CHAPTER – 4

$t$ AND $F$ DISTRIBUTIONS:
UNDER MODERATENESS ASSUMPTION
4.1 INTRODUCTION: -

It is well known that normality in the population is the underlying assumption in defining t and F distributions. Hence, if the assumption of normality is replaced by moderateness then it becomes interesting to know what type of changes take place in the corresponding t and F distributions. In this chapter, this aspect is considered.

4.2 t - DISTRIBUTION: -

It has been noted earlier that if

\[ X \sim M(\mu, \delta) \text{ then } \bar{x} \sim M\left(\mu, \frac{\delta}{\sqrt{n}}\right). \]

Hence,

\[ Z = \frac{\bar{x} - \mu}{\frac{\delta}{\sqrt{n}}} \sim M(0,1). \]

We have proved in chapter-3 that \( Y = \frac{(n-1)S^2}{\delta^2} \sim \psi_{(n-1)} \) where

\[ S^2 = \frac{1}{n-1} \sum \left| x - \bar{x} \right|^2. \]

Then, it becomes interesting to know the distribution of the statistic defined as \( \frac{Z}{\sqrt{Y/(n-1)}} \).
Let \( t' = \frac{Z}{\sqrt{\frac{y}{n-1}}} \).

Then, \( t' = \frac{\bar{x} - \mu}{\frac{\delta}{\sqrt{n}}} \cdot \frac{1}{\sqrt{\frac{(n-1)S^2}{\delta^2}} / n-1} = \frac{\bar{x} - \mu}{\frac{S}{\sqrt{n}}} \sim t_{(n-1)} \);

where \( S^2 = \frac{1}{n-1} \sum |x - \bar{x}|^2 \).

Thus, the distribution of \( t' \) is \( t \)-distribution having \((n-1)\) d.f. But we also know that if \( x \sim N(\mu, \sigma) \) and \( y = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)} \) then

\[
Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \quad \text{and} \quad \frac{Z}{\sqrt{\frac{y}{n}}} \sim t_{(n-1)}
\]

which means \( t \)-distribution remains same under the assumption of normality as well as moderateness. Therefore, all the properties and applications of \( t \)-distribution hold under the assumption of moderateness also.

### 4.3 F - DISTRIBUTION:

Let \( X_1 = \frac{(n_1-1)S_1^2}{\delta^2} \sim \psi^2_{(n_1-1)} \) and \( X_2 = \frac{(n_2-1)S_2^2}{\delta^2} \sim \psi^2_{(n_2-1)} \)

are independently distributed. Then it is interesting to know the distribution of the ratio of these two independent \( \psi \)-square variates divided by their respective degrees of freedom.
Let \( F' = \frac{\frac{X_1}{n_1 - 1}}{\frac{X_2}{n_2 - 1}} \).

Then \( F' = \frac{\frac{(n_1 - 1)S_1^2}{\delta^2}}{\frac{(n_2 - 1)S_2^2}{\delta^2}} = \frac{S_1^2}{S_2^2} \).

Where \( S_1^2 = \frac{1}{n_1 - 1} \sum (x - \bar{x})^2 \) and \( S_2^2 = \frac{1}{n_2 - 1} \sum (y - \bar{y})^2 \).

Thus, the distribution of \( F' \) is F-distribution having \([(n_1-1), (n_2-1)]\) d.f.. We know that if

\[
X = \frac{(n_1-1)S_1^2}{\sigma^2} \sim \chi^2_{(n_1)} \quad \text{and} \quad Y = \frac{(n_2-1)S_2^2}{\sigma^2} \sim \chi^2_{(n_2)}
\]

are independently distributed then the statistic

\[
F = \frac{X}{Y} \sim F_{[(n_1-1), (n_2-1)]}.
\]

Hence F-distribution remains same under the assumption of normality as well as moderateness. Therefore, all the properties and applications of F-distribution hold under the assumption of moderateness also.

**NOTE**: Since \( X_1 \sim M(0,1) \) and \( X_2 \sim M(0,1) \) are independent, \( X_1^2 \sim \psi^2_{(1)} \) and \( X_2^2 \sim \psi^2_{(1)} \) are also independent. Hence, F-statistic can be defined as

\[
\frac{X_1^2}{X_2^2} \sim F_{(1,1)}.
\]
4.4 EXACT SAMPLING TEST FOR TESTING HYPOTHESES $H_0$:

$\delta_1 = \delta_2$:

One of the main short-comings often highlighted against the use of mean deviation is that very little theory of inference related to it, particularly the tests of hypotheses related to it, is available. To consider this aspect of tests of hypothesis related to $\delta$ let us recall that, for normal distribution, the relationship between standard deviation, $\sigma$, and mean deviation, $\delta$, is $\delta = \sqrt{\frac{2}{\pi}} \sigma$ and for Laplace distribution it is $\delta = \sqrt{\frac{1}{2}} \sigma$.

Similarly, for any distribution, there always exists similar relationship between them and is of the form $\delta = c \sigma$, where $c \leq 1$ is a constant. Hence, with the crucial normality assumption involved in exact sampling tests, it is clear that testing $H_0: \delta = \delta_0$ is equivalent to test the $H_0: \sigma^2 = \sigma_0^2$ and testing $H_0: \delta_1 = \delta_2$ is equivalent to test $H_0: \sigma_1^2 = \sigma_2^2$, where $\delta_0 = \sqrt{\frac{2}{\pi}} \sigma_0$, $\delta_1 = \sqrt{\frac{2}{\pi}} \sigma_1$ and $\delta_2 = \sqrt{\frac{2}{\pi}} \sigma_2$. This means, Chi-square tests for testing $H_0: \sigma^2 = \sigma_0^2$ and F-test for testing $H_0: \sigma_1^2 = \sigma_2^2$ can always be used for testing $H_0: \delta = \delta_0$ and $H_0: \delta_1 = \delta_2$ respectively.

Since moderate distribution is an alternative normal distribution the above statement holds true for moderate distribution also.