CHAPTER VII
A COMPARISON OF HAAR WAVELET AND ADOMAIN DECOMPOSITION METHOD FOR SOLVING ONE-DIMENSIONAL REACTION-DIFFUSION EQUATIONS‡

Chapter 7

A comparison of Haar wavelet and Adomain Decomposition method for solving 1-D Reaction-Diffusion Equations

7.1 Introduction

Many reaction-diffusion problems in biology and chemistry are modeled by partial differential equations (PDEs). These problems have been extensively studied in the literature and their numerical solution can be accurately computed provided the diffusion coefficients, reaction excitation ...
intact living cell is based on a highly complex spatial organization of its constituents. The reactants mediating, and processed by the chemical pathways of cell are heterogeneously distributed through the cytoplasm and cell membranes. The diffusion of reactant species among localized reaction regions within the cell is therefore a central feature of biochemistry. For more details, we refer to [175] and further references are cited therein.

Reaction-diffusion equations are used to simulate a variety of different phenomena, from physics and engineering [17] to mathematical biology [145]. In the last decades, there have been great advances in the development of finite difference, finite element, spectral techniques, adaptive and non-adaptive algorithms and finite volume methods for the partial differential equations, especially for those of the advection-diffusion-reaction type. Reaction-diffusion equations also lead to many other interesting phenomena, such as, pulse splitting and shedding, reactions and competitions in excitable systems, and stability issues. Stable schemes for one-dimensional reaction-diffusion equation have demonstrated by Joao Teixeira [178]. Examples of this type of applications include numerical weather prediction and climate models [[24], [202]] where the time step and grid sizes are imposed from large-scale flow considerations, atmosperic chemistry models [103] or reactive flows in engineering [150]. Ramos [160] used a finite volume method for one-dimensional reaction-diffusion problems. Krishnan et al. [115] established Bifurcation analysis of nonlinear reaction-diffusion problems using wavelet-based reduction techniques.

This paper is devoted to study the linear single one-dimensional kinetic reaction-diffusion

\[
\frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial U}{\partial x} \right) - \lambda U, \tag{7.1}
\]
where $t$ and $x$ denote the time and spatial coordinate, respectively, $U$ is the dependent variable and $k$ is a constant diffusion coefficient.

The Adomain decomposition method (ADM) is a creative and effective method for exactly solving functional equations of various kinds. It is important to note that a large amount of research work has been devoted to the application of the ADM to a wide class of linear and nonlinear, ordinary or partial differential equations $[[6],[7],[16],[57],[193]]$. The decomposition method provides the solution as an infinite series in which each term can be easily determined. The rapid convergence of the series obtained by this method is thoroughly discussed by Cherruault et al. $[42]$ and the references therein. Wazwaz $[195]$ used the Adomain decomposition method for a reliable treatment of the Bratu-type equations.

It is somewhat surprising that among different solution techniques the wavelet method has not attained much attention. We found only one paper $[183]$ in which the wavelet method is applied for solving singularly perturbed reaction-diffusion problems; for this purpose the cubic spline adaptive wavelet functions are used. Lepik $[[121],[122],[124]]$ had solved higher order as well as nonlinear ODEs and some nonlinear evolution equations by Haar wavelet method. There are two possibilities for getting out of this situation. One way is to regularize the Haar wavelets with interpolating splines (e.g. B-splines or Deslaurier-Dabuc interpolating wavelets). This approach has been applied by Cattani $[34]$, but the regularization process considerably complicates the solution and the main advantage of the Haar wavelets—the simplicity gets to some extent lost. Hariharan et al. $[[77],[78]]$ had solved linear and nonlinear PDEs.

Among the different wavelet families mathematically most simple are the Haar wavelets $[76]$. Due to the simplicity the Haar wavelets are very effective for
solving ordinary differential and partial differential equations. Therefore the idea, to apply Haar wavelet technique also for solving one-dimensional linear reaction-diffusion problem, arises. This is the main aim of the present paper. The method with far less degrees of freedom and with smaller CPU time provides better solutions than classical ones. The accuracy and effectiveness of the method are analyzed; the results obtained are compared with the results of other authors (using classical numerical techniques) and with the exact solution, evaluating the error.

We introduce a Haar wavelet method for solving one-dimensional linear reaction-diffusion equations, which will exhibit several advantageous features:

i) Very high accuracy fast transformation and possibility of implementation of fast algorithms compared with other known methods.

ii) The simplicity and small computation costs, resulting from the sparsity of the transform matrices and the small number of significant wavelet coefficients.

iii) The method is also very convenient for solving the boundary value problems, since the boundary conditions are taken care of automatically.
7.2 The Adomain Decomposition Method (ADM) for solving one-dimensional reaction-diffusion equation

We consider the linear kinetic one-dimensional reaction-diffusion equation

\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial u}{\partial x} \right) - \lambda u, \]  
(7.2)

with the initial and boundary conditions, where the notations \( L_t = \frac{\partial}{\partial t} \) and \( L_x = \frac{\partial^2}{\partial x^2} \) symbolize the linear differential operators.

We assume the integration inverse operators \( L_t^{-1} \) and \( L_x^{-1} \) exist, and they are defined as \( L_t^{-1} = \int_0^t(\cdot)dt \) and \( L_x^{-1} = \int_0^x \int_0^x (\cdot)dx \) respectively. Therefore, we can write the solutions in \( t \) and \( x \) directions as [7].

\[ u(x,t) = u(x,0) + L_t^{-1} [kL_x(u(x,t)) + \phi(u)] \]  
(7.3)

\[ u(x,t) = u(0,t) + xu_x(0,t) + L_x^{-1} [kL_t(u(x,t)) - \phi(u)] \]  
(7.4)

respectively, where \( \phi(u) = -\lambda u \). By ADM [7] one can write the solution in series form as

\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \]

(7.5)

To find the solutions in \( t \) and \( x \) directions, one solves the recursive relations

\[ u_0 = u(x,0) \]

\[ u_0 = u(0,t) + xu_x(0,t) \]
\[ u_{n+1} = L_t^{-1} [kL_x(u_n) + B_n], n \geq 0, \]
\[ u_{n+1} = L_x^{-1} [kL_t(u_n) - B_n], n \geq 0. \]

respectively, where the Adomain polynomials are

\[ B_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \phi \left( \sum_{n=0}^{\infty} \lambda^n u_n \right) \right]_{\lambda=0}, n \geq 0 \]  

(7.6)

We obtain the first few Adomain polynomials for as, \( \phi(u) = -\lambda u \) as

\[ B_0 = -\lambda u_0 \]
\[ B_1 = -\lambda u_1 \]
\[ B_2 = \lambda u_2 \]

and so on. The convergence of the decomposition series (7.5) is studied in [43]. Then \( u(x, t) \) is the particular exact solution and \( \phi_n(x, t) \) is the partial sum

\[ \phi_n(x, t) = \sum_{k=0}^{\infty} u_k(x, t), n \geq 0 \]  

(7.7)

As it is clear from (7.5) and (7.7)

\[ u(x, t) = \lim_{n \to \infty} \phi_n(x, t). \]  

(7.8)

### 7.3 Method of solution

Consider the linear kinetic one-dimensional reaction-diffusion equation

\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial u}{\partial x} \right) - \lambda u, \]  

(9)

with the initial condition \( u(x, 0) = f(x), 0 \leq x \leq 1 \) and the boundary conditions
\begin{align*}
u (0,t) &= g_0 (t), \; u (1,t) = g_1 (t), \; 0 < t \leq T.
\end{align*}

Let us divide the interval \((0,1] \) into \( N \) equal parts of length \( \Delta = (0,1]/N \) and denote \( t_s = (s - 1) \Delta t, \; s = 1, 2, \cdots , N \). We assume that \( \dot{u}''(x,t) \) can be expanded in terms of Haar wavelets as formula

\begin{align*}
\dot{u}'' (x,t) &= \sum_{n=0}^{m-1} c_s (n) h_n (x) = c^T_{(m)} h_{(m)} (x), \quad (7.10)
\end{align*}

where \( . \) and \( ' \) means differentiation with respect to \( t \) and \( x \) respectively, the row vector \( c^T_{(m)} \) is constant in the subinterval \( t \in (t_s, t_{s+1}] \).

Integrating formula (7.10) with respect to \( t \) from \( t_s \) to \( t \) and twice with respect to \( x \) from \( 0 \) to \( x \), we obtain

\begin{align*}
u'' (x,t) &= (t - t_s) c^T_{(m)} h_{(m)} (x) + \dot{u}'' (x,t_s), \quad (7.11)
\end{align*}

\begin{align*}
u (x,t) &= (t - t_s) c^T_{(m)} Q_{(m)} h_{(m)} (x) + u (x,t_s) - u (0,t_s) + x [u' (0,t) - u' (0,t_s)] + u (0,t) \quad (7.12)
\end{align*}

\begin{align*}
\dot{u} (x,t) &= c^T_{(m)} Q_{(m)} h_{(m)} (x) + x \dot{u}' (0,t) + \dot{u} (0,t) \quad (7.13)
\end{align*}

Using the boundary conditions, we obtain

\begin{align*}
u (0,t_s) &= g_0 (t_s), \\
u (1,t_s) &= g_1 (t_s),
\dot{u} (0,t) &= g'_0 (t), \\
\dot{u} (1,t) &= g'_1 (t).
\end{align*}
Putting $x = 1$ in formulae (7.12) and (7.13), we have

$$u'(0, t) - u'(0, t) = -(t - t_s) c_{(m)}^T Q_{(m)} h_{(m)} + g_1(t) - g_0(t) - g_1(t_s) + g_0(t_s), \quad (7.14)$$

$$\dot{u}'(0, t) = g'_1(t) - c_{(m)}^T Q_{(m)} h_{(m)} - g'_0(t) \quad (7.15)$$

Substituting formulae (7.14) and (7.15) into formulae (7.11)-(7.13), and discretising the results by assuming $x \rightarrow x_l, t \rightarrow t_{s+1}$ we obtain

$$u''(x_l, t_{s+1}) = (t_{s+1} - t_s) c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + u''(x_l, t_s) \quad (7.16)$$

$$u(x_l, t_{s+1}) = (t_{s+1} - t_s) c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + u(x_l, t_s) - g_0(t_s) + g_0(t_{s+1}) x_l[-(t_{s+1} - t_s) C_{(m)}^T P_{(m)} f + g_1(t_{s+1}) - g_0(t_{s+1}) - g_1(t_s) + g_0(t_s)] \quad (7.17)$$

where the vector $f$ is defined as $f = \begin{bmatrix} 1, 0, \ldots, 0 \end{bmatrix}_{(m-1)\text{elements}}^T$.

There are several possibilities for treating the nonlinearity in Eq. (7.9). In the following scheme

$$\dot{u}(x_l, t_{s+1}) = u''(x_l, t_{s+1}) + ku''(x_l, t_{s+1}) - \lambda u(x_l, t_{s+1}) \quad (7.18)$$

which leads us from the time layer $t_s$ to $t_{s+1}$ is used.

Substituting the equations (7.16) and (7.17) into the equation (7.18), we gain

$$c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + x_l \left[ -c_{(m)}^T P_{(m)} f + g'_1(t_{s+1}) - g'_0(t_{s+1}) \right] + g'_0(t_{s+1})$$
\[ u''(x_l,t_{s+1}) + ku''(x_l,t_{s+1}) - \lambda u(x_l,t_{s+1}) \]

From the above formula, the wavelet coefficients \( c_T^{(m)} \) can be successively calculated.

In the following section we provide couple of examples and calculate the absolute errors by using the formula

\[ E_W = |u_{exact} - u_{Haar}| \]

and

\[ \delta ex = \|u(x,t) - u_{ex}(x,t)\| / 2M. \]

### 7.4 Applications and results

**Example:1**

If we take \( k = 1 \) and \( \lambda = -1 \) in the equation (7.9), we obtain the linear heat equation, namely

\[ u_t = u_{xx} + u \quad (7.19) \]

We impose the initial condition

\[ u(x,0) = \cos(\pi x) \quad (7.20) \]

and boundary conditions

\[ u(0,t) = e^{(1-\pi^2)t}, u_x(0,t) = 0. \quad (7.21) \]

To obtain the solution in \( t \) direction, we use the above mentioned recursive relation by simply taking \( u_0 = \cos(\pi x) \) In this case the Adomain polynomials are \( B_0 = u_0, B_1 = u_1, B_2 = u_2 \), and so on. Therefore, we have
and so on, in this manner the rest of the components of the series (7.5) have been calculated using Mathcad7. Putting these individual terms in (7.5) one gets the exact solution

\[ u(x, t) = \cos(\pi x) + (1 - \pi^2)t\cos(\pi x) + \frac{1}{2!}(1 - \pi^2)^2t^2\cos(\pi x) + \frac{1}{3!}(1 - \pi^2)^3t^3\cos(\pi x) + \ldots = e^{(1-\pi^2)t}\cos(\pi x). \]

which can be verified through substitution.

Similarly, to obtain the solution in \( x \) direction, we use the above mentioned recursive relation by taking \( u_0 = e^{(1-\pi^2)} \), where the \( B_n \)'s are the same as above. We therefore have

\[ u_1 = -\frac{(\pi x)^2}{2!}e^{(1-\pi^2)t}, \]
\[ u_2 = \frac{(\pi x)^4}{4!}e^{(1-\pi^2)t}, \]
\[ u_3 = \frac{(\pi x)^6}{6!}e^{(1-\pi^2)t}, \]

and so on, in this manner the rest of the components of the series (7.5) have been calculated. From the decomposition series (7.5), we gain obtain the exact solution

\[ u(x, t) = e^{(1-\pi^2)t}\cos(\pi x). \] (7.22)

In the following scheme

\[ \dot{u}(x_l, t_{s+1}) = u''(x_l, t_{s+1}) + u(x_l, t_{s+1}) \] (7.23)

which leads us from the time layer \( t_s \) to \( t_{s+1} \) is used.
\[ c^{T}(m)Q(m)h(m)(x_l + x_l[-c^{T}(m)P(m)\cos(\pi x) + g'(t_{s+1}) - g'(t_{s+1})] + g'(t_{s+1}) \]
\[ = u''(x_l, t_{s+1}) + u(x_l, t_{s+1}) \]

From the above formula the wavelet coefficients can be successively calculated. This process is started with

\[
\begin{align*}
  u(x_l, t_s) &= -\pi \sin(\pi x) \\
  u'(x_l, t_s) &= -\pi^2 \cos(\pi x) \\
  u''(x_l, t_s) &= \pi^3 \sin(\pi x)
\end{align*}
\]

Table 7.1: The absolute errors at different times and space locations for example 1

<table>
<thead>
<tr>
<th>( (x, t) )</th>
<th>ADM solution</th>
<th>Haar solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 16 )</td>
<td>( m = 32 )</td>
<td></td>
</tr>
<tr>
<td>(0.25,0.1)</td>
<td>6.9388E-016</td>
<td>3.2570E-005</td>
</tr>
<tr>
<td>(0.5,1.0)</td>
<td>1.3322E-015</td>
<td>5.5452E-005</td>
</tr>
<tr>
<td>(0.75,1.0)</td>
<td>1.8873E-015</td>
<td>4.3456E-005</td>
</tr>
<tr>
<td>(0.25,0.5)</td>
<td>1.2679E-009</td>
<td>1.9562E-004</td>
</tr>
<tr>
<td>(0.5,0.5)</td>
<td>2.4570E-009</td>
<td>5.4572E-004</td>
</tr>
<tr>
<td>(0.75,0.5)</td>
<td>3.4933E-009</td>
<td>7.7654E-004</td>
</tr>
<tr>
<td>(0.25,1.0)</td>
<td>6.1931E-007</td>
<td>1.6562E-004</td>
</tr>
<tr>
<td>(0.5,1.0)</td>
<td>1.2001E-006</td>
<td>3.7982E-003</td>
</tr>
<tr>
<td>(0.75,1.0)</td>
<td>1.7063E-006</td>
<td>4.4324E-003</td>
</tr>
</tbody>
</table>

Using Adomain decomposition method, the exact solution in a closed form is given by \( u(x, t) = e^{(1-\pi^2)t} \cos(\pi x) \) can be compared with the Haar solution. The accuracy of the results is estimated by the error function. In the case of error estimates, if the exact solution of the problem \( u = u(x, t) \) is known we shall calculate the differences \( \Delta_{ex}(l) = u(x_l, t_{s+1}) - u_{ex}(x_l, t_{s+1}), l = 1, 2, ..., 2M \) and we define the error estimates as \( \delta_{ex} = \max_l |\Delta_{ex}(l)| \) (local estimate) (or) \( \delta_{ex} = \frac{1}{2M} \|u(x, t) - u_{ex}(x, t)\| \) (global estimate). The convergence of Haar method is fast and it’s accuracy is height, as numerical examples show error [122].
Computer simulation was carried out in the cases $m = 16$ and $m = 32$, the computed results were compared with the exact solution, more accurate results can be obtained by using a larger $m$. The method with far less degrees of freedom and with a smaller CPU time provides better solutions than classical ones.

7.5 Features

The goal to obtain exact and Haar solutions for one-dimensional reaction-diffusion problems has been achieved. The theoretical elegance of the Haar wavelet approach can be appreciated from the simple mathematical relations and their compact derivations and proofs. It has been well demonstrated that while applying the nice properties of Haar wavelets, the partial differential equations can be solved conveniently and accurately by using Haar wavelet method systematically.

In this chapter only linear equations are considered, but the method are applicable also for nonlinear systems. The main advantages of the presented method are its simplicity and small computation costs: it is due to the sparcity of the transform matrices and to the small number of significant wavelet coefficients. An authentic conclusion can be drawn from the numerical results that the Adomain decomposition method provides more accurate numerical solutions than Haar wavelet method. In our opinion the Haar wavelet method is wholly competitive in comparison with the classical methods.

Future work will involve the extension of the scheme to two and three dimensions and to the advection-diffusion equation. Extending the scheme to higher dimensions and to the advection-reaction-diffusion equations.