CHAPTER - IV

ON GENERALISED INTEGRAL TRANSFORMS OF
ONE VARIABLE

4.1. SUMMARY

In this chapter We establish two general theorems on generalized integral transforms of one variable. A number of new and known Corollaries of these Theorems for the H-function transforms which are of interest in themselves will also be given. Several results lying scattered in the literature follow as simple special cases of our findings including the result given by Gupta [1, p.46].

4.2. NOTATIONS AND RESULTS USED

\[(a)_n = a(a + 1)(a + 2) \ldots \ldots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}, \quad n \geq 1\]

\[(a)_0 = 1, \quad a \neq 0\]

\[(a_p) = a_1, a_2, \ldots, a_p\]

\[1(a_j, A_j)_p = (a_1, A_1) \ldots \ldots (a_p, A_p)\]

The integral transform of a function \(f(x)\) belonging to a prescribed class of functions of one variable is defined and represented as follows:

\[T\{ f(x) ; \ a ; \ p ; \ \lambda \} = \int_0^\infty K \left[ a (p x)\lambda \right] f(x) dx \quad \text{(4.2.1)}\]

provided, the integral involved in (4.2.1) is absolutely convergent where \(\lambda\) is a real number.

When \(T\) is the H-function transform, (4.2.1) becomes

\[H\{ f(x) ; \ a ; \ p ; \ \lambda \}\]
\[
= \int_{0}^{\infty} H_{p,q}^{M,N} \left[ a(p,x) \right]_{1(a_j, \alpha_j)_p}^{1(b_j, \beta_j)_q} f(x) \, dx
\] (4.2.2)

provided the integral involved in (4.2.2) is absolutely convergent.

When \( T \) is the G-function transform, (4.2.1) becomes

\[
G\{ f(x) ; a ; p ; \lambda \} = \int_{0}^{\infty} G_{p,q}^{M,N} \left[ a(p,x) \right]_{1(a_j, \alpha_j)_p}^{1(b_j, \beta_j)_q} f(x) \, dx
\] (4.2.3)

provided, the integral involved in (4.2.3) is absolutely convergent.

On specializing the parameters involved in (4.2.3) it is possible to obtain various other transforms- for example: Verma transforms, Mainra transforms, Hankel transforms etc.

4.3. GENERALISED INTEGRAL TRANSFORMS OF ONE VARIABLE

**THEOREM. 1**

If \( T\{ f_1(x) ; a ; p ; \lambda \} = \phi_1(p) \)

and

\( T\{ f_2(x) ; a ; p ; \lambda \} = \phi_2(p), \)

then

\[
\int_{0}^{\infty} f_1(x) \phi_2(x) \, dx = \int_{0}^{\infty} f_2(x) \phi_1(x) \, dx
\] (4.3.1)

provided that the various integrals involved converge absolutely.
PROOF:

\[
\int_0^\infty f_1(x) \phi_2(x) \, dx = \int_0^\infty f_1(x) T\{ f_2(p) ; a ; p ; \lambda \} \, dx \\
= \int_0^\infty f_1(x) \left(\int_0^\infty K[a(px)^{\lambda}]f_2(p) \, dp\right) \, dx \\
= \int_0^\infty f_2(x) \left(\int_0^\infty K[a(px)^{\lambda}]f_1(p) \, dp\right) \, dx \\
= \int_0^\infty f_2(x) \phi_1(x) \, dx.
\]

When \( a = \lambda = 1 \), Theorem 1 reduces to:

COROLLARY. 1

If \( T \{ f_1(x) ; p \} = \phi_1(p) \) and \( T \{ f_2(x) ; p \} = \phi_2(p) \)

then

\[
\int_0^\infty f_1(x) \phi_2(x) \, dx = \int_0^\infty f_2(x) \phi_1(x) \, dx \tag{4.3.2}
\]

provided that the various integrals involved converge absolutely.

Corollary 1 agrees with the Theorem [1, p.42] which is an analogue of the well known Parseval-Goldstein Theorem.

The integral transforms \( T_1 \) and \( T_2 \) are defined below:

\[
T_1\{ f(x) ; a ; s ; \lambda \} = \int_0^\infty K_1[a(sx)^{\lambda}] \, f(x) \, dx,
\]
provided that the integrals involved converge absolutely.

**THEOREM 2**

If

\[ h_1(s^{\delta}) = T_2 \{ f(x) ; b ; s ; \mu \} = \int_{0}^{\infty} K_2[b(s x)^{\mu}] f(x) \, dx, \]

and

\[ h_2(s^{\delta/\sigma}) = T_2 \{ f(x) ; b ; s ; \mu \} = (1/\sigma) \int_{0}^{\infty} f(x) \, \phi(x, s) \, dx \]

then

\[ h_1(s^{\delta}) = (1/\sigma) \int_{0}^{\infty} f(x) \, \phi(x, s) \, dx \]

where

\[ \phi(s, \alpha) = T_2 \{ x^{(1/\sigma)-1} g(x^{1/\sigma}) K_1[a (\alpha x^{1/\sigma})^{\lambda}] ; b ; s ; \mu \} \]

provided that the integrals involved are absolutely convergent, \( \alpha \) is independent of \( s \) and \( \sigma, \delta, \lambda \) are non zero real numbers.

**PROOF:**

Applying (4.3.1) to the operational pairs (4.3.4) and (4.3.6) to get:

\[ \int_{0}^{\infty} x^{(1/\sigma)-1} g(x^{1/\sigma}) K_1[a (\alpha x^{1/\sigma})^{\lambda}] h_2(x^{\delta/\sigma}) \, dx \]

\[ = \int_{0}^{\infty} f(x) \, \phi(x, \alpha) \, dx \]
Now replacing \( \alpha \) by \( S \) in (4.3.7), changing the variable of integration on the left hand side and interpreting result thus obtained in terms of (4.3.3), the required result follows.

If in Theorem 2, take the transformations denoted by the symbols \( T_1 \) and \( T_2 \) to be \( H \)-function transforms and put \( g(x) = x^{\sigma - 1} \) there in, equation (4.3.6) reduces to the following result:

\[
\phi(s, \alpha) = \int_0^\infty x^{c-1} H\left[ \alpha \left( \alpha x^{1/\sigma} \right)^{\lambda} \right] H\left[ b (s x)^{\mu} \right] \, dx
\]

Now evaluating the integral (4.3.6) with the help of (2.3.7), replacing \( S, \alpha \) by \( x \) and \( S \) respectively, and putting the value of \( \phi(s, \alpha) \) thus obtained in (4.3.5) to get the following result:

**COROLLARY. 2**

If

\[
h_1(s^{1/\delta}) = \int_0^\infty x^{\sigma c-1} h_2(x^\delta) H^{m_1, n_1}_{p_1, q_1}\left[ \alpha (sx)^{\lambda} \right]_{1(c_j, C_j)_{p_1}}{1(d_j, D_j)_{q_1}} \, dx
\]

(4.3.8)

and

\[
h_2(s^{\delta/\sigma}) = \int_0^\infty H^{m_2, n_2}_{p_2, q_2}\left[ b (sx)^{\mu} \right]_{1(e_j, E_j)_{p_2}}{1(f_j, F_j)_{q_2}} f(x) \, dx,
\]

(4.3.9)

then
\[ h_1(s^{1/\delta}) = \rho \int_0^\infty x^{-c} f(x) \left[ \xi s^{2 \lambda} x^{-\eta} \right] \bigg|_{1(c_j, C_j)_n}^{1(d_j, D_j)_m} \left[ 1(1 - f_j - (c / \mu)F_j, (\eta / \mu)F_j)_{q_2}, n_{i+1}(c_j, C_j)_{p_1}, \right. \\
\left. 1(1 - e_j - (c / \mu)E_j, (\eta / \mu)E_j)_{p_2}, m_{i+1}(d_j, D_j)_{q_1} \right] dx \]

(4.3.10)

provided that

i) \( \eta = \frac{\lambda}{\sigma}, \ \rho = \frac{b^{-(c / \mu)}}{\mu}, \ \xi = a b^{-(\eta / \mu)}, \)

\( \sigma > 0, \delta > 0, \mu > 0, \lambda > 0 \)

ii) the H-function transform defined by (4.3.9) of \(| f(x) | \) exists.

iii) \( V_1 < 0, V_2 < 0, \Delta_1 > 0, \Delta_2 > 0, \)

\[ | \arg(\alpha s^\lambda) | < (1/2)\pi \Delta_1, | \arg(\beta s^\mu) | < (1/2)\pi \Delta_2, \]

where \( V_1, V_2, \Delta_1 \) and \( \Delta_2 \) are defined by (2.3.8), (2.3.9), (2.3.10) and (2.3.11) respectively.

iv) \( \Re \left[ c + \frac{\mu f_j}{F_j} + \frac{\eta d_i}{D_i} \right] > 0 \) for \( i = 1, \ldots, m_1, \ j = 1, \ldots, m_2, \)

\[ \Re \left[ c + \frac{\mu (e_j - 1)}{E_j} + \frac{\eta (c_i - 1)}{C_i} \right] < 0 \) for \( i = 1, \ldots, n_1, \ j = 1, \ldots, n_2. \)

and

v) the integral (4.3.10) converges absolutely.

When \( \delta = 1, \) Corollary.2 reduces to:
COROLLARY. 3

If

\[ h_1(s) = \int_0^\infty x^\sigma c^{-1} h_2(x) H_{p_1,q_1}^{m_1,n_1} a (sx)^\lambda \left| \begin{array}{c} 1(c_j, C_j)_{p_1} \\ 1(d_j, D_j)_{q_1} \end{array} \right| dx \]

(4.3.11)

and

\[ h_2(s^{1/\sigma}) = \int_0^\infty H_{p_2,q_2}^{m_2,n_2} b (sx)^\mu \left| \begin{array}{c} 1(e_j, E_j)_{p_2} \\ 1(f_j, F_j)_{q_2} \end{array} \right| f(x)dx \]

(4.3.12)

then

\[ h_1(s) = \rho \int_0^\infty x^{-c} f(x) H_{p_1,q_1}^{m_1+n_2,n_1+m_2} \left[ s^\lambda \xi \right] H_{p_2,q_2}^{m_2,n_2} a (sx)^\lambda \left| \begin{array}{c} 1(c_j, C_j)_{p_2} \\ 1(d_j, D_j)_{q_2} \end{array} \right| dx \]

(4.3.13)

provided \( \eta = \frac{\lambda}{\sigma} \), \( \rho = \frac{b^{-(c/\mu)}}{\mu} \), \( \xi = a b^{-(\eta/\mu)} \), \( \sigma > 0, \mu > 0, \lambda > 0 \) and the conditions similar to Corollary. 2 are satisfied with \( \delta = 1 \).

When \( a = b = \lambda = \mu = 1 \), and replacing \( \sigma \) by \( 1/\sigma \), Corollary. 3 reduces to the result given by Gupta [1, p.46].

When \( \sigma = \delta = 1 \), Corollary.2 reduces to:
COROLLARY. 4

If

\[ h_1(s) = \int_0^\infty x^c x^{-1} h_2(x) \ H_{p_1, q_1}^{m_1 n_1} \left[ a(sx)^\lambda \right] \frac{(cj, C_j)^{p_1}}{(ij, D_j)^{q_1}} \ dx \]

(4.3.14)

and

\[ h_2(s) = \int_0^\infty H_{p_2, q_2}^{m_2 n_2} \left[ b(sx)^\mu \right] \frac{(ej, E_j)^{p_2}}{(fj, F_j)^{q_2}} \ f(x) \ dx \]

(4.3.15)

then

\[ h_1(s) = \rho \int_0^\infty x^c f(x) H_{p_1+q_2, q_1+p_2}^{m_1+n_2, n_1+m_2} \left[ \xi x^{-\lambda} \right] \frac{(cj, C_j)^{n_1}}{(ij, D_j)^{m_1}} \]

\[ \frac{1 - f_j - (c / \mu) F_j, (\lambda / \mu) F_j; q_2, n_1+1 (c_j, C_j)^{p_1}}{1 - e_j - (c / \mu) E_j, (\lambda / \mu) E_j; p_2, m_1+1 (d_j, D_j)^{q_1}} \ dx \]

(4.3.16)

provided \( \rho = b^{-(c / \mu)} \), \( \xi = a \ b^{-(\lambda / \mu)} \), \( \mu > 0, \lambda > 0 \) and the conditions similar to Corollary. 2 are satisfied with \( \sigma = \delta = 1 \).

COROLLARY. 5

If

\[ h_1(s^{1/\delta}) = \int_0^\infty x^{-\sigma c -1} h_2(x^\delta) \ H_{p_1, q_1}^{m_1 n_1} \left[ a(sx)^\lambda \right] \frac{(cj, C_j)^{p_1}}{(ij, D_j)^{q_1}} \ dx \]

(4.3.17)
and

\[ h_2(s^{-\delta/\sigma}) = \int_0^\infty H^{m_2,n_2}_{p_2,q_2} \left[ b(sx)^\mu \begin{pmatrix} 1(e_j, E_j)_{p_2} \\ 1(f_j, F_j)_{q_2} \end{pmatrix} \right] f(x) dx, \]

(4.3.18)

then

\[ h_1(s^{1/\delta}) = \rho \int_0^\infty x^{-c} f(x) H^{m_1+m_2,n_1+n_2}_{p_1+p_2,q_1+q_2} \left[ \xi_1 s^\lambda x^\eta \begin{pmatrix} 1(c_j, C_j)_{n_1} \\ 1(d_j, D_j)_{m_1} \end{pmatrix} \right] dx \]

(4.3.19)

provided that

i) \( \eta = \frac{\lambda}{\sigma}, \rho = \frac{b^{-(c/\mu)}}{\mu}, \xi_1 = a b^{\eta/\mu}, \sigma > 0, \delta > 0, \mu > 0, \lambda > 0 \)

ii) the H-function transform defined by (4.3.18) of \( |f(x)| \) exists.

iii) \( V_1 < 0, V_2 < 0, \Delta_1 > 0, \Delta_2 > 0, \)

\[ |\arg(as^\lambda)| < (1/2)\pi \Delta_1, |\arg(bs^\mu)| < (1/2)\pi \Delta_2, \]

where \( V_1, V_2, \Delta_1 \) and \( \Delta_2 \) are defined by (2.3.8), (2.3.9), (2.3.10) and (2.3.11) respectively.

iv) \( \text{Re} \left[ c + \frac{\mu f_j}{F_j} + \frac{\eta (1 - c_j)}{C_i} \right] > 0 \) for \( i = 1,2,\ldots,n_1, j = 1,2,\ldots,m_2, \)

\[ \text{Re} \left[ c + \frac{\mu (e_j - 1)}{E_j} - \frac{\eta (c_i - 1)}{C_i} \right] < 0 \] for \( i = 1,2,\ldots,m_1, j = 1,2,\ldots,n_2 \)
and

v) the integral (4.3.19) converges absolutely.

Proof is similar to that of Corollary. 2, on using (2.3.21).

When \( \delta = 1 \), Corollary. 5 reduces to:

**COROLLARY. 6**

If

\[
h_1(s) = \int_0^\infty x^{-\sigma} \left( 1 - h_2(x) H_{p_1,q_1}^{m_1,n_1} \left[ a (sx)^\lambda \right] \right) \left[ 1(h_j,C_j)_{p_1} \right] \, dx
\]

(4.3.20)

and

\[
h_2(s^{-1/\sigma}) = \int_0^\infty H_{p_2,q_2}^{m_2,n_2} \left[ b (sx)^\mu \right] \left[ 1(h_j,E_j)_{p_2} \right] \left[ 1(f_j,F_j)_{q_2} \right] f(x) \, dx,
\]

(4.3.21)

then

\[
h_1(s) = \rho \int_0^\infty x^{-c} f(x) H_{p_1+p_2,q_1+q_2}^{m_1+m_2,n_1+n_2} \left[ \xi_1 s^\lambda \right] \left[ 1(h_j,C_j)_{n_1} \right] \left[ 1(d_j,D_j)_{m_1} \right] \, dx
\]

(4.3.22)

provided \( \eta = \frac{\lambda}{\sigma} \), \( \rho = \frac{b^{-(c/\mu)}}{\mu} \), \( \xi_1 = a b^{\eta/\mu} \), \( \sigma > 0, \mu > 0, \lambda > 0 \)

and the conditions similar to Corollary.5 are satisfied with \( \delta = 1 \).
When $\sigma = \delta = 1$, Corollary 5 reduces to:

**COROLLARY. 7**

If

$$h_1(s) = \int_0^\infty x^{-c} h_2(x) H_{p_1,q_1}^{m_1,n_1} \left[ a \left( sx \right)^{\lambda} \begin{bmatrix} 1(c_j, C_j)_{p_1} \\ 1(d_j, D_j)_{q_1} \end{bmatrix} \right] dx$$

(4.3.23)

and

$$h_2(s^{-1}) = \int_0^\infty H_{p_2,q_2}^{m_2,n_2}(sx)^{\mu} \begin{bmatrix} 1(e_j, E_j)_{p_2} \\ 1(f_j, F_j)_{q_2} \end{bmatrix} f(x) dx,$$

(4.3.24)

then

$$h_1(s) = \rho \int_0^\infty x^{-c} f(x) H_{p_1+p_2,q_1+q_2}^{m_1+m_2,n_1+n_2} \left[ \xi_1 s^{\lambda} x^{\lambda} \begin{bmatrix} 1(c_j, C_j)_{n_1} \\ 1(d_j, D_j)_{m_1} \end{bmatrix} \right]$$

$$+ (c / \mu) E_j, (\lambda / \mu) E_j)_{p_2}, n_{i+1}(c_j, C_j)_{p_1} \\ + (c / \mu) F_j, (\lambda / \mu) F_j)_{q_2}, m_{i+1}(d_j, D_j)_{q_2} \right] dx$$

(4.3.25)

provided

$$\rho = \frac{b^{-(c / \mu)}}{\mu}, \quad \xi_1 = a \cdot b^{\lambda / \mu}, \quad \mu > 0, \lambda > 0$$

and the conditions similar to Corollary 5 are satisfied with $\sigma = \delta = 1$. 
COROLLARY. 8

If

\[ h_1(s^{1/\delta}) = \int_0^\infty x^{-c} h_2(x^\delta) \int_0^\infty H_{p_1,q_1}^{m_1,n_1} \left[ a (sx)^{\lambda} \right] \left[ 1(c_j,C_j)_{p_1} \right] dx \]

and

\[ h_2(s^{\delta/\sigma}) = \int_0^\infty H_{p_2,q_2}^{m_2,n_2} \left[ b (sx)^{\mu} \right] \left[ 1(e_j,E_j)_{p_2} \right] f(x) dx, \]

then

\[ h_1(s^{1/\delta}) = \rho \int_0^\infty x^{\xi} f(x) \int_0^\infty H_{p_1+q_2,q_1+p_2}^{m_1+n_2,n_1+m_2} \left[ \xi^{s-\eta} \right] \left[ 1(c_j,C_j)_{n_1} \right] \left[ 1(d_j,D_j)_{m_1} \right] dx \]

provided that

i) \( \eta = \frac{\lambda}{\sigma}, \quad \rho = \frac{b^{c/\mu}}{\mu}, \quad \xi = a b^{-\left(\eta/\mu\right)}, \quad \sigma > 0, \delta > 0, \mu > 0, \lambda > 0 \)

ii) the H-function transform defined by (4.3.27) of \( |f(x)| \) exists.

iii) \( V_1 < 0, V_2 < 0, \Delta_1 > 0, \Delta_2 > 0, \)

\[ |\arg(a s^{\lambda})| < (1/2)\pi \Delta_1, \quad |\arg(b s^{\mu})| < (1/2)\pi \Delta_2 \]

where \( V_1, V_2, \Delta_1 \) and \( \Delta_2 \) are defined by (2.3.8), (2.3.9), (2.3.10) and (2.3.11)
respectively.

iv) \( \text{Re} \left[ c - \frac{\mu f_j}{F_j} - \frac{\eta d_i}{D_i} \right] < 0 \) for \( i = 1,2,\ldots,m_2, \ j = 1,2,\ldots,m_1, \)

\[ \text{Re} \left[ c + \frac{\mu (1 - e_j)}{E_j} + \frac{\eta (1 - c_i)}{C_i} \right] > 0 \) for \( i = 1,2,\ldots,n_2, \ j = 1,2,\ldots,n_1. \)

and

v) the integral (4.3.28) converges absolutely.

The proof is similar to that of Corollary 2, on using (2.3.7).

When \( \delta = 1, \) Corollary 8 reduces to:

**COROLLARY 9**

If

\[ h_1(s) = \int_0^\infty x^{-\sigma} h_2(x) H_{p_1,q_1}^{m_1,n_1} \left[ a (sx) \right] \left[ \left( c_j, C_j \right)_{p_1} \right] dx \]

(4.3.29)

and

\[ h_2(s^{1/\sigma}) = \int_0^\infty H_{p_2,q_2}^{m_2,n_2} \left[ b (sx)^\mu \right] \left[ \left( e_j, E_j \right)_{p_2} \right] f(x) dx, \]

(4.3.30)

then

\[ h_1(s) = \rho \int_0^\infty f(x) H_{p_1+q_2,q_1+p_2}^{m_1+n_2,n_1+m_2} \left[ \xi s^{\lambda} x^{-\eta} \right] \left[ \left( c_j, C_j \right)_{n_1} \right] \left[ \left( d_j, D_j \right)_{m_1} \right] dx, \]
\begin{equation}
1(1 - f_j + (c / \mu) F_j, (\eta / \mu) F_j)_{q_2}, \ n_{i+1}(c_j, C_j)_{p_i} \right) dx
\end{equation}

\begin{equation}
1(1 - e_j + (c / \mu) E_j, (\eta / \mu) E_j)_{p_2}, \ m_{i+1}(d_j, D_j)_{q_i} \right) dx
\end{equation}

provided \( \eta = \frac{\lambda}{\sigma}, \ \rho = \frac{b^{c/\mu}}{\mu}, \ \zeta = a b^{-(\eta/\mu)}, \ \sigma > 0, \mu > 0, \lambda > 0 \)

and the conditions similar to Corollary 8 are satisfied with \( \delta = 1 \).

When \( \sigma = \delta = 1 \), Corollary 8 reduces to:

**COROLLARY. 10**

If

\begin{equation}
h_1(s) = \int_0^\infty x^{-c-1} h_2(x) H^{m_i+n_i}_{p_i,q_i} \left[ a(sx) s^{\lambda} \right] 1(c_j, C_j)_{p_i} \right) dx
\end{equation}

and

\begin{equation}
h_2(s) = \int_0^\infty H^{m_2+n_2}_{p_2,q_2} \left[ b(sx) \right] 1(e_j, E_j)_{p_2} \right) f(x) \ dx
\end{equation}

then

\begin{equation}
h_1(s) = \rho \int_0^\infty x^{c} f(x) H^{m_1+n_2+n_1+m_2}_{p_1+q_2+q_i+p_2} \left[ \zeta \ s^{\lambda} x^{-\lambda} \right] 1(c_j, C_j)_{n_i}, \ \right) 1(d_j, D_j)_{m_i},
\end{equation}

\begin{equation}
1(1 - f_j + (c / \mu) F_j, (\lambda / \mu) F_j)_{q_2}, \ n_{i+1}(c_j, C_j)_{p_i} \right) dx
\end{equation}

\begin{equation}
1(1 - e_j + (c / \mu) E_j, (\lambda / \mu) E_j)_{p_2}, \ m_{i+1}(d_j, D_j)_{q_i} \right) dx
\end{equation}
provided \( \rho = \frac{b^{\lambda / \mu}}{\mu} \) , \( \xi = a \ b^{-\left(\lambda / \mu\right)} \) , \( \mu > 0, \lambda > 0 \) and the conditions similar to Corollary. 8 are satisfied with \( \sigma = \delta = 1. \)
REFERENCES


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