\text{Apr}}_J(I)$ is a sublattice of $L$.

Proposition 7.1.15. Let $I, J$ be two ideals of $L$, then $\overline{\text{Apr}}_I(J)$ and $\overline{\text{Apr}}_J(I)$ are ideals of $L$.

**Proof:** Let $a, b \in \overline{\text{Apr}}_I(J)$ and $l \in L$. Then $[a]_I \cap J \neq \emptyset$ and $[b]_I \cap J \neq \emptyset$. This implies $\exists x \in [a]_I \cap J$ and $y \in [b]_I \cap J$. So that $x \lor y \in J$, as $J$ is an ideal and $x \lor y \in [a]_I \lor [b]_I = [a \lor b]_I$. Hence $[a \lor b]_I \cap J \neq \emptyset$, so that $a \lor b \in \overline{\text{Apr}}_I(J)$. Also $x \in J$ and $l \in L$ implies $x \land l \in J$. Further $x \land l \in [a]_I \land [l]_I$. Therefore $x \land l \in [a \land l]_I \cap J$. Hence $[a \land l]_I \cap J \neq \emptyset$, which implies $a \land l \in \overline{\text{Apr}}_I(J)$. Hence $\overline{\text{Apr}}_I(J)$ is an ideal of $L$.

By proposition 7.1.14, $\overline{\text{Apr}}_J(I)$ is a sublattice of $L$. Next, let $a \in \overline{\text{Apr}}_J(I)$ and $l \in L$. Then $[a]_I \subseteq J$ which implies $[a \land l]_I \subseteq J$, otherwise, if $[a \land l]_I \not\subseteq J$, then $\exists x \in [a \land l]_I$ such that $x \not\in J$, so that $x \in [a]_I \land [l]_I$ with $x \not\in J$, hence $x = c \land d$, where $c \in [a]_I$ and $d \in [l]_I$, since $[a]_I \subseteq J$, we have $c \in J$, $d \in L$, thus $c \land d \in J$, by the definition of an ideal, that is $x \in J$, which is a contradiction. Thus $a \land l \in \overline{\text{Apr}}_J(I)$. Hence $\overline{\text{Apr}}_J(I)$ is an ideal of $L$. 
Definition 7.1.16. Let I be an ideal of L and $\text{Apr}_I(A) = (\text{Apr}_I(A), \overline{\text{Apr}_I(A)})$ a rough set in the approximation space $(L, I)$. If $\text{Apr}_I(A)$ and $\overline{\text{Apr}_I(A)}$ are sublattices (ideals) of L, then we call $\text{Apr}_I(A)$ a rough sublattice (rough ideal) of $L$.

Corrollary 7.1.17. (i) Let $I$ and $J$ be ideals of $L$. Then $\text{Apr}_I(J)$ and $\text{Apr}_J(I)$ are rough ideals of $L$. (ii) Let $I$ be an ideal and $J$ is a sublattice of $L$. Then $\text{Apr}_I(J)$ is a rough sublattice of $L$.

Proof: Follows from the above Propositions.

Definition 7.1.18. Let $I$ be an ideal of L and $A$ be a non-empty subset of L and $\text{Apr}_I(A) = (\text{Apr}_I(A), \overline{\text{Apr}_I(A)})$ is a rough set in the approximation space $(L, I)$. If $\text{Apr}_I(A)$ and $\overline{\text{Apr}_I(A)}$ are prime ideals of L, we call $\text{Apr}_I(A)$ a rough prime ideal.

Proposition 7.1.19. Let $I$ be an ideal and $J$ a prime ideal of $L$. Then $\overline{\text{Apr}_I(J)}$ and $\text{Apr}_I(J)$ are prime ideals of $L$.

Proof: By proposition 7.1.15, $\overline{\text{Apr}_I(J)}$ is an ideal of $L$. Next, we prove that it is a prime ideal, let $a \land b \in \overline{\text{Apr}_I(J)}$. Then $[a \land b] \cap J \neq \emptyset$. This implies $\exists x \in [a \land b]_I$ and $x \in J$. Now $x \in [a \land b] \Rightarrow x \in [a]_I \land [b]_I$. So that $x = x' \land y'$, where $x' \in [a]_I$ and $y' \in [b]_I$. Also $x' \land y' \in J$. Hence either $x' \in J$ or $y' \in J$, since $J$ is a prime ideal. Thus either $[a]_I \cap J \neq \emptyset$ or $[b]_I \cap J \neq \emptyset$. This implies $a \in \overline{\text{Apr}_I(J)}$ or $b \in \overline{\text{Apr}_I(J)}$. Thus $\overline{\text{Apr}_I(J)}$ is a prime ideal. By proposition 7.1.15, $\overline{\text{Apr}_I(J)}$ is an ideal of $L$. Next, we show that it is a prime ideal. If possible, suppose that $\overline{\text{Apr}_I(J)}$ is not a prime ideal. Then $\exists x, y, \in L$ such that $x \land y \in \overline{\text{Apr}_I(J)}$ but $x \notin \overline{\text{Apr}_I(J)}$ and $y \notin \overline{\text{Apr}_I(J)}$. Then $[x \land y] \subseteq J$ but $[x]_I \nsubseteq J$ and $[y]_I \nsubseteq J$. Hence $\exists x' \in [x]_I$ but $x' \notin J$ and $y' \in [y]_I$ but $y' \notin J$. Then $x' \land y' \in [x]_I \land [y]_I = [x \land y]_I \subseteq J$. 
Thus \( x' \land y' \in J \Rightarrow \) either \( x' \in J \) or \( y' \in J \), which is a contradiction. Hence \( \overline{Apr}_I(J) \) is a prime ideal.

**Corollary 7.1.20.** If \( I \) is an ideal and \( J \) a prime ideal of \( L \), then \( \overline{Apr}_I(J) \) is a rough prime ideal of \( L \).

**Proof:** Follow from Propositions 7.1.19.

Here, we express the lower and upper approximations of a non-empty set \( A \) of \( L \) in terms of the members of the quotient lattice \( L/I \). Let \( I \) be an ideal of \( L \) and \( A \) non-empty subset of \( L \). Then the lower and upper approximations of \( A \) can be presented in an equivalent form as follows:

\[
\overline{Apr}_I(A) = \{ [a]_I \in L/I \mid [a]_I \subseteq A \}, \quad \overline{Apr}_I(A) = \{ [a]_I \in L/I \mid [a]_I \cap A \neq \emptyset \}
\]

**Proposition 7.1.21.** Let \( I, J \) be two ideals of \( L \). Then \( \overline{Apr}_I(J) \) and \( \overline{Apr}_I(J) \) are ideals of \( L/I \).

**Proof:** Assume that \( [a]_I, [b]_I \in \overline{Apr}_I(J) \) and \( [r]_I \in L/I \). Then \( [a]_I \cap J \neq \emptyset \) and \( [b]_I \cap J \neq \emptyset \) ⇒ \( \exists x \in [a]_I \cap J \) and \( y \in [b]_I \cap J \).

Since \( J \) is an ideal of \( L \), we have \( x \lor y \in J \) and \( r \land x \in J \). Also, we have \( x \lor y \in [a]_I \lor [b]_I = [a \lor b]_I \) and \( r \land x \in [r]_I \cap [a]_I = [r \land a]_I \). Thus \( a \lor b]_I \cap J \neq \emptyset \) and \( [r \land a]_I \cap J \neq \emptyset \). Therefore \( [a \lor b]_I \in \overline{Apr}_I(J) \) and \( [r \land a]_I \in \overline{Apr}_I(J) \). That is \( [a]_I \lor [b]_I \in \overline{Apr}_I(J) \) and \( [r]_I \land [a]_I \in \overline{Apr}_I(J) \). Hence \( \overline{Apr}_I(J) \) is an ideal of \( L/I \).

Now let \( [a]_I, [b]_I \in \overline{Apr}_I(J) \) and \( [r]_I \in L/I \). Then \( [a]_I \subseteq J \) and \( [b]_I \subseteq J \). So that \( [a]_I \lor [b]_I \subseteq J \) and hence \( [a \lor b]_I \subseteq J \). Thus \( [a \lor b]_I \in \overline{Apr}_I(J) \). That is \( [a]_I \lor [b]_I \in \overline{Apr}_I(J) \). Also, since \( J \) is an ideal of \( L \), \( [r]_I \land [a]_I \subseteq J \) and hence \( [r \land a]_I \subseteq J \). Hence \( [r \land a]_I \in \overline{Apr}_I(J) \). Thus \( [r]_I \land [a]_I \in \overline{Apr}_I(J) \). Hence \( \overline{Apr}_I(J) \) is an ideal of \( L/I \).
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**Proposition 7.1.22.** If $I$ is an ideal and $J$ a sublattice of $L$. Then $\overline{\text{Apr}_I(J)}$ and $\text{Apr}_I(J)$ are sublattices of $L/I$.

**Proof:** Similar to the above Proposition 7.1.21.

**Theorem 7.1.23.** Let $I$ be an ideal of $L$ and $A$ be any non-empty subset of $L$. If $\text{Apr}^{(1)}_I(A) = (\overline{\text{Apr}_I(A)}, \overline{\text{Apr}_I(A)})$ is a rough ideal of $L$, then $\text{Apr}^{(2)}_I(A) = (\overline{\text{Apr}_I(A)}, \overline{\text{Apr}_I(A)})$ is a rough ideal of $L/I$.

**Proof:** Let $[a], [b] \in \overline{\text{Apr}_I(A)}$. Then $[a] \subseteq A$ and $[b] \subseteq A \Rightarrow a \in \overline{\text{Apr}_I(A)}$ and $b \in \overline{\text{Apr}_I(A)}$. Thus $a \lor b \in \overline{\text{Apr}_I(A)}$, since $\overline{\text{Apr}_I(A)}$ is an ideal. So that $[a \lor b] \subseteq A$. Hence $[a \lor b] \in \overline{\text{Apr}_I(A)}$. That is $[a] \lor [b] \in \overline{\text{Apr}_I(A)}$. Also let $[r] \in L/I$ and $[a] \in \overline{\text{Apr}_I(A)}$. Then $[a] \subseteq A$ and hence $a \in \overline{\text{Apr}_I(A)}$. This implies $r \land a \in \overline{\text{Apr}_I(A)}$, since $\overline{\text{Apr}_I(A)}$ is an ideal of $L$. Thus $[r] \land [a] \subseteq A$ and hence $[r] \land [a] \in \overline{\text{Apr}_I(A)}$. That is $[r] \land [a] \in \overline{\text{Apr}_I(A)}$. Thus $\overline{\text{Apr}_I(A)}$ is an ideal of $L/I$.

On the other hand, let $[a], [b] \in \overline{\text{Apr}_I(A)}$. Then $[a] \cap A \neq \emptyset$ and $[b] \cap A \neq \emptyset$. This implies $a \in \overline{\text{Apr}_I(A)}$ and $b \in \overline{\text{Apr}_I(A)}$. Hence $a \lor b \in \overline{\text{Apr}_I(A)}$, as $\overline{\text{Apr}_I(A)}$ is an ideal of $L$. Thus $[a \lor b] \cap A \neq \emptyset \Rightarrow [a \lor b] \in \overline{\text{Apr}_I(A)}$. That is $[a] \lor [b] \in \overline{\text{Apr}_I(A)}$. Also, let $[a] \in \overline{\text{Apr}_I(A)}$ and $[r] \in L/I$. Then $[a] \cap A \neq \emptyset$ and hence $a \in \overline{\text{Apr}_I(A)}$. This implies $r \land a \in \overline{\text{Apr}_I(A)}$, as $\overline{\text{Apr}_I(A)}$ is an ideal of $L$. Hence $[r] \land [a] \subseteq A$ and hence $[r] \land [a] \in \overline{\text{Apr}_I(A)}$. Thus $\overline{\text{Apr}_I(A)}$ is an ideal of $L/I$. Hence $\text{Apr}^{(2)}_I(A)$ is a rough ideal of $L/I$.

**Proposition 7.1.24.** If $I$ is an ideal of $L$ and $J$ a prime ideal of $L$. Then $\overline{\text{Apr}_I(J)}$ and $\text{Apr}_I(J)$ are prime ideals of $L/I$. 

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**Proof:** By Proposition 7.1.21, \( \overline{\text{Apr}}_I(J) \) and \( \overline{\text{Apr}}_I(J) \) are ideals of \( L/I \). Now, we prove that \( \overline{\text{Apr}}_I(J) \) is a prime ideal. For this, let \([a]_I \wedge [b]_I \in \overline{\text{Apr}}_I(J)\). Then \([a]_I \wedge [b]_I \subseteq J \neq \emptyset\). So that \( \exists x \in [a]_I \wedge [b]_I \) and \( x \in J\). That is \( x = x' \wedge y' \), where \( x' \in [a]_I \) and \( y' \in [b]_I \). Then \( x = x' \wedge y' \in J\) implies either \( x' \in J \) or \( y' \in J \), as \( J \) is a prime ideal. That is either \([a]_I \cap J \neq \emptyset \) or \([b]_I \cap J \neq \emptyset \). This implies either \([a]_I \in \overline{\text{Apr}}_I(J)\) or \([b]_I \in \overline{\text{Apr}}_I(J)\). Thus \( \overline{\text{Apr}}_I(J) \) is a prime ideal of \( L/I \). Next, we show that \( \overline{\text{Apr}}_I(J) \) is a prime ideal. If possible, assume that it is not a prime ideal. Then \( \exists [x]_I \wedge [y]_I \in \overline{\text{Apr}}_I(J) \) but \([x]_I \notin \overline{\text{Apr}}_I(J)\) and \([y]_I \notin \overline{\text{Apr}}_I(J)\). That is \([x]_I \wedge [y]_I \subseteq J \) but \([x]_I \not\subseteq J\) and \([y]_I \not\subseteq J\). This shows that \( \exists x' \in [x]_I \) such that \( x' \notin J \) and \( y' \notin J \). This contradicts the fact that \( J \) is a prime ideal of \( L \). Therefore our assumption is wrong. Hence \( \overline{\text{Apr}}_I(J) \) is a prime ideal of \( L/I \).

**Theorem 7.1.25.** Let \( I \) be an ideal of \( L \) and \( A \) be a non-empty subset of \( L \) and if 

\[
\text{Apr}^{(1)}_I(A) = (\text{Apr}_I(A), \overline{\text{Apr}}_I(A)) \quad \text{is a rough prime ideal of } L. \quad \text{Then } \text{Apr}^{(2)}_I(A) = (\overline{\text{Apr}}_I(A), \overline{\text{Apr}}_I(A)) \quad \text{is a rough prime ideal of } L/I.
\]

**Proof:** By Theorem 7.1.23, \( \text{Apr}^{(2)}_I(A) = (\overline{\text{Apr}}_I(A), \overline{\text{Apr}}_I(A)) \) is a rough ideal of \( L/I \). To prove \( \text{Apr}^{(2)}_I(A) \) is a rough prime ideal, it is enough to show that the ideals \( \overline{\text{Apr}}_I(A) \) and \( \overline{\text{Apr}}_I(A) \) are prime ideals of \( L/I \). Let \([a]_I \wedge [b]_I \in \overline{\text{Apr}}_I(A)\). Then \([a]_I \wedge [b]_I \subseteq A\). Hence \([a \wedge b]_I \subseteq A\). Thus \( a \wedge b \in \overline{\text{Apr}}_I(A)\). So that either \( a \in \overline{\text{Apr}}_I(A) \) or \( b \in \overline{\text{Apr}}_I(A)\), since \( \overline{\text{Apr}}_I(A) \) is a prime ideal. This implies either \([a]_I \subseteq A \) or \([b]_I \subseteq A\). Hence either \([a]_I \in \overline{\text{Apr}}_I(A) \) or \([b]_I \in \overline{\text{Apr}}_I(A)\). Thus \( \overline{\text{Apr}}_I(A) \) is a prime ideal of \( L/I \).
Next, we show that $\overline{Ap_{f_j}(A)}$ is a prime ideal of $L/I$. Let $[a]_I \land [b]_I \in \overline{Ap_{f_j}(A)}$. Then $([a]_I \land [b]_I) \cap A \neq \emptyset$. Thus $[a \land b]_I \cap A \neq \emptyset$ and hence $a \land b \in \overline{Ap_{f_j}(A)}$. This implies either $a \in \overline{Ap_{f_j}(A)}$ or $b \in \overline{Ap_{f_j}(A)}$, since $\overline{Ap_{f_j}(A)}$ is a prime ideal. Hence either $[a]_I \cap A \neq \emptyset$ or $[b]_I \cap A \neq \emptyset$. So that either $[a]_I \in \overline{Ap_{f_j}(A)}$ or $[b]_I \in \overline{Ap_{f_j}(A)}$. Thus $\overline{Ap_{f_j}(A)}$ is a prime ideal of $L/I$. Hence $Ap_{f_j}^{(2)}(A)$ is a rough prime ideal of $L/I$.

### 7.2. Rough Fuzzy Ideals and Homomorphism

Here we defined rough fuzzy sublattices, ideals and prime ideals and we prove that the level set corresponding to lower approximation [upper approximation] of a fuzzy set $\mu$ is same as lower approximation [upper approximation] of the level set $\mu_{\lambda}$ of $\mu$. With the help of this result we proved that if $\mu$ is a fuzzy sublattice [ideal or prime ideal] of $L$ then the rough fuzzy set $C(\mu)$ is a rough fuzzy sublattice [ideal or prime ideal] of $L$. Also we discussed the relationship between the upper (lower) rough fuzzy ideals (prime ideals) and the upper (lower) approximation of their homomorphic images.

**Definition 7.2.1.** Let $C$ be a congruence relation on $L$ and $\mu$ a fuzzy subset of $L$. Then we call the rough fuzzy set $C(\mu) = (C(\mu), \overline{C} (\mu))$ a **rough fuzzy sublattice (rough fuzzy ideal)** if both $C(\mu)$ and $\overline{C} (\mu)$ are fuzzy sublattices (fuzzy ideals) of $L$.

**Lemma 7.2.2.** Let $C$ be a congruence relation on $L$, $\mu$ is a fuzzy subset of $L$ and $\lambda \in [0, 1]$. Then

(i) $[C(\mu)]_\lambda = C(\mu_{\lambda})$ and (ii) $[\overline{C} (\mu)]_\lambda = \overline{C} (\mu_{\lambda})$.

**Proof:** (i) Let $x \in C(\mu_{\lambda}) \iff [x]_C \subseteq \mu_{\lambda} \iff \forall x' \in [x]_C, \mu(x') \geq \lambda$

$$\iff x' \in \bigwedge_{x \in [x]_C} \mu(x') \geq \lambda \iff C(\mu)(x) \geq \lambda \iff x \in [C(\mu)]_\lambda.$$
(ii) Let $x \in \overline{C}(\mu_\lambda^\prime) \iff [x]_C \cap \mu_\lambda^\prime \neq \Phi \iff \exists \ x^\prime \in [x]_C$ such that

$$
\mu(\ x^\prime) > \lambda \iff \bigvee_{x^\prime \in [x]_C} \mu(x^\prime) > \lambda \iff \overline{C}(\mu)(x) > \lambda \iff x \in \overline{C}(\mu)_{\lambda}^\prime .
$$

**Theorem 7.2.3.** If $\mu$ is a fuzzy sublattice (fuzzy ideal) of $L$ then the rough fuzzy set $C(\mu) = (C(\mu), \overline{C}(\mu))$ is a rough fuzzy sublattice (rough fuzzy ideal) of $L$.

**Proof:** Suppose $\mu$ is a fuzzy sublattice of $L$. Then we have $\mu_\lambda$ and $\mu_\lambda^\prime$ are sublattices of $L, \forall \lambda \in [0,1]$, by Theorem 2.1.17. Then by Proposition 7.1.14, $C(\mu_\lambda)$, and $\overline{C}(\mu_\lambda^\prime)$ are sublattices of $L$. By Lemma 7.2.2, $\overline{C}(\mu_\lambda) = [C(\mu)]_{\lambda}$ and $\overline{C}(\mu_\lambda^\prime) = [\overline{C}(\mu)]_{\lambda}$. Hence $[C(\mu)]_{\lambda}$ and $[\overline{C}(\mu)]_{\lambda}$ are sublattices of $L, \forall \lambda \in [0,1]$. Again by theorem 2.1.17, $C(\mu)$ and $\overline{C}(\mu)$ are fuzzy sublattices of $L$. Hence $C(\mu) = (C(\mu), \overline{C}(\mu))$ is a rough fuzzy sublattices of $L$. Proof for fuzzy ideal is similarly follows from Proposition 7.1.15.

**Definition 7.2.4.** The rough fuzzy set $C(\mu) = (C(\mu), \overline{C}(\mu))$ is called a rough fuzzy prime ideal if both $C(\mu)$ and $\overline{C}(\mu)$ are fuzzy prime ideals of $L$.

**Theorem 7.2.5.** If $\mu$ is a fuzzy prime ideal of $L$ then $C(\mu) = (C(\mu), \overline{C}(\mu))$ is a rough fuzzy prime ideal of $L$.

**Proof:** Since $\mu$ is a fuzzy prime ideal, we have $\mu_\lambda$ and $\mu_\lambda^\prime$ are prime ideals of $L, \forall \lambda \in [0,1]$, by Theorem 2.1.17. Then $C(\mu_\lambda)$ and $\overline{C}(\mu_\lambda^\prime)$ are prime ideals of $L$, by Proposition 7.1.19. Hence by Lemma 7.2.2, $[C(\mu)]_{\lambda}$ and $[\overline{C}(\mu)]_{\lambda}$ are prime ideals of $L$. Then by Theorem 2.1.17, $C(\mu)$ and $\overline{C}(\mu)$ are fuzzy prime ideals of $L$. Hence $C(\mu) = (C(\mu), \overline{C}(\mu))$ is a rough fuzzy prime ideal of $L$.

**Theorem 7.2.6.** Let $C$ be a congruence relation on $L$, and $\mu$ is a fuzzy subset of $L$. Then $\mu$ has a lower (upper) rough fuzzy prime ideal iff $\mu_\lambda$ and $\mu_\lambda^\prime$ have lower and upper rough prime ideals in $L$, respectively, $\forall \lambda \in [0,1]$. 
Proof: Firstly suppose that \( \mu \) has a lower (upper) rough fuzzy prime ideal. Then we have \( \mathbb{C}(\mu) \) and \( \mathbb{C}(\mu) \) are fuzzy prime ideals of \( L \).

\[ \iff [\mathbb{C}(\mu)], \text{ and } [\mathbb{C}(\mu)]^{\prime} \text{ are prime ideals of } L \]

\[ \iff \mathbb{C}(\mu) \text{ and } \mathbb{C}(\mu)^{\prime} \text{ are prime ideals of } L \text{ (by Lemma 7.2.2)} \]

\[ \iff \mu \text{ and } \mu^{\prime} \text{ have lower and upper rough prime ideals in } L, \text{ respectively.} \]

Lemma 7.2.7. Let \( f: L \to L' \) be a lattice epimorphism and \( C_2 \) be a congruence relation on \( L' \) and \( C_1 = \{(x_1, x_2) \in L \times L / (f(x_1), f(x_2) \in C_2 \} \) and \( A \) is any subset of \( L \). Then

(i) \( C_1 \) is congruence relation on \( L \)

(ii) \( f[\mathbb{C}_1(A)] = \mathbb{C}_2[f(A)] \)

(iii) \( f[\mathbb{C}_1(A)] = \mathbb{C}_2[f(A)], \text{ if } f \text{ is one-one.} \)

Proof: (i) Clearly \( C_1 \) is an equivalence relation. Also if \( (a, b) \in C_1 \) and \( (c, d) \in C_1 \)

Then \( (f(a), f(b)) \in C_2 \) and \( (f(c), f(d)) \in C_2 \).

Hence \( (f(a) \vee f(c), f(b) \vee f(d)) \in C_2 \) since \( C_2 \) is a congruence

\[ \Rightarrow (f(a \vee c), f(b \vee d)) \in C_2, \text{ since } f \text{ is a homomorphism} \]

\[ \Rightarrow (a \vee c, b \vee d) \in C_1, \text{ by definition of } C_1. \]

and

\[ f(a) \wedge f(c), f(b) \wedge f(d) \in C_2 \]

\[ \Rightarrow (f(a \wedge c), f(b \wedge d)) \in C_2 \]

\[ \Rightarrow (a \wedge c, b \wedge d) \in C_1 \]

Hence \( C_1 \) is a congruence relation on \( L \).
(ii) Let $y \in f[\overline{C}_1(A)]$. Then $\exists x \in \overline{C}_1(A)$ such that $f(x) = y$. Now, $x \in \overline{C}_1(A) \Rightarrow [x]_{C_1} \cap A \neq \emptyset$. So that $\exists x' \in [x]_{C_1} \cap A$. Hence $f(x') \in f(A)$ and $f(x') \in \overline{f(A)}$. Then $[f(x')]_{C_1} \cap A \neq \emptyset \Rightarrow f(x) \in \overline{C}_2(f(A)) \Rightarrow y \in \overline{C}_2(f(A))$

Hence $f[\overline{C}_1(A)] \subseteq \overline{C}_2(f(A))$ …………………………………(1)

Conversely, let $y \in \overline{C}_2(f(A))$. Then $[y]_{C_2} \cap f(A) \neq \emptyset$. Since $f$ is onto $\exists x' \in L$ such that $f(x') = y$. So that $[f(x')]_{C_2} \cap f(A) \neq \emptyset$. Then $\exists$ some $a \in A$ such that $f(a) \in f(A)$ and $f(a) \in [f(x')]_{C_2}$. Then by the definition of $C_1$,

$\exists x' \in \overline{C}_1(A)$. Thus $f(x') \in f(\overline{C}_1(A))$. That is $y \in f[\overline{C}_1(A)]$

Hence $\overline{C}_2[f(A)] \subseteq f[\overline{C}_1(A)]$ ………………………………………(2)

From (1) and (2) we get (ii).

(iii) Let $y \in f[\overline{C}_1(A)]$. Then $\exists x \in \overline{C}_1(A)$ such that $f(x) = y$. Thus $[x]_{C_1} \subseteq A$.

Let $y' \in [y]_{C_2}$ then since $f$ is onto $\exists x' \in L$ such that $f(x') = y'$. Thus we have $f(x') \in [f(x)]_{C_2}$. So by definition of $C_1$ we get $x' \in [x]_{C_1} \subseteq A$. Hence $y' = f(x') \in f(A)$. Thus $[y]_{C_2} \subseteq f(A)$. So that $y \in \overline{C}_2[f(A)]$.

Hence $f[\overline{C}_1(A)] \subseteq \overline{C}_2[f(A)]$ …………………(1)

Let $y \in \overline{C}_2[f(A)]$ then $[y]_{C_2} \subseteq f(A)$. Since $f$ is onto $\exists x' \in L$ such that $f(x') = y$. Then $[f(x')]_{C_2} \subseteq f(A)$. Let $x \in [x']_{C_1}$ then $f(x) \in [f(x')]_{C_1}$ \subseteq $f(A) \Rightarrow x \in A$, since $f$ is one -one. Thus $[x]_{C_1} \subseteq A \Rightarrow x' \in \overline{C}_1(A)$. Then $f(x') \in f[\overline{C}_1(A)]$. That is $y \in f[\overline{C}_1(A)]$.

Hence $\overline{C}_2[f(A)] \subseteq f[\overline{C}_1(A)]$………………………………(2)

From (1) and (2) we get (iii).
Theorem 7.2.8. Let \( f: L \to L' \) be a lattice isomorphism and \( C_2 \) be a congruence relation on \( L' \) and \( A \) any subset of \( L \). If \( C_1 = \{(x_1, x_2) \in L \times L / (f(x_1), f(x_2)) \in C_2\} \). Then

(i) \( C_1(A) \) is an ideal of \( L \) iff \( C_2 [f(A)] \) is an ideal of \( L' \);

(ii) \( C_1(A) \) is a prime ideal of \( L \) iff \( C_2 [f(A)] \) is a prime ideal of \( L' \).

Proof: (i) By Lemma 7.2.7, \( f [C_1 (A)] = C_2 [f (A)] \). Suppose \( C_1(A) \) is an ideal. To prove that \( C_2 [f(A)] \) is an ideal. It is enough to show that \( f [C_1 (A)] \) is an ideal.

Let \( a, b \in f [C_1 (A)] \). Then \( \exists x, y \in C_1(A) \) such that \( f(x) = a \) and \( f(y) = b \). So that \( x \lor y \in C_1(A) \), since \( C_1(A) \) is an ideal. Hence \( f(x \lor y) \in f [C_1 (A)] \). That is \( f(x) \lor f(y) \in f [C_1 (A)] \), since \( f \) is a homomorphism. That is \( a \lor b \in f [C_1 (A)] \). Let \( l' \in L' \) and \( a \in f [C_1 (A)] \). Since \( f \) is onto, \( \exists l \in L \) such that \( f(l) = l' \).

Also, \( a \in f [C_1 (A)] \) implies \( \exists x \in C_1(A) \) such that \( f(x) = a \). Now, \( a \land l' = f(x) \land f(l) = f(x \land l) \in f [C_1 (A)] \), here \( x \land l \in C_1(A) \), as \( C_1(A) \) is an ideal. Consequently, \( f [C_1 (A)] \) is an ideal. That is \( C_2 [f(A)] \) is an ideal.

Conversely, assumes that \( C_2 [f(A)] \) is an ideal. That is \( C_1(f(A)) \) is an ideal. We prove that \( C_1(A) \) is an ideal. For this, let \( x, y \in C_1(A) \). Then \( f(x), f(y) \in f [C_1 (A)] \) \( \Rightarrow f(x) \lor f(y) \in f [C_1 (A)] \). That is \( f(x \lor y) \in f [C_1 (A)] \). Since \( f \) is one-one, \( x \lor y \in C_1(A) \). Let \( l \in L \) and \( x \in C_1(A) \). Then \( f(l) \in L' \) and \( f(x) \in f [C_1 (A)] \). Hence \( f(l) \land f(x) \in f [C_1 (A)] \), since \( f [C_1 (A)] \) is an ideal. That is \( f(x \land l) \in f [C_1 (A)] \). Since \( f \) is one-one, \( x \land l \in C_1(A) \). Consequently \( C_1(A) \) is an ideal of \( L \).

(ii) Suppose \( C_1(A) \) is a prime ideal of \( L \). We have to prove that \( C_2 [f(A)] \) is a prime ideal of \( L' \). By result (i) \( C_1 [f(A)] \) is an ideal of \( L' \). Let \( a, b \in L' \) such that \( a \land b \in C_2 [f(A)] \). Then since \( f \) is onto \( \exists x, y \in L \) such that \( f(x) = a \) and \( f(y) = b \). Then
f(x) \land f(y) \in C_2[f(A)].

\Rightarrow f(x \land y) \in C_2[f(A)] = f[C_1(A)], by Lemma 7.2.7.

\Rightarrow x \land y \in C_1(A), since f is one-one

\Rightarrow either x \in C_1(A) or y \in C_1(A), since C_1(A) is a prime ideal

\Rightarrow either f(x) \in f[C_1(A)] or f(y) \in f[C_1(A)]

\Rightarrow either a \in f[C_1(A)] or b \in f[C_1(A)]

\Rightarrow either a \in C_2[f(A)] or b \in C_2[f(A)]

\Rightarrow C_2[f(A)] is a prime ideal of L'.

Conversely, assume that C_2[f(A)] is a prime ideal of L'. We have to prove that C_1(A) is a prime ideal of L. Clearly C_1(A) is an ideal of L by (i). Let x, y \in L such that x \land y \in C_1(A). Then f(x \land y) \in f[C_1(A)]. So that f(x) \land f(y) \in f[C_1(A)]. Hence either f(x) \in f[C_1(A)] or f(y) \in f[C_1(A)], since f[C_1(A)] = C_2[f(A)] is a prime ideal. This implies either x \in C_1(A) or y \in C_1(A), since f is one-one. This means that C_1(A) is a prime ideal of L.

Theorem 7.2.9. Let f: L \rightarrow L' be a lattice epimorphism and C_2 be a congruence relation on L' and A any subset of L. If C_1 = \{(x_1, x_2) \in L \times L / (f(x_1), f(x_2)) \in C_2\}. Then

(i) C_1(A) is an ideal of L iff C_2[f(A)] is an ideal of L'

(ii) C_1(A) is a prime ideal of L iff C_2[f(A)] is a prime ideal of L'.

Proof: (i) Suppose C_1(A) is an ideal. By Lemma 7.2.7, C_2[f(A)] = f[C_1(A)]. Therefore to prove C_2[f(A)] is an ideal, it is enough to show that f[C_1(A)] is an ideal. For this, let a, b \in f[C_1(A)]. Then \exists x, y \in C_1(A) such that f(x) = a and f(y)
= b. We have, \( x \lor y \in \overline{C}_1(A) \), since \( \overline{C}_1(A) \) is an ideal. Also \( f(x \lor y) \in f[\overline{C}_1(A)] \).

Since \( f \) is a homomorphism, \( f(x) \lor f(y) \in f[\overline{C}_1(A)] \). That is \( a \lor b \in f[\overline{C}_1(A)] \).

Let \( l' \) \in \( L' \) and \( a \in f[\overline{C}_1(A)] \). Since \( f \) is onto, \( l' \in L' \Rightarrow \exists l \in L \) such that \( f(l) = l' \). Also \( a \in f[\overline{C}_1(A)] \) \( \Rightarrow \exists x \in \overline{C}_1(A) \) such that \( f(x) = a \). Now \( a \land l' = f(x) \land f(l) = f(x \land l) \in f[\overline{C}_1(A)] \), here \( x \land l \in \overline{C}_1(A) \), as \( \overline{C}_1(A) \) is an ideal. Thus \( f[\overline{C}_1(A)] \) is an ideal of \( L' \). That is \( \overline{C}_2[f(A)] \) is an ideal of \( L'. \)

Conversely, assumes that \( \overline{C}_2[f(A)] \) is an ideal. Then \( f[\overline{C}_1(A)] \) is an ideal, by Lemma 7.2.7. We prove that \( \overline{C}_1(A) \) is an ideal. For this, let \( x, y \in \overline{C}_1(A) \). Then \( f(x), f(y) \in f[\overline{C}_1(A)] \). So that \( f(x) \lor f(y) \in f[\overline{C}_1(A)] \), since \( f[\overline{C}_1(A)] \) is an ideal. That is \( f(x \lor y) \in f[\overline{C}_1(A)] \). This implies \( \exists x' \in \overline{C}_1(A) \) such that \( f(x \lor y) = f(x') \). Then by the definition of \( C_1 \), \( x \lor y \in [x']_{C_1} \). But \( [x']_{C_1} \cap A \neq \phi \) and hence \( [x \lor y] \cap A \neq \phi \). This implies \( x \lor y \in \overline{C}_1(A) \). Next, let \( l \in L \) and \( x \in \overline{C}_1(A) \). Then \( f(l) \in L' \) and \( f(x) \in f[\overline{C}_1(A)] \). Hence \( f(l) \land f(x) \in f[\overline{C}_1(A)] \), since \( f[\overline{C}_1(A)] \) is an ideal. That is \( f(x \land l) \in f[\overline{C}_1(A)] \). This implies \( \exists y' \in \overline{C}_1(A) \) such that \( f(x \land l) = f(y') \). Thus \( x \land l \in [y']_{C_1} \). But \( [y']_{C_1} \cap A \neq \phi \). So \( [x \land l]_{C_1} \cap A \neq \phi \Rightarrow x \land l \in \overline{C}_1(A) \). Hence \( \overline{C}_1(A) \) is an ideal of \( L \).

(ii) Suppose \( \overline{C}_1(A) \) is a prime ideal of \( L \). We have to prove that \( \overline{C}_2[f(A)] \) is a prime ideal of \( L' \). By result (i) \( \overline{C}_2[f(A)] \) is an ideal of \( L' \). Let \( a, b \in L' \) such that \( a \land b \in \overline{C}_2[f(A)] \). Then \( \exists x, y \in L \) such that \( f(x) = a \) and \( f(y) = b \), since \( f \) is onto. Then

\[
\Rightarrow f(x \land y) \in \overline{C}_2[f(A)]
\]

\[
\Rightarrow f(x \land y) \in f[\overline{C}_1(A)], \text{by Lemma 7.2.7}
\]
\[ \Rightarrow \exists c \in \overline{C}_1(A) \text{ such that } f(x \wedge y) = f(c) \]
\[ \Rightarrow x \wedge y \in [c]_{C_i} \]

But [c]_{C_i} \cap A \neq \emptyset. So [x \wedge y]_{C_i} \cap A \neq \emptyset. This implies \( x \wedge y \in \overline{C}_1(A) \).

\[ \Rightarrow \text{either } x \in \overline{C}_1(A) \text{ or } y \in \overline{C}_1(A), \text{ since } \overline{C}_1(A) \text{ is a prime ideal.} \]

\[ \Rightarrow \text{either } f(x) \in f[\overline{C}_1(A)] \text{ or } f(y) \in f[\overline{C}_1(A)]. \]

\[ \Rightarrow \text{either } a \in f[\overline{C}_1(A)] \text{ or } b \in f[\overline{C}_1(A)] \]

\[ \Rightarrow \text{either } a \in \overline{C}_2[f(A)] \text{ or } b \in \overline{C}_2[f(A)] \]

\[ \Rightarrow \overline{C}_2[f(A)] \text{ is a prime ideal of } L'. \]

Conversely, assume that \( \overline{C}_2[f(A)] \) is a prime ideal of \( L' \). Clearly \( \overline{C}_1(A) \) is an ideal of \( L \), by result (i). Let \( x, y \in L \) such that \( x \wedge y \in \overline{C}_1(A) \). Then

\[ f(x \wedge y) \in f[\overline{C}_1(A)] \]

\[ \Rightarrow f(x) \wedge f(y) \in f[\overline{C}_1(A)] \]

\[ \Rightarrow \text{either } f(x) \in f[\overline{C}_1(A)] \text{ or } f(y) \in f[\overline{C}_1(A)], \text{ since } f[\overline{C}_1(A)] = \overline{C}_2[f(A)] \text{ is a prime ideal.} \]

Now, if \( f(x) \in f[\overline{C}_1(A)] \) then \( \exists x' \in \overline{C}_1(A) \) such that \( f(x) = f(x') \). Then \( x \in [x]_{C_i} \). But \( [x']_{C_i} \cap A \neq \emptyset \) and hence \( [x]_{C_i} \cap A \neq \emptyset \). This implies \( x \in \overline{C}_1(A) \). Similarly, if \( f(y) \in f[\overline{C}_1(A)] \). Then \( y \in \overline{C}_1(A) \). This means that \( \overline{C}_1(A) \) is a prime ideal of \( L \).

**Corollary 7.2.10.** Let \( f: L \rightarrow L' \) be a lattice isomorphism and \( C_2 \) be a congruence relation on \( L' \) and \( A \) any subset of \( L \). If \( C_1 = \{(x_1, x_2) \in L \times L / \}

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(\textit{f}(x_1), \textit{f}(x_2) \in C_2}. Then (i) \frac{C_1(A)}{C_1} is an ideal \textbf{[prime ideal]} of \(L/C_1\) iff \nabla\frac{C_2[f(A)]}{C_2} is an ideal \textbf{[prime ideal]} of \(L/C_2\).

(ii) \frac{C_1(A)}{C_1} is an ideal \textbf{[prime ideal]} of \(L/C_1\) iff \frac{C_2[f(A)]}{C_2} is an ideal \textbf{[prime ideal]} of \(L/C_2\).

\textbf{Proof:} Similar to above Theorems 7.2.8 and 7.2.9.

\textbf{Theorem 7.2.11.} Let \(\textit{f}: L \rightarrow L'\) be a lattice epimorphism and \(C_2\) be a congruence relation on \(L'\) and \(\mu\) any fuzzy subset of \(L\). If \(C_1 = \{(x_1, x_2) \in L \times L / (f(x_1), f(x_2)) \in C_2\}. Then

(i) \(\frac{C_1(\mu)}{C_1}\) is a fuzzy ideal\textbf{(fuzzy prime ideal)} of \(L\) iff \(\frac{C_2[f(\mu)]}{C_2}\) is a fuzzy ideal\textbf{(fuzzy prime ideal)} of \(L'\).

(ii) If \(\textit{f}\) is a one-one function, then \(\frac{C_1(\mu)}{C_1}\) is a fuzzy ideal \textbf{(fuzzy prime ideal)} of \(L\) iff \(\frac{C_2[f(\mu)]}{C_2}\) is a fuzzy ideal \textbf{(fuzzy prime ideal)} of \(L'\).

\textbf{Proof:} (i) We have by Theorem 2.1.17, \(\frac{C_1(\mu)}{C_1}\) is a fuzzy ideal\textbf{(fuzzy prime ideal)} of \(L\) iff \(\frac{[C_1(\mu)]}{C_1}\) is an ideal \textbf{(prime ideal)} of \(L\), \(\forall \lambda \in [0, 1]\). But by Lemma 7.2.2, we have \([C_1(\mu)] = \overline{C}(\mu'\lambda). By Theorem 7.2.9, we have \(\frac{C_1[\mu]}{C_1}\) is an ideal \textbf{(prime ideal)} of \(L\) iff \(\frac{C_2[f(\mu')]_{\lambda}}{C_2}\) is an ideal \textbf{(prime ideal)} of \(L'\). But

\(\frac{C_2[f(\mu')]_{\lambda}}{C_2} = \frac{C_2[f(\mu')]_{\lambda}}{C_2} = \frac{C_2[f(\mu')]_{\lambda}}{C_2} = \frac{C_2[f(\mu)]_{\lambda}}{C_2}\) and \(\frac{C_1[\mu]}{C_1} = \frac{C_1[\mu]}{C_1}\).

Hence again by Theorem 2.1.17, \(\frac{C_1(\mu)}{C_1}\) is a fuzzy ideal\textbf{(fuzzy prime ideal)} of \(L\) iff \(\frac{C_2[f(\mu)]}{C_2}\) is a fuzzy ideal\textbf{(fuzzy prime ideal)} of \(L'\).

(ii) We have by Theorem 2.1.17, \(\frac{C_1(\mu)}{C_1}\) is a fuzzy ideal \textbf{(fuzzy prime ideal)} of \(L\) iff \(\frac{[C_1(\mu)]}{C_1}\) is an ideal \textbf{(prime ideal)} of \(L\), \(\forall \lambda \in [0, 1]\). But by Lemma 7.2.2, we
have \( [C_{1}(\mu)]_{\lambda} = C_{2}(\mu_{\lambda}) \). By Theorem 7.2.8, we have \( C_{1}[\mu_{\lambda}] \) is an ideal (prime ideal) of \( L \) iff \( C_{2}[f(\mu_{\lambda})] \) is an ideal (prime ideal) of \( L' \). But \( C_{2}[f(\mu_{\lambda})] = C_{2}[f(\mu)] = [C_{2}(f(\mu))]_{\lambda} \).

Hence again by Theorem 2.1.17, \( C_{1}(\mu) \) is a fuzzy ideal (fuzzy prime ideal) of \( L \) iff \( C_{2}[f(\mu)] \) is a fuzzy ideal (fuzzy prime ideal) of \( L' \).

**Corollary 7.2.12.** Let \( f : L \rightarrow L' \) be a lattice isomorphism and \( C_{2} \) be a congruence relation on \( L' \) and \( \mu \) any fuzzy subset of \( L \). If \( C_{1} = \{(x_{1}, x_{2}) \in L \times L / (f(x), f(x_{2})) \in C_{2}\} \), then (i) \( \frac{C_{1}(\mu)}{C_{1}} \) is an ideal [prime ideal] of \( \frac{L}{C_{1}} \) iff \( \frac{C_{2}[f(\mu)]}{C_{2}} \) is an ideal [prime ideal] of \( \frac{L'}{C_{2}} \).

(ii) \( \frac{C_{1}(\mu)}{C_{1}} \) is an ideal [prime ideal] of \( \frac{L}{C_{1}} \) iff \( \frac{C_{2}[f(\mu)]}{C_{2}} \) is an ideal [prime ideal] of \( \frac{L'}{C_{2}} \).

**Proof:** Follow from Theorem 2.1.17 and Corollary 7.2.10.

### 7.3. Rough Intuitionistic Fuzzy Sets in a Lattice

In this section we define Rough intuitionistic fuzzy sets and define some operations on them. Rough intuitionistic fuzzy lattices (RIFL) and ideals (RIFI) are introduced and certain properties are studied. Also we introduce intuitionistic fuzzy rough sublattices and ideals, and certain characterization of intuitionistic fuzzy rough sublattice (ideal) in terms of level rough set.

**Definition 7.3.1.** Let \( A = \{< x, \mu_{A}(x), \nu_{A}(x) > | x \in X \} \) be an IFS of \( X \). Then the **Rough intuitionistic fuzzy set of \( A \) (RIFS)** is denoted by \( \text{Apr}(A) = (\text{Apr}(A), \text{Apr}x(A)) \) and defined as

\[
\text{Apr}(A) = \langle x, \mu_{A}(x), \nu_{A}(x) \rangle \quad \text{where} \quad \mu_{A}(x) = \bigwedge_{x \leq x' \leq x_{e}} \mu(x') \quad \text{and} \quad \nu_{A}(x) = \bigvee_{x \leq x' \leq x_{e}} \nu(x')
\]
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And $\text{Apr}(A) = (x, \mu_A(x), \nu_A(x))$ where $\mu_A(x) = \bigvee_{x' \in [x]_\theta} \mu(x')$ and $\nu_A(x) = \bigwedge_{x' \in [x]_\theta} \nu(x')$.

Here $\theta$ is an equivalence relation on $X$ and $[x]_\theta$ is the equivalence class containing $x$.

**Definition 7.3.2.** Let $\text{Apr}(A) = (\overline{\text{Apr}}(A), \text{Apr}(A))$, $\text{Apr}(B) = (\overline{\text{Apr}}(B), \text{Apr}(B))$ are two RIFS, Then

1. $\text{Apr}(A) \cup \text{Apr}(B) = (\overline{\text{Apr}}(A) \cup \overline{\text{Apr}}(B), \text{Apr}(A) \cup \text{Apr}(B))$.
2. $\text{Apr}(A) \cap \text{Apr}(B) = (\overline{\text{Apr}}(A) \cap \overline{\text{Apr}}(B), \text{Apr}(A) \cap \text{Apr}(B))$.
3. $\text{Apr}(A) \subseteq \text{Apr}(B)$ if and only if $\overline{\text{Apr}}(A) \subseteq \overline{\text{Apr}}(B)$, $\text{Apr}(A) \subseteq \text{Apr}(B)$.
4. The complement of $\text{Apr}(A)$ denoted as $\text{Apr}^c(A) = (\overline{\text{Apr}^c}(A), \overline{\text{Apr}}(A))$.

**Proposition 7.3.3.** For every approximation space $(X, \theta)$ and for every IFS’s $A$, $B \subseteq X$, we have

(i) $\text{Apr}(A) \subseteq A \subseteq \overline{\text{Apr}(A)}$,

(ii) If $A \subseteq B$ then $\text{Apr}(A) \subseteq \text{Apr}(B)$ and $\overline{\text{Apr}(A)} \subseteq \overline{\text{Apr}(B)}$,

(iii) $\text{Apr} \left( \overline{\text{Apr}(A)} \right) = \text{Apr}(A)$,

(iv) $\overline{\text{Apr} \left( \overline{\text{Apr}(A)} \right)} = \text{Apr}(A)$,

(v) $\overline{\text{Apr} \left( \text{Apr}(A) \right)} = \text{Apr}(A)$,

(vi) $\text{Apr} \left( \overline{\text{Apr}(A)} \right) = \text{Apr}(A)$,

(vii) $\text{Apr}(A) = [\overline{\text{Apr}(A')}]^c$,

(viii) $\overline{\text{Apr}(A)} = [\overline{\text{Apr}(A')}]^c$,

(ix) $\overline{\text{Apr}(A \cap B)} = \text{Apr}(A) \cap \text{Apr}(B)$,

(x) $\overline{\text{Apr}(A \cap B)} \subseteq \overline{\text{Apr}(A)} \cap \overline{\text{Apr}(B)}$. 
(xi) \[ \text{Apr} (A \cup B) \supseteq \text{Apr} (A) \cup \text{Apr} (B), \]

(xii) \[ \text{Apr} (A \cup B) = \text{Apr} (A) \cup \text{Apr} (B). \]

**Proof:** Follow from Definitions 2.2.2, 7.3.1 and 7.3.2.

**Proposition 7.3.4.** Let L be a lattice and A is an IFL (IFI) of L. Then \( \text{Apr}(A) \) and \( \text{Apr}(A) \) are also IFLs (IFI) of L.

**Proof:** Let \( x, y \in L \). Then we have

\[
\mu_A(x \lor y) = \bigwedge_{x \lor y \leq \mu_A(x \lor y)} \mu(x' \lor y') \\
\geq \bigwedge_{x \lor y \leq \mu_A(x \lor y)} \min\{\mu(x'), \mu(y')\}, \text{ since } A \text{ is an IFL of } L. \\
= \min\left( \bigwedge_{x \leq x'} \mu(x'), \bigwedge_{y \leq y'} \mu(y') \right) \\
= \min(\mu_A(x), \mu_A(y)).
\]

Also

\[
\mu_A(x \land y) = \bigwedge_{x \land y \leq \mu_A(x \land y)} \mu(x' \land y') \\
\geq \bigwedge_{x \land y \leq \mu_A(x \land y)} \min\{\mu(x'), \mu(y')\}, \text{ since } A \text{ is an IFL of } L. \\
= \min\left( \bigwedge_{x \leq x'} \mu(x'), \bigwedge_{y \leq y'} \mu(y') \right) \\
= \min(\mu_A(x), \mu_A(y)).
\]

Similarly

\[
\nu_A(x \lor y) = \bigvee_{x \lor y \leq \nu_A(x \lor y)} \nu(x' \lor y') \\
\leq \bigvee_{x \lor y \leq \nu_A(x \lor y)} \max\{\nu(x'), \nu(y')\}, \text{ since } A \text{ is an IFL of } L. \\
= \max\left( \bigvee_{x \leq x'} \nu(x'), \bigvee_{y \leq y'} \nu(y') \right) \\
= \max(\nu_A(x), \nu_A(y)).
\]
And
\[ V_d(x \wedge y) = \bigvee_{x', y' \in [x \wedge y]}^A \nu(x' \wedge y') \leq \bigvee_{x', y' \in [x \wedge y]}^A \{\max(\nu(x'), \nu(y'))\} \text{, since } A \text{ is an IFL of } L. \]
\[ = \max(\bigvee_{x' \in [x]}^A \nu(x'), \bigvee_{y' \in [y]}^A \nu(y')) \]
\[ = \max(V_d(x), V_d(y)). \]

Hence \( \text{Apr}(A) \) is an IFL of \( L \). Now
\[ \mu_d(x \vee y) = \bigvee_{x' \vee y' \in [x \vee y]}^A \mu(x' \vee y') \geq \bigvee_{x' \vee y' \in [x \vee y]}^A \{\min(\mu(x'), \mu(y'))\} \text{, since } A \text{ is an IFL of } L. \]
\[ = \min(\bigvee_{x' \in [x]}^A \mu(x'), \bigvee_{y' \in [y]}^A \mu(y')) \text{, since } x' \text{ and } y' \text{ vary independently} \]
\[ = \min(\mu_d(x), \mu_d(y)) \]

Similarly,
\[ \mu_d(x \wedge y) = \min(\mu_d(x), \mu_d(y)). \]

Also
\[ V_d(x \vee y) = \bigwedge_{x' \vee y' \in [x \vee y]}^A \nu(x' \vee y') \leq \bigwedge_{x' \vee y' \in [x \vee y]}^A \{\max(\nu(x'), \nu(y'))\} \text{, since } A \text{ is an IFL of } L. \]
\[ = \max(\bigwedge_{x' \in [x]}^A \nu(x'), \bigwedge_{y' \in [y]}^A \nu(y')) \text{, since } x' \text{ and } y' \text{ vary independently} \]
\[ = \max(V_d(x), V_d(y)). \]

Similarly,
\[ V_d(x \wedge y) \leq \max(V_d(x), V_d(y)). \]

So \( \text{Apr}(A) \) is an IFL of \( L \). Proof for IFI is similar.

**Definition 7.3.5.** A rough intuitionistic fuzzy set \( \text{Apr}(A) \) of \( L \) is called a **rough intuitionistic fuzzy lattice** (RIFL) [**rough intuitionistic fuzzy ideal** (RIFI)] if both \( \text{Apr}(A) \) and \( \text{Apr}(A) \) are IFLs [IFIs] of \( L \).

**Theorem 7.3.6.** If \( A \) is an IFL[IFI] of \( L \) then \( \text{Apr}(A) \) is a RIFL[RIFI].
**Proof:** Follow from Proposition 7.3.4.

**Theorem 7.3.7.** If $\text{Apr}(A)$ and $\text{Apr}(B)$ are RIFLs (RIFIs), then $\text{Apr}(A) \cap \text{Apr}(B)$ is a RIFL (RIFI).

**Proof:** We have $\text{Apr}(A) \cap \text{Apr}(B) = \left( \overline{\text{Apr}(A) \cap \text{Apr}(B)} \right)$ since $\text{Apr}(A)$ and $\text{Apr}(B)$ are RIFLs (RIFIs). Then $\overline{\text{Apr}(A) \cap \text{Apr}(B)}$ and $\overline{\text{Apr}(A) \cap \text{Apr}(B)}$ are IFLs (IFIs) by Theorem 3.1.6. So that $\text{Apr}(A) \cap \text{Apr}(B)$ is a RIFL (RIFI), by Definition 7.3.5.

**Result 7.3.8.** The union of two RIFIs need not be a RIFI.

Consider the lattice $L = \{1, 2, 3, 4, 6, 12\}$ of divisors of 12. Let $\theta = \{1, 2\}, (3, 6), (4), (12)\}$ be the equivalence class. We define $A = \{ \langle x, \mu(x) \rangle : x \in L \}$ by $A = \{ \langle 1, .7, .2 \rangle, \langle 2, .5, .5 \rangle, \langle 3, .6, .3 \rangle, \langle 4, .4, .5 \rangle, \langle 6, .5, .5 \rangle \}$ and $B = \{ \langle x, \mu(x) \rangle : x \in L \}$ by $B = \{ \langle 1, .6, .2 \rangle, \langle 2, .6, .4 \rangle, \langle 3, .5, .5 \rangle, \langle 4, .5, .4 \rangle, \langle 6, .4, .5 \rangle, \langle 12, .4, .5 \rangle \}$. Here $A$ and $B$ are IFI’s of $L$. Now $\text{Apr}(A) = \left( \overline{\text{Apr}(A)} \right)$ where $\overline{\text{Apr}(A)} = \langle x, \mu(x) \rangle$ is $\{ \langle 1, .5, .5 \rangle, \langle 2, .5, .5 \rangle, \langle 3, .5, .5 \rangle, \langle 4, .4, .5 \rangle, \langle 6, .5, .5 \rangle, \langle 12, .4, .5 \rangle \}$ and $\overline{\text{Apr}(A)} = \langle x, \mu(x), \nu(x) \rangle$ is $\{ \langle 1, .7, .2 \rangle, \langle 2, .7, .2 \rangle, \langle 3, .6, .3 \rangle, \langle 4, .4, .5 \rangle, \langle 6, .6, .3 \rangle, \langle 12, .4, .5 \rangle \}$. Also $\text{Apr}(B) = \left( \overline{\text{Apr}(B)} \right)$, where $\overline{\text{Apr}(B)} = \langle x, \mu(x), \nu(x) \rangle$ is $\{ \langle 1, .6, .4 \rangle, \langle 2, .6, .4 \rangle, \langle 3, .4, .5 \rangle, \langle 4, .5, .4 \rangle, \langle 6, .4, .5 \rangle, \langle 12, .4, .5 \rangle \}$ and $\overline{\text{Apr}(B)} = \langle x, \mu(x), \nu(x) \rangle$ is $\{ \langle 1, .6, .2 \rangle, \langle 2, .6, .2 \rangle, \langle 3, .5, .5 \rangle, \langle 4, .5, .4 \rangle, \langle 6, .5, .5 \rangle, \langle 12, .4, .5 \rangle \}$. Clearly $\text{Apr}(A)$ and $\text{Apr}(B)$ are RIFIs.

Now $\text{Apr}(A) \cup \text{Apr}(B) = \left( \overline{\text{Apr}(A) \cup \text{Apr}(B)} \right)$ is given by, $\overline{\text{Apr}(A) \cup \text{Apr}(B)} = \{ \langle 1, .6, .4 \rangle, \langle 2, .6, .4 \rangle, \langle 3, .5, .5 \rangle, \langle 4, .5, .4 \rangle, \langle 6, .5, .5 \rangle,$
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\[
\langle 12, .4, .5 \rangle. \quad \overline{\text{Apr}(A)} \cup \overline{\text{Apr}(B)} = \{ \langle 1, .7, .2 \rangle, \langle 2, .7, .2 \rangle, \langle 3, .6, .3 \rangle, \langle 4, .5, .4 \rangle, \langle 6, .6, .3 \rangle, \langle 12, .4, .5 \rangle. \]

Here \( \mu_{\overline{\text{Apr}(A)} \cup \overline{\text{Apr}(B)}}(3 \lor 4) \neq \min \{ \mu_{\overline{\text{Apr}(A)} \cup \overline{\text{Apr}(B)}}(3), \mu_{\overline{\text{Apr}(A)} \cup \overline{\text{Apr}(B)}}(4) \} \).

Hence \( \text{Apr} (A) \cup \text{Apr} (B) \) is not an RIFI.

**Result 7.3.9.** Every RIFI is a RIFL. But the converse is not true.

Consider the lattice and the equivalence relation given in the Result 7.3.8. Let

\[ B = \{ \langle x, \mu_b(x), \nu_b(x) \rangle \mid x \in L \} \]

be given by, \( \{ \langle 1, .2, .7 \rangle, \langle 2, .4, .4 \rangle, \langle 3, .2, .5 \rangle, \langle 4, .3, .6 \rangle, \langle 6, .2, .5 \rangle, \langle 12, .4, .5 \rangle \} \). Now \( \text{Apr}(B) = (\overline{\text{Apr}(B)}, \overline{\text{Apr}(B)}) \), where \( \overline{\text{Apr}(B)} = \{ \langle x, \mu_b(x), \nu_b(x) \rangle \mid \langle 1, .2, .7 \rangle, \langle 2, .2, .7 \rangle, \langle 3, .2, .5 \rangle, \langle 4, .3, .6 \rangle, \langle 6, .2, .5 \rangle, \langle 12, .6, .3 \rangle \} \) and \( \overline{\text{Apr}(B)} = \{ \langle x, \mu_b(x), \nu_b(x) \rangle \mid \langle 1, .4, .4 \rangle, \langle 2, .4, .4 \rangle, \langle 3, .5, .5 \rangle, \langle 4, .3, .6 \rangle, \langle 6, .5, .5 \rangle, \langle 12, .6, .3 \rangle \} \). It can be easily verified that \( \text{Apr}(B) \) is a RIFL. But not RIFI, because \( \mu_b(6 \land 4) \neq \max \{ \mu_b(6), \mu_b(4) \} \).

**Result 7.3.10.** If \( \text{Apr} (A) \) is a RIFI and \( \text{Apr} (B) \) is a RIFL. Then \( \text{Apr} A \cap \text{Apr} B \) is a RIFL, but not a RIFI.

Consider the RIFI \( \text{Apr}(A) \) given in Result 7.3.8, and the RIFL \( \text{Apr}(B) \) given in Result 7.3.9. Then \( \text{Apr} (A) \cap \text{Apr} (B) = (\overline{\text{Apr}(A)} \cap \overline{\text{Apr}(B)}, \overline{\text{Apr}(A)} \cap \overline{\text{Apr}(B)}) \) is given by, \( \overline{\text{Apr}(A)} \cap \overline{\text{Apr}(B)} = \{ \langle 1, .2, .7 \rangle, \langle 2, .2, .7 \rangle, \langle 3, .2, .5 \rangle, \langle 4, .3, .6 \rangle, \langle 6, .2, .5 \rangle, \langle 12, .4, .5 \rangle \} \) and \( \overline{\text{Apr}(B)} \cap \overline{\text{Apr}(B)} = \{ \langle 1, .4, .4 \rangle, \langle 2, .4, .4 \rangle, \langle 3, .5, .5 \rangle, \langle 4, .3, .6 \rangle, \langle 6, .5, .5 \rangle, \langle 12, .4, .5 \rangle \} \).

Here it is easily verified that \( \text{Apr} (A) \cap \text{Apr} (B) \) is a RIFL, but not a RIFI because \( \mu_{\overline{\text{Apr}(A)} \cap \overline{\text{Apr}(B)}}(6 \land 4) \neq \max \{ \mu_{\overline{\text{Apr}(A)} \cap \overline{\text{Apr}(B)}}(6), \mu_{\overline{\text{Apr}(A)} \cap \overline{\text{Apr}(B)}}(4) \} \).

**Definition 7.3.11.** Let \( \text{Apr}(A) \) is a roughest in \( (U, \theta) \) then an intuitionistic fuzzy rough set (IFRS) \( \text{Apr}(A) = (\overline{\text{Apr}(A)}, \overline{\text{Apr}(A)}) \) in \( \text{Apr}(X) \) is obtained by the maps
\[ \mu_{\text{Apr}(A)} : \text{Apr}(X) \to [0,1], \quad \nu_{\text{Apr}(A)} : \text{Apr}(X) \to [0,1] \] where \( 0 \leq \mu_{\text{Apr}(A)} + \nu_{\text{Apr}(A)} \leq 1 \) and

also with property \( \mu_{\text{Apr}(A)}(x) \leq \mu_{\text{Apr}(A)}(x) \) and \( \nu_{\text{Apr}(A)}(x) \geq \nu_{\text{Apr}(A)}(x) \),

\( \forall x \in \text{Apr}(X) \).

**Definition 7.3.12.** Let \( \text{Apr}(X) \) be a rough set and \( \text{Apr}(A) = (\text{Apr}(A), \text{Apr}(A)) \) is an IFRS in \( \text{Apr}(X) \). Then we can define an **interval valued intuitionistic fuzzy set**

\[ A = \{ (x, [\mu_{\text{Apr}(A)}(x), \nu_{\text{Apr}(A)}(x)]) / \forall x \in \text{Apr}(X) \} \]

where

\[
\mu_{\text{Apr}(A)}(x) = \begin{cases} 
\mu_{\text{Apr}(A)}(x), & \text{if } x \in \text{Apr}(X) \\
0, & \text{if } x \in \text{Apr}(X)
\end{cases}
\]

and

\[
\nu_{\text{Apr}(A)}(x) = \begin{cases} 
\nu_{\text{Apr}(A)}(x), & \text{if } x \in \text{Apr}(X) \\
1, & \text{if } x \in \text{Apr}(X)
\end{cases}
\]

Here \( \text{Apr}(X) = \text{Apr}(X) - \text{Apr}(X) \). We denote \( \tilde{\mu}_A(x) = [\mu_{\text{Apr}(A)}(x), \mu_{\text{Apr}(A)}(x)] \) and \( \tilde{\nu}_A(x) = [\nu_{\text{Apr}(A)}(x), \nu_{\text{Apr}(A)}(x)] \).

**Definition 7.3.13.** Let \( \text{Apr}(X) \) be a rough lattice and \( \text{Apr}(A) = (\text{Apr}(A), \text{Apr}(A)) \) is an IFRS in \( \text{Apr}(X) \). Then \( \text{Apr}(A) \) is called a **intuitionistic fuzzy rough sublattice (IFRL)** if the following conditions hold, \( \forall x, y \in \text{Apr}(X) \):

(i) \( \tilde{\mu}_A(x \lor y) \geq \min\{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \} \)

(ii) \( \tilde{\nu}_A(x \lor y) \geq \min\{ \tilde{\nu}_A(x), \tilde{\nu}_A(y) \} \)

(iii) \( \tilde{\mu}_A(x \land y) \leq \max\{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \} \)

(iv) \( \tilde{\nu}_A(x \land y) \leq \max\{ \tilde{\nu}_A(x), \tilde{\nu}_A(y) \} \)
If conditions (ii) and (iv) are replaced by \( \tilde{\mu}_A(x \wedge y) \geq \max\{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \} \) and \( \tilde{\nu}_A(x \wedge y) \leq \min\{ \tilde{\nu}_A(x), \tilde{\nu}_A(y) \} \). Then \( \text{Apr}(A) \) is called an **intuitionistic fuzzy rough ideal (IFRI)**.

**Definition 7.3.14.** Let \( \text{Apr}(X) \) be a rough lattice and \( \text{Apr}(A) = (\text{Apr}(A), \text{Apr}(A)) \) a IFRS in \( \text{Apr}(X) \). Then we define

\[
\widetilde{A}_{(\alpha, \beta)} = \{ x \in \text{Apr}(X) / \mu_{\text{Apr}(A)}(x) \geq \alpha \text{ and } \nu_{\text{Apr}(A)}(x) \leq \beta \} \quad \text{and}
\]

\[
\overline{A}_{(\alpha, \beta)} = \{ x \in \text{Apr}(X) / \mu_{\text{Apr}(A)}(x) \geq \alpha \text{ and } \nu_{\text{Apr}(A)}(x) \leq \beta \} .
\]

Here \( (A_{(\alpha, \beta)}, \overline{A}_{(\alpha, \beta)}) \) is called a **level rough set**.

**Theorem 7.3.15.** Let \( \text{Apr}(X) \) be a rough lattice and \( \text{Apr}(A) = (\text{Apr}(A), \text{Apr}(A)) \) is an IFRS in \( \text{Apr}(X) \). Then \( \text{Apr}(A) \) is an IFRL iff \( (A_{(\alpha, \beta)}, \overline{A}_{(\alpha, \beta)}) \) is a rough sublattice of \( \text{Apr}(X) \).

**Proof:** First assume that \( (A_{(\alpha, \beta)}, \overline{A}_{(\alpha, \beta)}) \) is a rough sublattice in \( \text{Apr}(X) \). We have to prove that \( \text{Apr}(A) \) is a IFRL of \( \text{Apr}(X) \).

Let \( x, y \in \text{Apr}(X) \). Then we set

\[
\min\{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \} = [\alpha_0, \alpha_1] \text{ and } \max\{ \tilde{\nu}_A(x), \tilde{\nu}_A(y) \} = [\beta_0, \beta_1].
\]

Then

\[
\min\{ \overline{\mu}_{\text{Apr}(A)}(x), \overline{\mu}_{\text{Apr}(A)}(y) \} = \alpha_0 \text{ and } \min\{ \overline{\mu}_{\text{Apr}(A)}(x), \overline{\mu}_{\text{Apr}(A)}(y) \} = \alpha_1 \text{ and}
\]

\[
\max\{ \overline{\nu}_{\text{Apr}(A)}(x), \overline{\nu}_{\text{Apr}(A)}(y) \} = \beta_0 \text{ and } \max\{ \overline{\nu}_{\text{Apr}(A)}(x), \overline{\nu}_{\text{Apr}(A)}(y) \} = \beta_1 .
\]

Hence

\[
\overline{\mu}_{\text{Apr}(A)}(x) \geq \alpha_1 , \quad \overline{\nu}_{\text{Apr}(A)}(x) \geq \alpha_1 , \quad \overline{\nu}_{\text{Apr}(A)}(x) \leq \beta_0 \text{ and } \overline{\nu}_{\text{Apr}(A)}(y) \leq \beta_0 .
\]

Thus

\[
\overline{A}_{(\alpha_0, \beta_0)} \Rightarrow x \vee y, x \wedge y \in \overline{A}_{(\alpha_0, \beta_0)} , \text{ since } \overline{A}_{(\alpha_0, \beta_0)} \text{ is a sublattice. Thus}
\]

\[
\overline{\mu}_{\text{Apr}(A)}(x \vee y) \geq \alpha_1 , \quad \overline{\nu}_{\text{Apr}(A)}(x \vee y) \geq \alpha_1 , \quad \overline{\nu}_{\text{Apr}(A)}(x \vee y) \leq \beta_0 \text{ and } \overline{\nu}_{\text{Apr}(A)}(x \wedge y) \leq \beta_0 .
\]

Let \( x, y \in \text{Apr}(X) \Rightarrow \text{either } x \text{ or } y \in \text{Apr}(X) \) or \( x \text{ and } y \notin \text{Apr}(X) \). If \( x \text{ or } y \in \text{Apr}(X) \) then \( \alpha_0 = 0 \text{ and } \beta_1 = 1 \). So that
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If \( x \) and \( y \in A_{\alpha, \beta} \) then we have

\[
\begin{align*}
\mu_{\text{Apr}(A)}(x \lor y) &\geq 0 = \alpha_0, \\
\overline{\mu}_{\text{Apr}(A)}(x \land y) &\geq 0 = \alpha_0, \\
\nu_{\text{Apr}(A)}(x \lor y) &\leq 1 = \beta_1, \\
\overline{\nu}_{\text{Apr}(A)}(x \land y) &\leq 1 = \beta_1.
\end{align*}
\]

This implies \( x \in A_{[\alpha_0, \beta_1]} \) and \( y \in A_{[\alpha_0, \beta_1]} \Rightarrow x \lor y, x \land y \in A_{[\alpha_0, \beta_1]} \), since \( A_{[\alpha_0, \beta_1]} \) is a sublattice. So that

\[
\begin{align*}
\mu_{\text{Apr}(A)}(x \lor y) &\geq \alpha_0, \\
\mu_{\text{Apr}(A)}(x \land y) &\geq \alpha_0, \\
\nu_{\text{Apr}(A)}(x \lor y) &\leq \beta_1, \\
\nu_{\text{Apr}(A)}(x \land y) &\leq \beta_1.
\end{align*}
\]

Hence

\[
\begin{align*}
\overline{\mu}_{\text{Apr}(A)}(x \lor y) &= \left[ \overline{\mu}_{\text{Apr}(A)}(x \lor y), \mu_{\text{Apr}(A)}(x \lor y) \right] \geq \alpha_0, \\
\overline{\nu}_{\text{Apr}(A)}(x \lor y) &= \left[ \nu_{\text{Apr}(A)}(x \lor y), \overline{\nu}_{\text{Apr}(A)}(x \lor y) \right] \leq \beta_1,
\end{align*}
\]

and

\[
\begin{align*}
\overline{\nu}_{\text{Apr}(A)}(x \land y) &= \left[ \nu_{\text{Apr}(A)}(x \land y), \overline{\nu}_{\text{Apr}(A)}(x \land y) \right] \leq \beta_1, \\
\overline{\nu}_{\text{Apr}(A)}(x \land y) &= \left[ \overline{\nu}_{\text{Apr}(A)}(x \land y), \nu_{\text{Apr}(A)}(x \land y) \right] \geq \alpha_0.
\end{align*}
\]

Thus \( \text{Apr}(A) \) is an IFRL of \( \text{Apr}(X) \).

Conversely, assume that \( \text{Apr}(A) \) is an IFRL of \( \text{Apr}(X) \). We have to prove that \( A_{[\alpha, \beta]} \) and \( \overline{A}_{[\alpha, \beta]} \) are sublattices of \( L \). We have \( \forall x, y \in A_{[\alpha, \beta]} \)
\[ \mu_{\text{Apr}^A}(x) \geq \alpha, \quad \mu_{\text{Apr}^A}(y) \geq \alpha, \quad \nu_{\text{Apr}^A}(x) \leq \beta \text{ and } \nu_{\text{Apr}^A}(y) \leq \beta. \]

So that
\[
\min \left\{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \right\} = [\alpha, \min \{ \mu_{\text{Apr}^A}(x), \mu_{\text{Apr}^A}(y) \}] \text{ and }
\max \left\{ \tilde{\nu}_A(x), \tilde{\nu}_A(y) \right\} = [\max \{ \nu_{\text{Apr}^A}(x), \nu_{\text{Apr}^A}(y) \}, \beta].
\]

But \( \tilde{\mu}_A(x \lor y) \geq \min \{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \}, \quad \tilde{\nu}_A(x \lor y) \leq \max \{ \tilde{\nu}_A(x), \tilde{\nu}_A(y) \} \) and
\[
\tilde{\mu}_A(x \land y) \geq \min \{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \}, \quad \tilde{\nu}_A(x \land y) \leq \max \{ \tilde{\nu}_A(x), \tilde{\nu}_A(y) \},
\]
since \( \text{Apr}^A \) is fuzzy rough sublattice of \( \text{Apr}^X \). Hence
\[
\tilde{\mu}_A(x \lor y) \geq [\alpha, \min \{ \mu_{\text{Apr}^A}(x), \mu_{\text{Apr}^A}(y) \}]], \tilde{\nu}_A(x \lor y) \leq [\max \{ \nu_{\text{Apr}^A}(x), \nu_{\text{Apr}^A}(y) \}],
\]
\[
\beta \text{ and } \tilde{\mu}_A(x \land y) \geq [\alpha, \min \{ \mu_{\text{Apr}^A}(x), \mu_{\text{Apr}^A}(y) \}], \tilde{\nu}_A(x \land y) \leq [\max \{ \nu_{\text{Apr}^A}(x), \nu_{\text{Apr}^A}(y) \}],
\]
\[
\nu_{\text{Apr}^A}(y), \beta]. \text{ From these inequalities we get }
\]
\[
\tilde{\mu}_{\text{Apr}^A}(x \lor y) \geq \alpha, \quad \tilde{\nu}_{\text{Apr}^A}(x \lor y) \leq \beta, \quad \tilde{\mu}_{\text{Apr}^A}(x \land y) \geq \alpha \quad \text{and} \quad \tilde{\nu}_{\text{Apr}^A}(x \land y) \leq \beta.
\]

Since \( x \lor y, x \land y \in \text{Apr}^X \),
\[
\tilde{\mu}_{\text{Apr}^A}(x \lor y) = \mu_{\text{Apr}^A}(x \lor y), \quad \tilde{\mu}_{\text{Apr}^A}(x \land y) = \mu_{\text{Apr}^A}(x \land y), \quad \text{and}
\]
\[
\tilde{\nu}_{\text{Apr}^A}(x \lor y) = \nu_{\text{Apr}^A}(x \lor y), \quad \tilde{\nu}_{\text{Apr}^A}(x \land y) = \nu_{\text{Apr}^A}(x \land y).\quad \text{Hence}
\]
\[
\mu_{\text{Apr}^A}(x \lor y) \geq \alpha, \quad \mu_{\text{Apr}^A}(x \land y) \geq \alpha, \quad \nu_{\text{Apr}^A}(x \lor y) \leq \beta \text{ and } \nu_{\text{Apr}^A}(x \land y) \leq \beta.
\]

So that \( x \lor y \) and \( x \land y \in A(\alpha, \beta) \). Thus \( A(\alpha, \beta) \) is a sublattice of \( L \). Similarly, let \( x, y \in A(\alpha, \beta) \). Then
Thus \( \min \{ \mu_A(x), \mu_A(y) \} \geq [0, \alpha] \), \( \max \{ \nu_A(x), \nu_A(y) \} \leq [\beta, 1] \). Since \( \text{Apr}_I(A) \) is fuzzy rough sublattice of \( \text{Apr}_I(X) \),

\[
\tilde{\mu}_A(x \lor y) \geq \min \{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \} \geq [0, \alpha],
\]

\[
\tilde{\mu}_A(x \land y) \geq \min \{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \} \geq [0, \alpha],
\]

\[
\tilde{\nu}_A(x \lor y) \leq \max \{ \tilde{\nu}_A(x), \tilde{\nu}_A(y) \} \leq [\beta, 1],
\]

\[
\tilde{\nu}_A(x \land y) \leq \max \{ \tilde{\nu}_A(x), \tilde{\nu}_A(y) \} \leq [\beta, 1].
\]

Hence

\[
\mu_{\text{Apr}_I(A)}(x \lor y) \geq \alpha, \quad \mu_{\text{Apr}_I(A)}(x \land y) \geq \alpha, \quad \nu_{\text{Apr}_I(A)}(x \lor y) \leq \beta \quad \text{and} \quad \nu_{\text{Apr}_I(A)}(x \land y) \leq \beta.
\]

Therefore \( x \lor y \) and \( x \land y \in \overline{A}_{(\alpha, \beta)} \). Thus \( \overline{A}_{(\alpha, \beta)} \) is a sublattice of \( L \). Consequently, \( (\overline{A}_{(\alpha, \beta)}, \overline{A}_{(\alpha, \beta)}) \) is a rough sublattice of \( \text{Apr}(X) \).