CHAPTER X
HAAR WAVELET SOLUTION
FOR TRAVELING WAVE
EQUATIONS-A PLAU成果IBLE
SOLUTION FOR LIGHTNING
STROKE PROBLEMS‡

‡ The part of this chapter has been accepted as “Haar wavelet solution for traveling wave
Chapter 10

Haar wavelet solution for traveling wave equations-A plausible solution for lightning stroke problems

10.1 Introduction

Any lightning model is an approximate mathematical construct designed to reproduce certain aspects of the physical processes involved in the lightning discharge. The basic assumptions of the model should be consistent with both the expected outputs of the model and the availability of quantities required as inputs to the model.

Knowledge of the characteristics of electric and magnetic fields produced by lightning discharges is needed for studying the effects of the potentially deleterious
coupling of lightning fields to various circuits and systems. Sensitive electronic circuits are particularly vulnerable to such effects. The computation of lightning electric and magnetic fields requires the use of a model that specifies current as a function of time at all points along the radiating lightning channel. The computed fields can be used as an input to electromagnetic coupling models, the latter, in turn, being used for the calculation of lightning induced voltages and currents in various circuits and systems.

Lightning is a momentary, atmospheric, transient, high current electrical discharge whose path length is measured in kilometers from clouds to earth. It carries high voltage current via a huge arc to ground. For many years, a lot of studies have been made on this subject. Some have been concerned simply with collecting statistics, some with making measurements, and others have tried to probe more deeply into the physical nature of the problems. From their efforts we can understand more about this mysterious process. It is now generally accepted that a typical lightning stroke begins with the propagation of a negatively charged channel, called a stepped leader, from cloud to the ground. But before this downward leader reaches the ground, an upward leader begins to proceed from the ground and meets the downward-moving leader at the junction point. Once a stepped leader has established a connection to earth, the so-called return stoke moves swiftly up the ionized channel prepared by the stepped leader like a traveling wave on a high-voltage transmission line and a heavy current occurs. However, the physical models derived from the experimental data or from the information determined directly from experimental data have often been obtained more on the basis of intuition than on the basis of detailed quantitative analysis. They are all well documented in the literature [[98],[158]].
The weather mapping system is used to detect lightning discharges and display them graphically to the pilot. Lightning produces an electromagnetic (EM) field by stripping electrons from atoms in the air. This process emits a broad spectrum of electromagnetic energy as well as a great deal of light and sound. The process starts with transient collisions of ice crystals with riming graupel pellets thus transferring charges within the maturing cloud as the heavier (more negative) particles fall and resulting in a vertical electric field. The net effect of this self-propagating lightning is the transfer of a negative charge from the atmosphere to the earth (Cloud to Ground). When the stepped leader hits the ground, the return stroke is triggered, producing a sharp voltage rise. This specific signature distinguishes a cloud-to-ground stroke from other electromagnetic noise.

The lightning return stroke is a shock of hot air produced by the electric discharge of a lightning event when it touches the ground. It involves a complex mix of compressible hydrodynamics and electrodynamics, chemistry, plasma physics, and radiative transfer that is not widely understood (Lowke 2004; Cooray 2003). This phenomena has been studied in the past, for instance in Plooster (1971), Paxton et al. (1986), and is not reviewed here (see, for instance, Cooray 2003; Lowke (2004)). The lightning return stroke model we have developed is fully described by Zinn et al. (2006), where other numerical experiments are also conducted. The shock propagation and its radiation have also been studied in Ripoll et al. (2006).
Beginning from 1980’s, wavelets have been used for solution of partial differential equations (PDE). The good features of this approach are possibilities to detect singularities, irregular structure and transient phenomena exhibited by the analyzed equations. Evidently all attempts to simplify the wavelet solutions for PDE are welcome. One possibility for this is to make use of the Haar wavelet family. Haar wavelets (which are Daubechies of order 1) consists of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. A drawback of the Haar wavelets is their discontinuity. Since the derivatives do not exist in the breaking points it is not possible to apply the Haar wavelets for solving PDE directly. There are two possibilities for getting out of this situation. One way is to regularize the Haar wavelets with interpolating splines (for example, B-splines or Deslaurier-Dabuc interpolating wavelets). The other way is proposed by Chen and Hsiao [37], which has expanded the highest derivative in the differential equation into Haar series. Other derivatives are obtained through integrations. The whole system is discretized by collocation method. The collocation method here is actually referred to segmentation process. In this chapter a traveling-wave model is used to describe the lightning stroke behavior [164], and then the Haar wavelet method is employed to solve it.

10.2 A Traveling-wave model

Basically, lightning stroke is an electromagnetic wave and can be determined by the use of traveling wave equations [164]. Naturally the discharge channel within a thundercloud is fully developed, then it will behave like the spark discharge between two flat plates forming a condenser that is in effect shorted out by a central conductor. Hence, a useful concept is to think of the cloud and earth
as forming a vast capacitor, which is being discharged by the stroke [157],[164]. Propagation of electromagnetic wave along the path can be treated as a circuit problem [164], and voltage $V(x,t)$ and current $i(x,t)$ satisfy the well-known telegraphist’s equations

$$-\frac{\partial V(x,t)}{\partial x} = L \frac{\partial i(x,t)}{\partial t} + R i(x,t) \quad (10.1)$$

$$-\frac{\partial i(x,t)}{\partial x} = C \frac{\partial V(x,t)}{\partial t} + G V(x,t) \quad (10.2)$$

Where $R,G,L$ and $C$ represent the unit per length resistance, conductance, inductance and capacitance of the path respectively.

Assume that the energy loss due to the conductance term is small compared to the other equations, and (10.1) and (10.2) are combined, then both voltage wave and current wave satisfy the following hyperbolic equation

$$\frac{\partial^2 \Phi(x,t)}{\partial x^2} = RC \frac{\partial \Phi(x,t)}{\partial t} + LC \frac{\partial^2 \Phi(x,t)}{\partial t^2} \quad (10.3)$$

We can see that the first term on the right-hand side represents the energy dissipation associated with the wave propagation process, and the second term represents the inertia of the time-dependent motion. For the problem statement to be complete, the boundary and initial conditions need to be specified. Let us consider the voltage wave equation of equation of (10.3). The voltage at the ground is assumed to be zero while the current at the cloud is assumed to be a small constant value, which is taken to be zero [164]. Hence, the boundary conditions are
At $t=0$, the voltage distribution is assumed to be a known constant value except at the ground where it is zero. Hence the initial conditions are

\[ V(x, 0) = V_0 \] \hspace{1cm} (10.6) \\
\[ \partial i(x, 0)/\partial t = 0 \] \hspace{1cm} (10.7)

Once the voltage distribution has been solved, the solution of current involves the integration of (10.2) using (10.4) as an initial condition.

### 10.3 Method of solution

The telegraphist’s equations (10.1) and (10.2) satisfy the hyperbolic equations

\[ \partial^2 v/\partial x^2 = LC \left( \partial^2 v/\partial t^2 \right) + (RC + LG) \left( \partial v/\partial t \right) + RGv \] \hspace{1cm} (10.8) \\
\[ \partial^2 i/\partial x^2 = LC \left( \partial^2 i/\partial t^2 \right) + (RC + LG) \left( \partial i/\partial t \right) + RGi \] \hspace{1cm} (10.9)

In the Haar domain, we assume that $\partial^2 v/\partial t^2$ can be expanded as a Haar series as

\[ \partial^2 v/\partial t^2 = a'(x)H(t) \] \hspace{1cm} (10.10)
where \( a(x) \) is an \( m \)-vector function of \( x \), \( x \) denotes the space distance. The \( x \) in \( a(x) \) will be dropped to simplify the notation

\[
\frac{\partial v}{\partial t} = \int \frac{\partial^2 v}{\partial t^2} dt + \frac{\partial v(x, 0)}{\partial t} = a^t PH(t) \tag{10.11}
\]

\[
v(x, t) = \int \frac{\partial v}{\partial t} + v(x, 0) = a^t P^2 H(t) \tag{10.12}
\]

where \( \frac{\partial v(x, 0)}{\partial t} \) and \( v(x, 0) \) have been set to zero for the initially relaxed system.

Equations (10.10), (10.11) and (10.12) can be applied to equation (10.8)

\[
\ddot{a}^t - a^t M^2 = 0 \tag{10.13}
\]

\[
\ddot{a} \approx d^2 a / dx^2 \tag{10.14}
\]

\[
M \approx \sqrt{LCI + (RC + LG)P + RGP^2}(1/P) \tag{10.15}
\]

Solving equation (10.13), we have

\[
a^t = a^t_1 \exp(-Mx) + a^t_2 \exp(Mx) \tag{10.16}
\]

where \( a_1 \) and \( a_2 \) are two constant vectors to satisfy the boundary conditions. The voltage \( v(x, t) \) must be finite when \( x \rightarrow \infty \). It is necessary that \( a_2 = 0 \).

\[
v(x, t) = a^t_1 \exp(-Mx)P^2 H(t) \tag{10.17}
\]

When a unit voltage is applied at \( x=0 \), it means \( v(0, t) = 1 \).

\[
v(0, t) = a^t_1 P^2 H(t) = [1, 1, ..., 1] \tag{10.18}
\]
\[ a_1^t P^2 = [1, 0, ..., 0] \approx a_0^t \] (10.19)

\[ a^t = [1, 0, ..., 1](1/P^2) \] (10.20)

Applying the Haar transform to eqn.(10.1) at the driving point, namely x=0

\[ di(0, t)/dt = b^t H(t) \] (10.21)

\[ i(0, t) = \int (di(0, t)/dt)dt + i(0, 0) = b^t PH(t) \] (10.22)

Substituting equations (10.17), (10.21) and (10.22) into equation (10.1) with x=0, we have

\[ b^t [RP + LI] = a_1^t MP^2 \] (10.23)

\[ b^t = a_1^t M(RP + LI)^{-1}P^2 \] (10.24)

\[ b^t = \sqrt{C/L}a_1^t \sqrt{[G/C]P + I} \sqrt{(R/L)P + I} P^{-1} \] (10.25)

where the commutative property has been applied. Finally, we have

\[ i(0, t) = \sqrt{C/L}a_0^t \sqrt{[G/C]P + I} \sqrt{(R/L)P + I}^{-1} H(t) \] (10.26)

where \( a_0 \) is defined by equation (10.19). The analytic solution can be found in [41], or

\[ i(0, t) = \sqrt{C/L} \exp(-\alpha t) I_0(\beta t) + (G/C) \int \exp(-\alpha t) I_0(\beta t)dt \] (10.27)

where \( \alpha = 1/2[(R/L) + (G/C)] \), \( \beta = 1/2[(R/L) - (G/C)] \)

Here \( I_0 \) is the modified Bessel function of the first kind of the zeroth order.

To illustrate and validate the method described above, results of an example
study are presented. Suppose we represent the cloud and earth as a capacitor having parallel plates of circular shape with a radius of 1 km, and let discharge channel be 20 cm in diameter. Three heights of cloud are considered. Table 10.1 shows the inductances and capacitances for each case [72]. If the resistance of the channel were greater than $Z$, then it would damp the discharge [164]. It is seen that the magnitude of voltage waveform is directly proportional to the cloud height obtained by using a larger $m$.

<table>
<thead>
<tr>
<th>Height (x)</th>
<th>Inductance (I)</th>
<th>Capacitance (C)</th>
<th>$2\sqrt{L/C}\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>250 m</td>
<td>1.85 mH</td>
<td>270.80 pF</td>
<td>516</td>
</tr>
<tr>
<td>500 m</td>
<td>2.02 mH</td>
<td>13.90 pF</td>
<td>762</td>
</tr>
<tr>
<td>1000 m</td>
<td>2.18 mH</td>
<td>13.90 pF</td>
<td>762</td>
</tr>
</tbody>
</table>
Figure 10.1: Comparison between exact and Haar solution for $m=16$.

Haar solutions can be compared with exact solutions, more accurate solution can be obtained by larger values of $m$. (That is $m = 16, m = 32, m = 64$)
10.4 Features

The traveling wave equation for modeling the lightning stroke by Haar wavelet method has been presented. It has been well demonstrated that in applying the nice properties of Haar wavelets, the partial differential equation can be solved conveniently by using Haar wavelet method systematically. The accuracy and effectiveness of the method are analyzed; the results obtained are compared with the results of other authors (using classical numerical techniques) evaluating the error. The benefits of Haar wavelet approach are sparse matrices of representation, fast transformation and designing of fast algorithms and the merits of the method lie in its simplicity, computational economy and easy implementation. The method with far less degrees of freedom and with smaller CPU time provides better solutions than classical ones. For better solutions, instead of increasing the value of one can use other type of wavelets such Mexican wavelets, Spline wavelets etc. Use of Spline wavelets for solving other type of traveling wave equation is presently ongoing.