CHAPTER V

A COMPARATIVE STUDY OF
A HAAR WAVELET METHOD
AND A RESTRICTIVE
TAYLOR’S SERIES METHOD
FOR SOLVING
CONVECTION-DIFFUSION
EQUATIONS‡

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Chapter 5

A comparative study of a Haar wavelet method and a restrictive Taylor’s series method for solving convection-diffusion equations

5.1 Introduction

The convection-diffusion equation applies to problems in such areas as mass transport, momentum transport, energy transport and neutron transport. In this paper we consider the numerical solution of the parabolic partial differential equations governing two of the most basic processes in physical systems, namely convection and diffusion.

Consider the one-dimensional convection-diffusion equation

\[ \frac{\partial U}{\partial t} + a \frac{\partial U}{\partial x} = \alpha \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < 1, \quad 0 \leq t \leq T \] (5.1)
with initial condition
\[ U(x, 0) = f(x), \quad 0 \leq x \leq 1 \quad (5.2) \]
and boundary conditions
\[ U(0, t) = g_1(t), \quad U(1, t) = g_2(t), \quad 0 < t \leq T \quad (5.3) \]

Accurate solution of the time-dependent convection-diffusion equation on large spatial domains and large times is desirable. This requires the development of numerical schemes that can remain accurate even for large nodes sizes and large time steps. Several effective schemes have been proposed to achieve this goal [65]. Each has its own advantages and disadvantages. Operator splitting schemes have become popular [144]. The disadvantage with the operator splitting method is that the boundary conditions are available for the actual operator, and not for the split-operators. Numerical schemes that have been successfully applied to other problems [49, 199, 203, 206] are also being developed to solve the convection-diffusion equation.

There are many numerical research results about convection-diffusion equations [50, 126, 143, 209]. Several common numerical methods have been developed to solve them, such as the FDM [112, 155, 206] and FAM [105] and so on. Tao [176] described several schemes of the FVM for solving convection-diffusion equations. Dou [50] proposed an explicit finite volume-finite element method for nonlinear convection-diffusion problems. Lin [127] found a local linear analytical solution for the nonlinear convection-diffusion equation, and Lu [129] gave a scheme for the two-dimensional convection diffusion. Enokizono and Nagata [55] used the boundary element method for convection-diffusion analysis at

The mathematical properties of Convection-diffusion equation (CD) have been studied extensively and there have been numerous discussions in the literature. Excellent summaries have been provided in [20],[53],[55], [56],[143],[179].

We introduce a Haar wavelet method for solving convection-diffusion (CD) equations with the initial and boundary conditions, which will exhibit several advantageous features:

i) Very high accuracy fast transformation and possibility of implementation of fast algorithms compared with other known methods.

ii) The simplicity and small computation costs, resulting from the sparsity of the transform matrices and the small number of significant wavelet coefficients.

iii) The method is also very convenient for solving the boundary value problems, since the boundary conditions are taken care of automatically.

Beginning from 1980’s, wavelets have been used for solution of partial differential equations (PDE). The good features of this approach are possibility to detect singularities, irregular structure and transient phenomena exhibited by the analyzed equations. Most of the wavelet algorithms can handle exactly periodic boundary conditions. The wavelet algorithms for solving PDE are based on the Galerkin techniques or on the collocation method.
Evidently all attempts to simplify the wavelet solutions for PDE are welcome. One possibility for this is to make use of the Haar wavelet family. Haar wavelets (which are Daubechies of order 1) consists of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. A drawback of the Haar wavelets is their discontinuity. Since the derivatives do not exist in the breaking points it is not possible to apply the Haar wavelets for solving PDE directly. There are two possibilities for getting out of this situation. One way is to regularize the Haar wavelets with interpolating splines (for example, B-splines or Deslaurier-Dubuc interpolating wavelets). This approach has been applied by Cattani [34]. The other way is to make use of the integral method, which was proposed by Chen and Hsiao [37]. Recently, Lepik [[120],[121] [122]] established the Haar wavelet method for solving some ODEs and PDEs. There are some useful discussions by other researchers [[77],[78],[85],[92],[123]].

The advantages of computational and memory requirements of the Haar transform make it use of a considerable interest to VLSI designers as well. The use of this set of CAD tools allowed the derivation of strategies for testing MOS circuits when memory states were encountered as a consequence of some type of faults.
5.2 The solution of Convection-Diffusion equation using Restrictive Taylor’s series method

The Restrictive Taylor’s method (RTM) approximation [[101],[102]] of the function $f(x)$ at the point $a$ can be written in the form

$$RT_{n,f(x)}(x,a) = \sum_{k=0}^{n-1} \frac{f^i(a)}{i!}(x-a)^i + \epsilon \frac{f^n(a)}{n!}(x-a)^n,$$  \hspace{1cm} (5.4)

where the parameter $\epsilon$ is to be determined such that

$$RT_{n,f(x)}(x_0,a) = f(x_0)$$  \hspace{1cm} (5.5)

It means that the considered approximation is exact at two points $x = a$ and $x = x_0$. In this case we have [102]

$$f(x) = RT_{n,f(x)}(x_0,a) + R_{n+1}(x)$$  \hspace{1cm} (5.6)

in which

$$R_{n+1}(x) = \epsilon \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(\xi) - n(\xi - 1) \frac{(x-a)^n}{x-\epsilon} \frac{(x-a)^{n+1}}{(n+1)!} f^n(\xi),$$  \hspace{1cm} (5.7)

where $\xi \in (a,x)$. Now, the formula of the RTs approximation of a single function can be used for the RTs approximation of a matrix function if the parameter $\epsilon$ is replaced by the square restrictive matrix $\Gamma = \epsilon I$, where $I$ is the unit matrix, for example

$$RT_{1,e^{kA}}(x,0) = I + k\Gamma A = I + \epsilon kA$$  \hspace{1cm} (5.8)
5.3 Restrictive Taylor’s approximation of the exponential matrix

The exponential matrix $e^{xA}$ can be defined by the power series

$$e^{xA} = I + xA + \frac{x^2}{2!} A^2 + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!} A^n, A^0 = I,$$  \hspace{1cm} (5.9)

where $A$ is a $(N-1) \times (N-1)$ matrix. In case of RT approximation of single function the term $\epsilon$ in equation of the type (5.4) can be reduced to the square restrictive matrix $\Gamma$.

5.4 Restrictive Taylor’s approximation for convection-diffusion equation

Consider the convection-diffusion equation (5.17) with the initial and boundary conditions (5.18) and (5.19). The open rectangular domain is covered by a rectangular grid with spacing $h$ and $k$ in the $x,t$ directions respectively (i.e., $h = \Delta x$, $k = \Delta t$, the grid point $(x,t)$ denoted by $(ih,jk)$ and $u(ih,jk) = u_{i,j}$ where $i = 0(1)N$ and $j$ is a non-negative integer.

The exact solution of grid representation of equation (5.17) is given by

$$u_{i,j+1} = e^{k(aD^2_x-aD_x)}u_{i,j}$$ \hspace{1cm} (5.10)

The approximation of the partial derivative $D^2_x$ and $D_x$ at the grid point $(ih,jk)$
will take the usual forms

\[ D^2_x u = \frac{1}{h^2} (u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \] (5.11)

and

\[ D_x u = \frac{1}{2h} (u_{i+1,j} - u_{i-1,j}) \] (5.12)

The result of making this approximation is to replace equation (5.10) by the following equation

\[ U_{j+1} = \exp(rA)U^j, r = \frac{k}{h^2} \] (5.13)

\[ U^j = (u_{1,j}, u_{2,j}, ..., u_{N-1,j})^T, Nh = 1 \] (5.14)

\[ A = \begin{bmatrix}
-2\alpha & (\alpha - \frac{ah}{2}) & 0 & \cdots & 0 & 0 \\
(\alpha + \frac{ah}{2}) & -2\alpha & (\alpha - \frac{ah}{2}) & 0 & \cdots & 0 \\
0 & (\alpha + \frac{ah}{2}) & -2\alpha & (\alpha - \frac{ah}{2}) & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & (\alpha + \frac{ah}{2}) & -2\alpha & (\alpha - \frac{ah}{2}) \\
0 & 0 & 0 & \cdots & (\alpha + \frac{ah}{2}) & -2\alpha \\
\end{bmatrix} \]

Here the order of the matrix is \((N - 1) \times (N - 1)\).

We use equation (5.8) to approximate the exponential matrix in equation (5.13), then

\[ U^{j+1} = (1 + r\Gamma A)U^j = BU^j \] (5.15)

or in the scalar form

\[ u_{i,j+1} = r \in (\alpha + \frac{ah}{2})u_{i-1,j} + (1 - 2r\epsilon\alpha)u_{i,j} + r \in (\alpha - \frac{ah}{2}) \] (5.16)
where \( i = 1(1)N - 1 \), \( j \) non-negative integer.

5.5 Stability analysis

The matrix \( B \) is of the form

\[
B = \begin{bmatrix}
1 - 2r\epsilon\alpha & r\epsilon(\alpha - \frac{ah}{2}) & 0 & \ldots & 0 & 0 \\
r\epsilon(\alpha + \frac{ah}{2}) & 1 - 2r\epsilon\alpha & r\epsilon(\alpha - \frac{ah}{2}) & 0 & \ldots & 0 \\
0 & r\epsilon(\alpha + \frac{ah}{2}) & 1 - 2r\epsilon\alpha & r\epsilon(\alpha - \frac{ah}{2}) & 0 & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & r\epsilon(\alpha + \frac{ah}{2}) & 1 - 2r\epsilon\alpha & r\epsilon(\alpha - \frac{ah}{2}) \\
0 & 0 & \ldots & 0 & \ldots & r\epsilon(\alpha + \frac{ah}{2}) & 1 - 2r\epsilon\alpha
\end{bmatrix}
\]

Here the order of the matrix is \((N - 1) \times (N - 1)\). Hence for \( 1 - 2r\epsilon\alpha > 0 \), we get

\[ \|\beta\|_{\infty} = 1, \] which will ensure stability for \( r\epsilon\alpha < \frac{1}{2} \) [161].

5.6 Haar wavelet solutions for Convection-Diffusion equation

Consider the Convection-Diffusion equation

\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}, 0 < x < 1, 0 < t \leq T \quad (5.17)
\]
with initial condition

\[ u(x, 0) = f(x), \quad 0 \leq x \leq 1 \quad (5.18) \]

and boundary conditions

\[ u(0, t) = g_1(t), \quad u(1, t) = g_2(t), \quad 0 < t \leq T \quad (5.19) \]

Let us divide the interval (0,1] into \( N \) equal parts of length \( \Delta t = (0,1]/N \) and denote \( t_s = (s - 1)\Delta t, s = 1, 2, ..., N \). We assume that \( \dot{u}''(x, t) \) can be expanded in terms of Haar wavelets as formula

\[ \dot{u}''(x, t) = \sum_{n=0}^{m-1} c_s(n)h_n(x) = c^{T}_{(m)}h_{(m)}(x) \quad (5.20) \]

where . and ' means differentiation with respect to \( t \) and \( x \) respectively, the row vector \( c^{T}_{(m)} \) is constant in the subinterval \( t \in (t_s, t_{s+1}] \)

Integrating formula (5.20) with respect to \( t \) from \( t_s \) to \( t \) and twice with respect to \( x \) from 0 to \( x \), we obtain

\[ u''(x, t) = (t - t_s)c^{T}_{(m)}h_{(m)}(x) + u''(x, t_s) \quad (5.21) \]

\[ u'(x, t) = (t - t_s)c^{T}_{(m)}P_h_{(m)}(x) + u'(x, t_s) - u'(0, t_s) + u'(0, t) \quad (5.22) \]

\[ u(x, t) = (t - t_s)c^{T}_{(m)}P^{2}h_{(m)}(x) \]

\[ +u(x, t_s) - u(0, t_s) + x[u'(0, t) - u'(0, t_s)] + u(0, t) \quad (5.23) \]

\[ \dot{u}(x, t) = c^{T}_{(m)}P^{2}h_{(m)}(x) + x\dot{u}'(0, t) + \dot{u}(0, t) \quad (5.24) \]
By the boundary conditions, we obtain

\[ u(0, t) = g_1(t) \]
\[ u(1, t) = g_2(t) \]
\[ \dot{u}(0, t) = g'_1(t), \]
\[ \dot{u}(1, t) = g'_2(t) \]

Putting \( x = 1 \) in formulae (5.23) and (5.24), we have

\[ u'(0, t) - u'(0, t_s) = -(t - t_s)c^T_{(m)}P_{(m)} h_{(m)}(x) \]
\[ + g_2(t) - g_1(t) - g_2(t_s) + g_1(t_s) \]  \hspace{1cm} (5.25)

\[ \dot{u}'(0, t) = -c^T_{(m)}p^2_{(m)}f + g'_2(t) - g'_1(t) \]  \hspace{1cm} (5.26)

Substituting formulae (5.25) and (5.26) into formulae (5.21)-(5.24), and discretizising the results by assuming \( x \to x_i, t \to t_{s+1} \) we obtain

\[ u''(x_i, t_{s+1}) = (t_{s+1} - t_s)c^T_{(m)}h_{(m)}(x_i) + u''(x_i, t_s) \]  \hspace{1cm} (5.27)

\[ u(x_i, t_{s+1}) = (t_{s+1} - t_s)c^T_{(m)}P_{(m)}h_{(m)}(x_i) - u(x_i, t_s) - g_1(t_s) + g_1(t_{s+1}) + x_i[-(t_{s+1} - t_s)c^T_{(m)}P_{(m)}f \]
\[ + g_2(t_{s+1}) - g_1(t_{s+1}) - g_2(t_s) + g_1(t_s)] \]  \hspace{1cm} (5.28)

\[ u'(x_i, t_{s+1}) = (t_{s+1} - t_s)c^T_{(m)}P_{(m)}h_{(m)}(x_i) + u'(x_i, t_s) \]
\[ - (t_{s+1} - t_s)c^T_{(m)}P_{(m)}f + g_2(t_{s+1}) - g_1(t_{s+1}) - g_2(t_s) + g_1(t_s) \]  \hspace{1cm} (5.29)

\[ \dot{u}(x_i, t_{s+1}) = c^T_{(m)}P^2_{(m)}h_{(m)}(x) + g_1(t_{s+1}) \]
\[ + x_i[-c^T_{(m)}P_{(m)}f + g'_2(t_{s+1}) - g'_1(t_{s+1})] \]  \hspace{1cm} (5.30)
Where the vector \( f \) is defined as
\[
f = [1, 0, \ldots, 0]^{\top}_{(m-1) \text{elements}}
\]
For clarity in the illustrative example, we assume that \( a = 0.8 \) and \( \alpha = 0.1 \)
\[
\frac{\partial u}{\partial t} + 0.8 \frac{\partial u}{\partial x} = 0.1 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1, \quad t \geq 0 \quad (5.31)
\]
\[
f(x) = e^{-\frac{(x-2)^2}{8}} \quad (5.32)
\]
\[
g_0(t) = \sqrt{\frac{20}{20 + t}} e^{-\frac{(5+4t)^2}{40(t+20)}} \quad (5.33)
\]
\[
g_1(t) = \sqrt{\frac{20}{20 + t}} e^{-\frac{2(5+2t)^2}{40(t+20)}} \quad (5.34)
\]
for which the exact solution is
\[
u(x, t) = \sqrt{\frac{20}{20 + t}} e^{-\frac{(x-2-0.8t)^2}{0.4(t+20)}} \quad (5.35)
\]
In the following scheme, the time layer \( t_s \) has been replaced by \( t_{s+1} \)
\[
\dot{u}(x_l, t_{s+1}) = -0.8u(x_l, t_s)u'(x_l, t_{s+1}) + 0.1u''(x_l, t_{s+1}) \quad (5.36)
\]
Substituting equations (5.27)-(5.30) into the equation (5.36), we obtain
\[
c^T(m) P^2(m) h(m)(x_l)
+ x_l[-c^T(m) P(m)f + g_2(t_{s+1}) - g_1'(t_{s+1})] + g_1'(t_{s+1}) = -0.8u'(x_l, t_s) - 0.1u''(x_l, t_s)
\]
From equation (5.37) the wavelet coefficients \( c^T(m) \) can be successively calculated.
This process is started with

\[ u(x_l, t_s) = e^{-\frac{(x_l-2)^2}{8}} \] (5.37)

\[ u'(x_l, t_s) = \left( -\frac{x_l - 2}{4} \right) e^{-\frac{(x_l-2)^2}{8}} \] (5.38)

\[ u''(x_l, t_s) = \left( -\frac{x_l^2 - 4x_l}{16} \right) e^{-\frac{(x_l-2)^2}{8}} \] (5.39)

Computer simulation was carried out for \( m = 32 \); the computed results were compared with the exact solution. More accurate results can be obtained for large values of \( m \).

### 5.7 Numerical Test

**Problem:1**

Consider the non-homogeneous wall problem

\[ \frac{d}{dx} \left[ x^2 \frac{du}{dx} \right] = 0 \] (5.40)

For solving this problem by the Haar wavelet method, we assume that \( u^{(2)}(x) \) can be expanded in terms of Haar wavelets as formula (1.27). That is

\[ u^{(2)}(x) = \sum_{n=0}^{m-1} c(n)h_{n}(x) = c^T(m)h_{m}(x) \] (5.41)

Integrating formula (5.41) from 0 to \( x \), the variables \( u'(x) \) and \( u(x) \) can be expressed as

\[ u'(x) = \int_0^x u^2(t)dt + u'(0) \]

\[ = \left[ c^T(m)P^3(m) + f^T u^{(3)}(0) P^2(m) f^T u^{(2)}(0) P(m) + f^T u'(0) \right] h_{m}(x) \]
\[ u(x) = \int_0^x u'(t) \, dt + u(0) \]

\[ = \left[ c^T_{(m)} P^4_{(m)} + f^T u^{(3)}(0) P^3_{(m)} f^T u^{(2)}(0) P_{(m)} + f^T u'(0) P_{(m)} f^T u(0) \right] h_m(x) \]

Where the vector \( f \) is defined as

\[ f = [1, \underbrace{0, \ldots, 0}_{(m-1)\text{ elements}}]^T \]

Substituting the above formulae into Eq. (5.42), we transfer the equation

\[ x^2 d^2 u \, dx^2 + 2x \frac{du}{dx} = 0 \] (5.42)

into a matrix equation, that is,

\[ x^2 c^T_{(m)} h_{(m)}(x) + 2x \left[ c^T_{(m)} P^3_{(m)} + f^T u^{(3)}(0) P^2_{(m)} f^T u^{(2)}(0) P_{(m)} + f^T u'(0) P_{(m)} \right] h_{(m)}(x) = 0 \] (5.43)

From the above formula the wavelet coefficients \( c^T_{(m)} \) can be successively calculated.

Analytical solution of Eq. (5.40) is given by

\[ u(x) = 2 - \left[ 2 \right] \frac{1}{x} \] (5.44)

in the domain \( x = (1, 2) \)

**Problem:**

Consider the nonlinear diffusivity problem

\[ \frac{d}{dx} \left[ \frac{1}{u} \frac{du}{dx} \right] = 0 \] (5.45)
The exact solution of Eq. (5.45) in a closed form is given by

\[ u(x) = 2^x \]  

(5.46)

For solving Eq. (5.45) by the Haar wavelet method, we assume that \( u^{(2)}(x) \) can be expanded in terms of Haar wavelets as formula (1.27). That is,

\[ \dot{u}^{(2)}(x, t) = \sum c_s(n) h_n(x) = c_{(m)}^T h_{(m)}(x) \]  

(5.47)

Integrating formula (5.47) from 0 to \( x \), the variables \( u'(x) \) and \( u(x) \) can be expressed as

\[ u'(x) = \int_0^x u^2(t) dt + u'(0) \]

\[ u(x) = \left[ c_{(m)}^T P_3^{(m)} + f^T u^{(3)}(0) P_2^{(m)} f^T u^{(2)}(0) P_{(m)} + f^T u'(0) \right] h_{(m)}(x) \]

From the above formula, the wavelet coefficients \( c_{(m)}^T \) can be successively calculated.
Problem: 3
We consider the problem

$$-2 \left( 2x^2 + \tan(x^2) \right) u = \frac{d^2 u}{dx^2} \quad (5.49)$$

It is worth noting that applying the scheme proposed above for the Eq.(5.49),
the solution in a closed form $u(x) = \cos(x^2)$, $x \in (0, \frac{\pi}{4})$ can be compared with
the Haar solution.

Problem: 4
We consider the entry flow problem

$$Pe \frac{du}{dx} = \frac{d^2 u}{dx^2}, \quad x \in (0, 1) \quad (5.50)$$

An analytical solution of Eq. (5.50) in a closed form is given by

$$u(x) = 1 - \left[ \frac{1 - e^{Pe x}}{1 - e^{Pe}} \right] \quad (5.51)$$

$$\frac{du}{dx} = \left[ \frac{Pe * e^{Pe x}}{1 - e^{Pe}} \right] \quad (5.52)$$

For solving Eq. (5.50) by the Haar wavelet method, we assume that $u^{(2)}(x)$ can
be expanded in terms of Haar wavelets as formula (1.27). That is

$$u^{(2)}(x) = \sum_{n=0}^{m-1} c_{(m)} h_{(n)}(x) = c^T_{(m)} h_{(m)}(x) \quad (5.53)$$

Integrating formula (5.53) from 0 to $x$ and using the formula (1.27), the variables
$u'(x)$ and $u(x)$ can be expressed as
\[ u'(x) = \int_0^x u^2(t) \, dt + u'(0) \]
\[ = \left[ c^T_{(m)} P^3_{(m)} + f^T u^{(3)}(0) P^2_{(m)} f^T u^{(2)}(0) P_{(m)} + f^T u'(0) \right] h_m(x) \]
\[ u(x) = \int_0^x u'(t) \, dt + u(0) \]
\[ = \left[ c^T_{(m)} P^4_{(m)} + f^T u^{(3)}(0) P^3_{(m)} f^T u^{(2)}(0) P_{(m)} + f^T u'(0) P_{(m)} f^T u(0) \right] h_m(x) \]

Where the vector \( f \) is defined as
\[ f = [1, 0, ..., 0]^{(m-1)\text{elements}} \]

Substituting the above formulae into Eq. (5.54), we transfer the equation

\[ Pe(u') = u'' \] (5.54)

into a matrix equation, that is

\[ Pe \left\{ \left[ c^T_{(m)} P^3_{(m)} + f^T u^{(3)}(0) P^2_{(m)} f^T u^{(2)}(0) P_{(m)} + f^T u'(0) \right] h_{(m)}(x) \right\} = c^T_{(m)} h_{(m)}(x) \] (5.55)

From the above formula (5.55) the wavelet coefficients \( c^T_{(m)} \) can be successively calculated. It is worthwhile to apply the scheme proposed above for the Eq. (5.50), the solution in a closed form

\[ u(x) = 1 - \left[ \frac{1-e^{Pe}}{1-e^{Pe}} \right] \] can be compared with the Haar solution. One increases the peclet number \( Pe \) from 1 to 10, 50, 100 and compares the \( \frac{du}{dx} = \left[ \frac{Pe e^{-Pe}}{1-e^{Pe}} \right] \) at \( x=1 \).

In numerical analysis, a high peclet number brings various problems. With respect to the domain type method such as FDM or FEM, the spurious oscillation appears in the numerical solutions when the peclet number of an element (cell peclet number) exceeds 1. In order to inhibit the spurious oscillation, upwinding schemes were developed [186]. However, the applicable upper limit of the upwinding scheme is about a hundred, because the numerical error increases as the peclet number becomes higher.

On the other hand, Haar wavelet method provides a stable and acceptably
accurate solution.

We present a numerical example Eq. (5.31) to compare the RT approximation with Haar method of a local truncation error $O(h^4, k^2)$. The accuracy of Haar and RT method are compared in Table 5.3 for various values of the time $t$. Table 5.3 gives the absolute error along $x = 0.1, 0.5$ and $0.9$ as this is the middle and end points of the domain, where $h = 0.1$ and $k = 0.1$.

The execution time of calculation 500 steps of Haar method is 1.25 second while that of Restrictive Taylor method is 1.65 second.

**Problem:5**

Consider the steady-state convection-diffusion equation

$$-\alpha u_{xx} + \beta u_x = 0, \quad u(0) = 0, \quad u(1) = 1 \quad (5.56)$$

where $\alpha$ and $\beta$ are constants.

The exact solution of Eq. (5.56) is given as

$$u(x) = \frac{e^{\beta x}}{e^{\frac{\beta}{\alpha}} - 1}$$

**Problem :6**

Consider the convection-diffusion equation

$$u_t + 0.1u_x = 0.02u_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0 \quad (5.58)$$

where the initial and boundary conditions are defined such that the exact solution is $u(x, t) = e^{1.17712434 - 0.09t}$. Let $h = 0.025$ and $k = 0.001$. The logarithm of the absolute error of the three methods at the time level $t = 0.1$ have been displayed in Figure 5.2. In the RTs method we used relation (14) with $\epsilon = 0.992866$ [102]
and in the Krylov subspace method (KSM) with $m = 5$. The proposed Haar Wavelet method can be compared with Salkuyeh [112] results. As the Figure 5.2 shows the results of the Haar method (HWM) are more effective than that of two other methods.

Table 5.1: Errors between the exact, Haar and the numerical solution of Eq. (5.55) with $\alpha = 0.1$ and $\beta = 1$.

<table>
<thead>
<tr>
<th>$\Delta x$</th>
<th>upwind</th>
<th>RT method</th>
<th>Haar method ($m=32$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.32E-01</td>
<td>3.45E-02</td>
<td>7.41E-04</td>
</tr>
<tr>
<td>0.05</td>
<td>7.64E-02</td>
<td>7.90E-03</td>
<td>4.74E-05</td>
</tr>
<tr>
<td>0.025</td>
<td>4.17E-02</td>
<td>1.90E-03</td>
<td>2.98E-06</td>
</tr>
<tr>
<td>0.0125</td>
<td>2.81E-02</td>
<td>4.79E-04</td>
<td>1.87E-07</td>
</tr>
<tr>
<td>0.00625</td>
<td>1.12E-02</td>
<td>1.20E-04</td>
<td>1.17E-08</td>
</tr>
</tbody>
</table>

Table 5.2: Error estimation of convection-diffusion equation (5.36) for various values of spatial variable $x$ and time $t$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x=0.1$</th>
<th>$x=0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2.2205 \times 10^{-16}$</td>
<td>$1.215 \times 10^{-20}$</td>
</tr>
<tr>
<td>2</td>
<td>$2.2205 \times 10^{-16}$</td>
<td>$1.308 \times 10^{-20}$</td>
</tr>
<tr>
<td>3</td>
<td>$3.3307 \times 10^{-16}$</td>
<td>$1.102 \times 10^{-18}$</td>
</tr>
<tr>
<td>4</td>
<td>$3.3307 \times 10^{-16}$</td>
<td>$1.102 \times 10^{-18}$</td>
</tr>
<tr>
<td>5</td>
<td>$3.3307 \times 10^{-16}$</td>
<td>$1.102 \times 10^{-18}$</td>
</tr>
<tr>
<td>10</td>
<td>$2.7756 \times 10^{-16}$</td>
<td>$1.004 \times 10^{-20}$</td>
</tr>
<tr>
<td>20</td>
<td>$1.1102 \times 10^{-16}$</td>
<td>$0.102 \times 10^{-21}$</td>
</tr>
<tr>
<td>30</td>
<td>$4.1633 \times 10^{-17}$</td>
<td>$0.421 \times 10^{-19}$</td>
</tr>
<tr>
<td>40</td>
<td>$1.3878 \times 10^{-17}$</td>
<td>$0.302 \times 10^{-20}$</td>
</tr>
<tr>
<td>50</td>
<td>$1.3878 \times 10^{-17}$</td>
<td>$0.012 \times 10^{-20}$</td>
</tr>
</tbody>
</table>

Table 5.1 clearly shows that to achieve the same accuracy the Haar method is much faster than both upwind and the RT method.

Tables 5.3, 5.4, 5.5 and 5.6 have shown the accuracy of the Haar wavelet method of Eq. (5.31) for various time values. Computer simulation was carried out in the cases $m=32$ and $m=64$, the computed results were compared with the
Table 5.3: Error estimation of convection-diffusion equation (5.36) for various values of spatial variable $x$ and time $t$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x=0.9$</th>
<th>RT method</th>
<th>Haar method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$0.0515 \times 10^{-20}$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$4.4409 \times 10^{-16}$</td>
<td>$0.0515 \times 10^{-20}$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$4.4409 \times 10^{-16}$</td>
<td>$0.0215 \times 10^{-20}$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$4.4409 \times 10^{-16}$</td>
<td>$0.0214 \times 10^{-20}$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$8.8818 \times 10^{-16}$</td>
<td>$0.0215 \times 10^{-20}$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>$6.6613 \times 10^{-16}$</td>
<td>$0.315 \times 10^{-20}$</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>$3.3307 \times 10^{-16}$</td>
<td>$0.0115 \times 10^{-20}$</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>$1.3878 \times 10^{-16}$</td>
<td>$0.0115 \times 10^{-20}$</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>$1.1102 \times 10^{-16}$</td>
<td>$0.012 \times 10^{-20}$</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>$3.4695 \times 10^{-17}$</td>
<td>$0.012 \times 10^{-20}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.4: Comparison of the exact solution and the Haar solution of Convection-diffusion equation (5.36) for $t=0.85$

<table>
<thead>
<tr>
<th>$x$</th>
<th>Exactsolution</th>
<th>Haar solution</th>
<th>$m = 32$</th>
<th>$m = 64$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.44089</td>
<td>0.44062</td>
<td>0.44087</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.46847</td>
<td>0.46818</td>
<td>0.46843</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.49658</td>
<td>0.49622</td>
<td>0.49655</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.52512</td>
<td>0.52498</td>
<td>0.52510</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.55397</td>
<td>0.55376</td>
<td>0.55389</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.58300</td>
<td>0.58288</td>
<td>0.58301</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.61208</td>
<td>0.61187</td>
<td>0.61207</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.64108</td>
<td>0.64114</td>
<td>0.64102</td>
<td></td>
</tr>
</tbody>
</table>

exact solution, more accurate results can be obtained by using a larger $m$. 

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Table 5.5: Comparison of the exact solution and the Haar solution of Convection-diffusion equation (5.36) for t=0.25

<table>
<thead>
<tr>
<th>x</th>
<th>Exact solution</th>
<th>m = 32</th>
<th>m = 64</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.57657</td>
<td>0.57625</td>
<td>0.57652</td>
</tr>
<tr>
<td>0.2</td>
<td>0.60650</td>
<td>0.60627</td>
<td>0.60648</td>
</tr>
<tr>
<td>0.3</td>
<td>0.63642</td>
<td>0.63611</td>
<td>0.63639</td>
</tr>
<tr>
<td>0.4</td>
<td>0.66616</td>
<td>0.66582</td>
<td>0.66609</td>
</tr>
<tr>
<td>0.5</td>
<td>0.69558</td>
<td>0.69537</td>
<td>0.69558</td>
</tr>
<tr>
<td>0.6</td>
<td>0.72450</td>
<td>0.72423</td>
<td>0.72450</td>
</tr>
<tr>
<td>0.7</td>
<td>0.75277</td>
<td>0.75258</td>
<td>0.75252</td>
</tr>
<tr>
<td>0.8</td>
<td>0.78021</td>
<td>0.78000</td>
<td>0.78019</td>
</tr>
</tbody>
</table>

Table 5.6: Comparison of the exact solution and the Haar solution of Convection-diffusion equation (5.36) for t=0.48

<table>
<thead>
<tr>
<th>x</th>
<th>Exact solution</th>
<th>m = 32</th>
<th>m = 64</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.52274</td>
<td>0.52246</td>
<td>0.52272</td>
</tr>
<tr>
<td>0.2</td>
<td>0.55205</td>
<td>0.55177</td>
<td>0.55197</td>
</tr>
<tr>
<td>0.3</td>
<td>0.58157</td>
<td>0.58135</td>
<td>0.58152</td>
</tr>
<tr>
<td>0.4</td>
<td>0.61118</td>
<td>0.61099</td>
<td>0.61110</td>
</tr>
<tr>
<td>0.5</td>
<td>0.64073</td>
<td>0.64057</td>
<td>0.64078</td>
</tr>
<tr>
<td>0.6</td>
<td>0.67007</td>
<td>0.67045</td>
<td>0.67015</td>
</tr>
<tr>
<td>0.7</td>
<td>0.69905</td>
<td>0.69885</td>
<td>0.69909</td>
</tr>
<tr>
<td>0.8</td>
<td>0.72750</td>
<td>0.72728</td>
<td>0.72756</td>
</tr>
</tbody>
</table>

5.8 Features

The theoretical elegance of the Haar wavelet approach can be appreciated from the simple mathematical relations and their compact derivations and proofs. It has been well demonstrated that in applying the nice properties of Haar wavelets, the partial differential equations can be solved conveniently and accurately by using Haar wavelet method systematically. The main advantage of
the method is its simplicity and small computation costs: it is due to the sparsity of the transform matrices and to the small number of significant wavelet coefficients. In comparison with existing numerical schemes used to solve the convection-diffusion (CD) equations, the scheme in this paper is an improvement over other methods in terms of accuracy. It is worth mentioning that Haar solution provides excellent results even for small values of m, (that is, m=16). For large values of m, we can obtain the results closer to the real values. The method with far less degrees of freedom and with smaller CPU time provides better solutions than classical ones.

The main goal of this chapter is to compare the Haar wavelet method and the restrictive Taylor’s series (RT) approximation method for some well-known
convection-diffusion (CD) equations that appear in many scientific applications. The results prove that the Haar method is more accurate than the previous one. Also Haar wavelet method has been compared with other numerical methods...
Krylov subspace method (KSM). The execution time for Haar wavelet method is less than that RT method and also the KS method. This chapter also confirmed the power of the Haar wavelet method in handling nonlinear equations in general. This method can be easily extended to find the solution of all other nonlinear differential equations. Another benefit of our method is that the scheme presented here, with some modifications, seems to be easily extended to solve model equations including more mechanical, physical or biophysical effects, such as nonlinear convection, reaction, linear diffusion and dispersion.