SECTION-THREE

INTEGRAL EQUATIONS
CHAPTER VII

INTEGRAL EQUATIONS INVOLVING THE GENERALIZED BATEMAN FUNCTIONS AS KERNELS

7.1. SUMMARY

This chapter gives two integral equations involving the generalized Bateman function as the kernel. They are solved by using the Laplace transformation.

Special cases include the result given by Nair [5, p.132].

7.2. NOTATIONS AND RESULTS USED

The Laplace transform

\[ F(p) = \int_{0}^{\infty} e^{-pt} f(t) dt, \quad \text{Re}(p) > 0 \]  \hspace{0.5cm} (7.2.1)

is represented by \( F(p) \triangleq f(t) \).

Erdelyi [1]

If \( F(p) \triangleq f(t) \) then

\[ e^{-at} f(t) \triangleq F(p + a). \] \hspace{0.5cm} (7.2.2)

If \( f(t) \triangleq F(p), f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0 \) and \( f^{(n)}(t) \) is continuous then

\[ f^{(n)}(t) \triangleq p^n F(p) \] \hspace{0.5cm} (7.2.3)

If \( f_1(t) \triangleq F_1(p) \) and \( f_2(t) \triangleq F_2(p) \), then


Theorem.2 was accepted for publication in the “Applied Science Periodical”.

\[ \int_0^t f_1(u) f_2(t-u) \, du = F_1(p) F_2(p) \]  
(7.2.4)

If \( f(t) \) is a continuous function of \( t \) and \( f(t) = g(p) \), the integral

\[ \int_0^\infty e^{-pt} t^n f(t) \, dt \]  
Converges,

then

\[ t^n f(t) \equiv \left( -\frac{d}{dp} \right)^n g(p) \]  
(7.2.5)

\[ K_{2n+2}(t) \equiv 2(1-p)^n (1+p)^{-n-2}, \]  
(7.2.6)

provided \( \text{Re}(p) > 0 \).

Gupta [3, p.245]

If \( f(t) = F(p), g(t) = G(p) \) then

\[ F(t)G(t) \equiv \int_0^\infty f(x) g(p+x) \, dx \]  
(7.2.7)

Srivastava [6]

\[ K_{2n}^{2m,2l}(x) \equiv 2(-1)^n p^{2m} (1-p)^{n-l-m-1} (1+p)^{-(n+l+m+1)} \]  
(7.2.8)

provided \( \text{Re}(p) > 0, (n-1) \geq 0 \) and \( (n-l-m-1) > 0 \),

where \( K_{2n}^{2m,2l} \) is the generalized Bateman function.

Gupta [2, p.280]

\[ K_{2n}^{2l}(a \, t) \equiv (-1)^{n-l-1} (2a)^{2l+1} (p-a)^{n-l-1} (p+a)^{-(n+l+1)} \]  
(7.2.9)

provided \( \text{Re}(p) > 0, (n-1) \geq 0 \) and \( (n-l-m-1) > 0 \).
THEOREM 1

Each of the integral equations

\[ f(t) = A \int_0^t \left\{ \frac{(a + \beta + D)^{n+l+m+1-r} \ (b + \alpha + D)^{g+h+c+1-s}}{(D-a+\beta)^{n-l-m-1-\rho} \ (D-b+\alpha)^{g-h-c-1-\sigma}} \right\} \]

\[ \frac{1}{(\beta + D)^{2m-\lambda} \ (\alpha + D)^{2c-\mu}} \ g(u) \}
\]

\[ \times \ e^{-\alpha(t-u)} K^{2c,2h}_{2g} \left[ b(t-u) \right] \ du \]  \hspace{1cm} (7.3.1)

and

\[ g(t) = B \int_0^t \left\{ \frac{(a + \beta + D)^{r} \ (b + \alpha + D)^{s}}{(D-a+\beta)^{\rho} \ (D-b+\alpha)^{\sigma}} \right\} \]

\[ \frac{1}{(\beta + D)^{\lambda} \ (\alpha + D)^{\mu}} \ f(u) \}
\]

\[ \times \ e^{-\beta(t-u)} K^{2m,2l}_{2n} \left[ a(t-u) \right] \ du \]  \hspace{1cm} (7.3.2)

is the solution of the other, provided

(i) \( f(0) = f^1(0) = \ldots = f^{i-1}(0) = 0, \ f^{(i)}(t) \) is continuous

(ii) \( g(0) = g^1(0) = \ldots = g^{j-1}(0) = 0, \ g^{(j)}(t) \) is continuous

(iii) \( m, n, l, r, g, h, c, s, \rho, \sigma, \lambda, \mu \) are all integers.

(iv) \( (n-1) \geq 0 \quad \text{and} \quad (n-l-m-1) > 0 \)

(v) \( (g-1) \geq 0 \quad \text{and} \quad (g-h-c-1) > 0 \)
(vi) \( \text{Re}(p) > 0 \), \( A = \frac{1}{(-1)^{d-h-1} 2(b)^{2h+1}} \)

and

\[ B = \frac{1}{(-1)^{n-l-1} 2(a)^{2l+1}} \]

(vii) \( i \) is the number of times \( f(u) \) is differentiated in (7.3.2), \( j \) is the number of times \( g(u) \) is differentiated in (7.3.1)

(viii) \( D = \frac{d}{du}, \quad \frac{1}{D} f(u) = \int_{0}^{u} f(u) du \)

and

\[ \frac{1}{D + a} f(u) = e^{-au} \int_{0}^{u} e^{au} f(u) du. \]

(ix) \( K_{p}^{m,n} \) is the generalized Bateman function.

**PROOF:**

Using (7.2.2), (7.2.3), (7.2.4) and (7.2.8) the first integral equation (7.3.1) becomes

\[
F(p) = (a + \beta + p)^{n+l+m+1-r} (b + \alpha + p)^{-s} (p + \alpha)^{\mu} (p + \beta)^{-2m+\lambda} (p + \alpha - b)^{\sigma} \\
\times (p + \beta - a)^{-n+m+l+1+\rho} G(p)
\]

(7.3.3)

Similarly the second integral equation (7.3.2) gives

\[
G(p) = (a + \beta + p)^{d+n+l+m+1+r} (b + \alpha + p)^{s} (p + \alpha)^{-\mu} (p + \beta)^{2m-\lambda} (p + \alpha - b)^{-\sigma} \\
\times (p + \beta - a)^{n-m-l-1-\rho} F(p)
\]

(7.3.4)
The equations (7.3.3) and (7.3.4) can be obtained from each other. Hence by Lerch’s theorem [4, p.5] it follows that each of the integral equations (7.3.1) and (7.3.2) is the solution of the other.

SPECIAL CASES

In the theorem put \( c = m = \lambda = \mu = 0 \) and using (7.2.9) to get:

**COROLLARY. 1**

Each of the integral equations

\[
f(t) = A \int_0^t \left\{ \frac{(a + \beta + D)^{n+l+1-r} (b + \alpha + D)^{g+h+1-s}}{(D-a+\beta)^{n-l-1-\rho} (D-b+\alpha)^{g-h-1-\sigma}} g(u) \right\}
\]

\[\times e^{-\alpha(t-u)} K_{2g}^{2h} \left[ b(t-u) \right] du \tag{7.3.5}\]

and

\[
g(t) = B \int_0^t \left\{ \frac{(a + \beta + D)^r (b + \alpha + D)^s}{(D-a+\beta)^{\rho} (D-b+\alpha)^{\sigma}} f(u) \right\}
\]

\[\times e^{-\beta(t-u)} K_{2l}^{2l} \left[ a(t-u) \right] du \tag{7.3.6}\]

is the solution of other, provided that the conditions are similar to that of the Theorem.1.

When \( c = m = \lambda = \mu = h = l = 0 \), the theorem reduces to:
COROLLARY. 2

Each of the integral equations

\[ f(t) = A \int_0^t \left\{ \frac{(a + \beta + D )^{n+1-r} (b + \alpha + D )^{g+1-s}}{(D - a + \beta)^{n-1-\rho} (D - b + \alpha)^{s-1-\sigma}} g(u) \right\} \]

\times e^{-\alpha(t-u)} K_{2g} [b(t-u)] \, du \quad (7.3.7) \]

and

\[ g(t) = B \int_0^t \left\{ \frac{(a + \beta + D )^r (b + \alpha + D )^s}{(D - a + \beta)^{\rho} (D - b + \alpha)^{\sigma}} f(u) \right\} \]

\times e^{-\beta(t-u)} K_{2n} [a(t-u)] \, du \quad (7.3.8) \]

is the solution of the other provided that the conditions are similar to that of Theorem.1 with

\[ A = \frac{1}{2b(-1)^{g-1}} \quad \text{and} \quad B = \frac{1}{2\alpha(-1)^{n-1}} . \]

Replacing \( g \) by \((g+1)\) and \( n \) by \( n+1 \), Corollary.2 reduces to:

COROLLARY. 3

Each of the integral equations

\[ f(t) = A \int_0^t \left\{ \frac{(a + \beta + D )^{n+2-r} (b + \alpha + D )^{g+2-s}}{(D - a + \beta)^{n-\rho} (D - b + \alpha)^{s-\sigma}} g(u) \right\} \]

\times e^{-\alpha(t-u)} K_{2g+2} [b(t-u)] \, du \quad (7.3.9) \]

and
\[ g(t) = B \int_0^t \left\{ \frac{(a + \beta + D)^r (b + \alpha + D)^s}{(D - a + \beta)^\rho (D - b + \alpha)^\sigma} f(u) \right\} \]

\[ \times e^{-\beta(t-u)} K_{2n+2} \left[ a(t-u) \right] du \quad (7.3.10) \]

is the solution of the other provided that the conditions are similar to that of Theorem 1 with

\[ A = \frac{1}{2b(-1)^g} \quad \text{and} \quad B = \frac{1}{2a(-1)^n}. \]

Corollary 3 agrees with the theorem proved by Nair [5, p.132].

**THEOREM 2**

Each of the integral equations

\[ F(p) = A \int_0^\infty \left\{ \frac{(a + \beta - D)^{n \pm l + m + 1 - r} (b + \alpha - D)^{g + h + c + 1 - s}}{(-a + \beta - D)^{n \pm l - m - 1 - \rho} (-b + \alpha - D)^{g - h - c - 1 - \sigma}} \right\} \]

\[ \frac{1}{(\beta - D)^{2m - \lambda} (\alpha - D)^{2c - \mu}} G(p + x) \left\} \right. \]

\[ \times e^{-\alpha x} K_{2g}^{2c,2h} \left[ b \ x \right] dx \quad (7.3.11) \]

and

\[ G(p) = B \int_0^\infty \left\{ \frac{(a + \beta - D)^r (b + \alpha - D)^s}{(-a + \beta - D)^\rho (-b + \alpha - D)^\sigma} \right\} \frac{1}{(\beta - D)^2 (\alpha - D)^\mu} \]

\[ F(p + x) \left\} \right. \]

\[ \times e^{-\beta x} K_{2n}^{2m,2l} \left[ a \ x \right] dx \quad (7.3.12) \]
is the solution of the other, provided

(i) The integrals are convergent.

(ii) \( m, n, l, r, g, h, c, s, \rho, \sigma, \lambda, \mu \) are all integers.

(iii) \((n - 1) \geq 0\) and \((n - l - m - 1) > 0\)

(iv) \((g - 1) \geq 0\) and \((g - h - c - 1) > 0\)

(v) \(\text{Re}(p) > 0\), \( A = \frac{1}{(-1)^{g-h-1}2(b)^{2h+1}} \)

and

\( B = \frac{1}{(-1)^{n-l-1}2(\alpha)^{2l+1}} \)

(vi) \( D \) represents differentiation with respect to \((p + x)\).

(vii) \( K_{m,n}^{p} \) is the generalized Bateman function.

**PROOF:**

Using (7.2.2), (7.2.3), (7.2.5), (7.2.7) and (7.2.8), the first integral equation (7.3.11) becomes

\[
f(t) = (a + \beta + t)^{n+m+1+r}(b + \alpha + t)^{-s}(t + \alpha)^{\mu}(t + \beta)^{-2m+\lambda}(t + \alpha - b)^{\sigma}
\]

\[
\times (t + \beta - a)^{-n+m+l+1+\rho}g(t)
\]

Similarly the second integral equation (7.3.12) gives

\[
g(t) = (a + \beta + t)^{-n+m+1+r}(b + \alpha + t)^{s}(t + \alpha)^{-\mu}(t + \beta)^{2m-\lambda}(t + \alpha - b)^{-\sigma}
\]

\[
\times (t + \beta - a)^{n-l-m-1-\rho}f(t)
\]
The equations (7.3.13) and (7.3.14) can be obtained from each other. Hence by Lerch’s theorem [4, p.5] it follows that each of the integral equations (7.3.11) and (7.3.12) is the solution of the other.

SPECIAL CASES

In the theorem put \( c = m = \lambda = \mu = 0 \) and using (7.2.9) to get:

**COROLLARY. 4**

Each of the integral equations

\[
F(p) = A \int_0^\infty \left\{ \frac{(a + \beta - D)^{n+l-1-r} (b + \alpha - D)^{g+h+s}}{(-a + \beta - D)^{n-l-1-r} (-b + \alpha - D)^{g-h-1-s}} \right\} G(p + x) \left\{ \frac{e^{-\alpha x} K^{2h}_{2g} [b \ x]}{x} \right\} \ dx
\]

and

\[
G(p) = B \int_0^\infty \left\{ \frac{(a + \beta - D)^{r} (b + \alpha - D)^{s}}{(-a + \beta - D)^{\rho} (-b + \alpha - D)^{\sigma}} \right\} F(p + x) \left\{ \frac{e^{-\beta x} K^{2l}_{2n} [a \ x]}{x} \right\} \ dx
\]

is the solution of other provided that the conditions are similar to that of Theorem.2 with \( c = m = \lambda = \mu = 0 \).

When \( c = m = \lambda = \mu = h = l = 0 \) the Theorem reduces to:

**COROLLARY. 5**

Each of the integral equations
\[ F(p) = A \int_0^\infty \left\{ \frac{(a + \beta - D)^{n+1-r} (b + \alpha - D)^{x+1-s}}{(-a + \beta - D)^{n-1-p} (-b + \alpha - D)^{x-1-\sigma}} \right\} G(p + x) \, dx \]
\[ \times e^{-\alpha x} K_{2g} [b, x] \, dx \quad (7.3.17) \]

and
\[ G(p) = B \int_0^\infty \left\{ \frac{(a + \beta - D)^{r} (b + \alpha - D)^{x}}{(-a + \beta - D)^{p} (-b + \alpha - D)^{x}} \right\} F(p + x) \, dx \]
\[ \times e^{-\beta x} K_{2n} [a, x] \, dx \quad (7.3.18) \]

is the solution of the other provided that the conditions are similar to that of Theorem 2 with 
\[ A = \frac{1}{2b(-1)^{g-1}} \quad \text{and} \quad B = \frac{1}{2\alpha(-1)^{n-1}} . \]
REFERENCES


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