CHAPTER – 3

$\psi^2$ (PSI-SQUARE)-DISTRIBUTION:
AN ALTERNATIVE CHI-SQUARE DISTRIBUTION
3.1 INTRODUCTION: -

Chi-square distribution, t- distribution and F- distribution are the most widely used distributions for testing various hypotheses. These distributions are basically derived from the normal distribution. Hence, if there is an alternative of normal distribution, it is natural to look for the parallel alternatives of $\chi^2$, t and F distribution. In this chapter we are looking to derive an alternative of $\chi^2$ distribution.

It is well known that if $X \sim N(0,1)$ then $X^2 \sim \chi^2(1)$. Therefore, when $X \sim M(0,1)$, it is natural to think about the distribution of $X^2$ and consider that distribution as an alternative of $\chi^2$ distribution. In this chapter an attempt is made to explore this possibility. A $\chi^2$ type distribution is defined and is called $\psi^2$ (psi-square) distribution. Various properties of this proposed $\psi^2$ distribution are discussed and the non-central psi-square distribution is derived.

3.2 PSI-SQUARE DISTRIBUTION WITH 1 D.F. : -

3.2.1 DERIVATION AND DEFINITION: -

Let $X \sim M(0,1)$. 

Then the m.g.f. of $X^2$ is
\[ M_{X^2}(t) = (1 - \pi t)^{-\frac{1}{2}}, \left|t\right| < \frac{1}{\pi} \]

= m.g.f. of Gamma distribution with parameters $\frac{1}{\pi}$ and $\frac{1}{2}$.

= m.g.f. of $G\left(\frac{1}{\pi}, \frac{1}{2}\right)$.

Hence by uniqueness theorem,

$X^2$ follows gamma distribution having parameters $\frac{1}{\pi}$ and $\frac{1}{2}$.

Further, just as square of standard normal variate follows $G\left(\frac{1}{2}, \frac{1}{2}\right)$ and is known as chi-square distribution having degree of freedom (d.f.) one, the square of standard moderate variate also follows $G\left(\frac{1}{\pi}, \frac{1}{2}\right)$ and therefore it can also be considered as chi-square($\chi^2$) type distribution having d.f. one. This $\chi^2$- type variable may be denoted by psi-square $(\psi^2)$ variable and the $\chi^2$- type distribution may be called psi-square $(\psi^2)$ distribution having d.f. one.

In notations, this may be written as $X \sim \psi^2(1)$.

3.2.2 **ANOTHER DEFINITION OF $\psi^2(1)$**:

Let $X$ be a random variable such that its p.d.f. is
\[
f(x) = \frac{1}{\pi} e^{-\frac{x}{\pi} x^{\frac{1}{2}}}, 0 \leq x < \infty
\]

\[
f(x) = 0, \text{ otherwise}
\]

Then X is said to follow psi-square distribution with 1 d.f.

**NOTE:** If \( X \sim M(\mu, \delta) \) then \( Z = \frac{X - \mu}{\delta} \sim M(0,1) \) and \( Z^2 = \left( \frac{X - \mu}{\delta} \right)^2 \) is a psi-square variate with 1 d.f.

The following are some properties of \( \psi^2_{(1)} \) which can easily proved.

3.2.3 **MEAN:** \( \mu_1 = \frac{\pi}{2} \).

3.2.4 **VARIANCE:** \( \mu_2 = \frac{\pi^2}{2} \).

3.2.5 \( \mu_3 = \pi^3, \quad \mu_4 = \frac{15\pi^4}{4} \).

3.2.6 \( \beta_1 = 8, \quad \beta_2 = 15 \).

3.3 **PSI-SQUARE DISTRIBUTION WITH n D.F. :**

3.3.1 **DERIVATION AND DEFINITION:**

It is well known that if \( (x_1, x_2, ..., x_n) \) is a random sample of size n drawn from \( N(0,1) \) then the sampling distribution of \( \sum_{i=1}^{n} x_i^2 \sim G\left(\frac{1}{2}, \frac{n}{2}\right) \) and it is known as chi-square distribution having n d.f.
in notation it is written as \( \sum_{i=1}^{n} x_i^2 \sim \chi^2_{(n)} \). Similarly, let \((x_1, x_2, ..., x_n)\) be a random sample of size \(n\) drawn from \(M(0,1)\). Then, the sampling distribution of \( \sum_{i=1}^{n} x_i^2 \) can be derived as follows.

The m.g.f. of \( \sum_{i=1}^{n} x_i^2 \) is

\[
M_{\sum_{i=1}^{n} x_i^2}(t) = (1 - \pi t)^{-\frac{n}{2}}, |t| < \frac{1}{\pi}
\]

= m.g.f. of Gamma distribution with parameters \( \frac{1}{\pi} \) and \( \frac{n}{2} \),

by uniqueness theorem,

\( \sum_{i=1}^{n} x_i^2 \) follows gamma distribution having parameters \( \frac{1}{\pi} \) and \( \frac{n}{2} \).

Thus, just as \( G\left(\frac{1}{2}, \frac{n}{2}\right) \) is known called \( \chi^2 \)-distribution, \( G\left(\frac{\pi}{2}, \frac{n}{2}\right) \) is also a \( \chi^2 \)-type distribution and this distribution may be called \( \psi^2 \) (psi-square) distribution having d.f. \( n \) and may be denoted by \( \psi^2_{(n)} \).

### 3.3.2 Another Definition of \( \psi^2_{(n)} \):

Let \( X \) be a random variable such that its p.d.f. is

\[
f(x) = \begin{cases} 
\frac{1}{\pi^{\frac{n}{2}}} e^{-\frac{x}{\pi/\frac{n}{2}}} \left(\frac{n}{2}\right)^{-1}, & 0 < x < \infty \\
0, & \text{otherwise}
\end{cases}
\]

\[
= 0, \quad \text{otherwise}
\]
then X is said to follow psi-square distribution with n d.f.

In notations this may be written as $X \sim \psi^2_{(n)}$.

**NOTE**: -(i) If $X_i \ (i = 1, 2, \ldots, n)$ are n independent moderate variates with means $\mu_i$ and mean deviations $\delta_i, (i = 1, 2, \ldots, n)$, then $\psi^2 = \sum_{i=1}^{n} \left( \frac{X_i - \mu_i}{\delta_i} \right)^2$,

is a psi-square variate with n d.f.

(ii) If $X \sim \chi^2_{(n)}$, then $Y = \frac{\pi}{2} X$ follows psi-square distribution with n d.f. i.e. $Y \sim \psi^2_{(n)}$.

(iii) If $X \sim \psi^2_{(n)}$, then $Y = \frac{2}{\pi} X$ follows chi-square distribution with n d.f. i.e. $Y \sim \chi^2_{(n)}$.

Thus, like moderate distribution and normal distribution, the coefficient $\frac{2}{\pi}$ (along with its reciprocal $\frac{\pi}{2}$) serves as golden bridge between the psi-square distribution and chi-square distribution through which we can commute between the two distributions for various purposes.

**3.4 SOME PROPERTIES AND RESULTS: -**

**3.4.1 CHARACTERISTIC FUNCTION: -**

It can easily be seen that if $X \sim \psi^2_{(n)}$ then
\[ \phi_X(t) = (1 - \pi it)^{-\frac{n}{2}}. \]

**NOTE:** - Let \( Y = \frac{2}{\pi} X \). Then, it can easily be seen that

\[ \phi_Y(t) = (1 - 2it)^{-\frac{n}{2}} \]

= Characteristic function of \( \chi^2_{(n)} \).

### 3.4.2 MEAN: -

\[ \mu_1 = \frac{n\pi}{2}. \]

**NOTE:** - If \( Y = \frac{2}{\pi} X \). Then, it can easily be seen that

\[ E(Y) = \frac{2}{\pi} E(X) = n = \text{Mean of } \chi^2_{(n)}. \]

#### 3.4.3

\[ \mu_2 = n(n + 2)^{\frac{\pi^2}{4}}, \]

\[ \mu_3 = n(n + 2)(n + 4)^{\frac{\pi^3}{8}}, \]

\[ \mu_4 = n(n + 2)(n + 4)(n + 6)^{\frac{\pi^4}{16}}. \]

### 3.4.4 VARIANCE:-

\[ \mu_2 = \frac{n\pi^2}{2}. \text{ i.e. Variance} = \pi \text{ Mean} \]

**NOTE:** - If \( Y = \frac{2}{\pi} X \). Then, it can easily be seen that

\[ V(Y) = \frac{4}{\pi} V(X) = 2n = \text{Variance of } \chi^2_{(n)}. \]

**REMARK:** - For \( X \sim \chi^2_{(n)} \), Variance = 2 Mean.
3.4.5 $\mu_3 = n\pi^3$

3.4.6 $\mu_4 = \frac{3}{4} n(n + 4)\pi^4$

3.4.7 $r^{th}$ ROW MOMENT: -

$$\mu'_r = \left(\frac{\pi}{2}\right)^r n(n + 2)(n + 4)\ldots(n + 2r - 2).$$

3.4.8 MOMENT GENERATING FUNCTION: -

$$M_X(t) = (1 - \pi t)^\frac{-n}{2}, \quad |t| < \frac{1}{\pi}.$$ 

3.4.9 CUMULANT GENERATING FUNCTION: -

$$K_X(t) = -\frac{n}{2} \log(1 - \pi t).$$

3.4.10 CUMULANT: -

$$k_r = \frac{n\pi^r (r-1)!}{2}.$$ 

3.4.11 $\beta_1 = \frac{8}{n}, \quad \beta_2 = \frac{12}{n} + 3.$

**NOTE**: As $n \to \infty$, $\beta_1 \to 0$ and $\beta_2 \to 3.$

3.4.12 $\gamma_1 = \sqrt[8]{n}, \quad \gamma_2 = \frac{12}{n}.$

3.4.13 MODE: -

Let $X \sim \psi^2_{(n)},$

so that
\[ f(x) = \frac{1}{\frac{n}{\pi^2 \Gamma(n/2)}} \cdot \frac{e^{-x/n}}{x} ; 0 \leq x < \infty \]

Mode of the distribution is the solution of \( f'(x) = 0 \) and \( f''(x) < 0 \).

Differentiation of Logarithm of \( f(x) \) with respect to \( x \) gives:

\[
\frac{f'(x)}{f(x)} = 0 - \frac{1}{\pi} + \frac{(n/2 - 1)}{x}.
\]

Since \( f(x) \neq 0 \), \( f'(x) = 0 \Rightarrow x = \frac{\pi(n-2)}{2} \).

It can be easily seen that at the point \( x = \frac{\pi(n-2)}{2} \), \( f''(x) < 0 \).

\[
\therefore \text{Mode of the psi-square distribution with n d.f. is } x = \frac{\pi(n-2)}{2}.
\]

Thus, mode of the psi-square distribution with n d.f. is \( \text{Mean} - \pi \).

Clearly, Mean - mode = \( \pi \).

**REMARK**: - For \( X \sim \chi^2_{(n)} \), Mode = Mean - 2.

**3.4.14 KARL-PEARSON’S COEFFICIENT OF SKEWNESS**: -

Karl-Pearson’s coefficient of skewness is defined as

\[
\frac{\text{Mean} - \text{Mode}}{\text{S.d.}} = \sqrt{\frac{2}{n}}.
\]

**REMARK**: -(i) For \( X \sim \chi^2_{(n)} \), Karl-Pearson’s coefficient of skewness is also \( \sqrt{\frac{2}{n}} \).
(ii) Since Karl Pearson’s coefficient of skewness is greater than zero for \( n \geq 1 \), like chi-square distribution, the psi-square distribution is also positively skewed. Further since skewness is inversely proportional to the square root of d.f., it rapidly tends to symmetry as the d.f. increases.

3.4.15 ADDITIVE PROPERTY: -

The sum of independent psi-square variates is also a psi-square variate. i.e. If \( X_i, (i = 1, 2, \ldots, k) \) are independent psi-square variates with \( n_i \) d.f. respectively, then \( \sum_{i=1}^{k} X_i \) is also a psi-square variate with \( \sum_{i=1}^{k} n_i \) d.f.

NOTE: - (i) Converse is also true, i.e. if \( X_i, i = 1, 2, \ldots, k \) are psi-square variates with \( n_i, i = 1, 2, \ldots, k \) d.f. respectively and if \( \sum_{i=1}^{k} X_i \) is a psi-square variate with \( \sum_{i=1}^{k} n_i \) d.f. then \( X_i \)'s are independent.

(ii) If \( X \) and \( Y \) are independent non-negative variates such that \( X + Y \) follows psi-square distribution with \( n_1 + n_2 \) d.f. and if \( X \) is a psi-square variate with \( n_1 \) d.f. then \( Y \) is a psi-square variate with \( n_2 \) d.f.

3.4.16 PROBABILITY CURVE: -

We have

\[
f(x) = \frac{1}{\pi^{n/2} \Gamma(n/2)} e^{-\frac{x}{\pi} \left( \frac{n}{2} \right)}; 0 \leq x < \infty.
\]
\[ : f'(x) = \left[ \frac{\pi(n-2)}{2} - x \right] \frac{x}{\pi x} f(x). \]

Since \( X > 0 \) and \( f(x) \geq 0, f'(x) < 0 \) if \( n \leq 2 \), for all values of \( X \). Thus the psi-square curve for 1 and 2 d.f. is monotonically decreasing.

When \( n > 2 \),

\[
\begin{align*}
   f'(x) &= \begin{cases} 
   > 0, & \text{if } X < \frac{\pi}{2} (n-2) \\
   = 0, & \text{if } X = \frac{\pi}{2} (n-2) \\
   < 0, & \text{if } X > \frac{\pi}{2} (n-2).
   \end{cases}
\end{align*}
\]

i.e.

\[
\begin{align*}
   f'(x) &= \begin{cases} 
   > 0, & \text{if } X < (\text{mean} - \pi) \\
   = 0, & \text{if } X = (\text{mean} - \pi) \\
   < 0, & \text{if } X > (\text{mean} - \pi).
   \end{cases}
\end{align*}
\]

This implies that for \( n > 2 \), \( f(x) \) is monotonically increasing for \( 0 < X < (\text{mean} - \pi) \) and monotonically decreasing for \( (\text{mean} - \pi) < X < \infty \), while at \( X = (\text{mean} - \pi) \), it attains the maximum value.

For \( n \geq 1 \), as \( x \) increases, \( f(x) \) decreases rapidly and finally tends to zero as \( X \to \infty \).
Thus for $n > 1$, the psi-square probability curve is positively skewed towards higher values of $X$. Moreover $X$-axis is an asymptote to the curve.

For $n = 1$, it will be an inverted J-shaped curve.

For $n = 2$, the curve will meet $Y = f(x)$ axis at $X = 0$.

i.e. at $f(x) = \frac{1}{\pi}$.

![Psi-square curve with various d.f.](image1)

Fig. 3.4.1 Psi-square curve with various d.f.

**REMARK**: - Comparison between chi square and psi square curves for various d.f. is shown in Figure 3.4.2.
Fig. 3.4.2 Comparison of Chi-square curve and Psi-square curve with various d.f.

In the above Figure 3.4.2, the curve having higher peak is chi-square curve while the curve having lower peak is psi-square curve. For 

\( n = 2 \), the chi-square curve will meet Y axis at \( X = 0 \) for \( f(x) = 0.5 \) and the psi-square curve will meet Y axis at \( X = 0 \) for \( f(x) = 0.3183 \). It may be noted that as \( n \) increases the skewness disappears rapidly and the curves assume normal curve type symmetry for \( n \geq 30 \).

### 3.4.17 MEAN DEVIATION:

If \( X \sim \chi^2_{(n)} \), then mean deviation of \( X \) is

\[
\frac{n}{2} e^{-\frac{n}{2}} \frac{-\frac{n}{2}}{\Gamma\left(\frac{n}{2}\right)}
\]


Let \( y = \frac{\pi}{2} x \) : M.D.(\( y \)) = \[\frac{\pi}{2}\] M.D.(\( x \)). [See : Appendix – (C)(6)]

\[
\therefore \text{Mean deviation of psi-square distribution with n d.f. is}
\]

\[
\pi \left(\frac{n}{2}\right)^{\frac{n}{2}} e^{-\frac{n}{2}} \frac{-\frac{n}{2}}{\Gamma\left(\frac{n}{2}\right)} \text{ i.e. } \pi \left(\frac{n}{2e}\right)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)
\]
We know that \( \Gamma(az + b) \approx \sqrt{2\pi}e^{-az}(az)^{az+b-\frac{1}{2}}. \)

[See: Appendix-C(11)].

Take \( a = \frac{1}{2}, z = n \) and \( b = 0 \), we get

\[
\Gamma\left(\frac{n}{2}\right) \approx \sqrt{2\pi}e^{-\frac{n}{2}}\left(\frac{n}{2}\right)^{n-\frac{1}{2}}.
\]

\[
\therefore \frac{n^{n/2}}{\Gamma(n/2)} \approx \frac{1}{2} \sqrt{n/\pi}.
\]

\[
\therefore \text{For large value of } n, \text{ mean deviation of psi-square distribution with } n \text{ d.f. is } \frac{1}{2} \sqrt{n\pi}.
\]

**NOTE**: For large value of \( n \), \( \sigma = \sqrt{2\pi}\delta \). i.e. \( \frac{\delta}{\sigma} = \frac{1}{\sqrt{2\pi}}. \)

**3.4.18** If \( X_1 \) and \( X_2 \) are two independent psi-square variates with \( n_1 \) and \( n_2 \) d.f. respectively, then \( \frac{X_1}{X_2} \) is a \( \beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \) variate.

**3.4.19** If \( X_1 \) and \( X_2 \) are two independent psi-square variates with \( n_1 \) and \( n_2 \) d.f. respectively, then \( U = \frac{X_1}{X_1 + X_2} \) and \( V = X_1 + X_2 \) are independently distributed, \( U \) as a \( \beta_1\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \) variate and \( V \) as a psi-square variate with \( n_1 + n_2 \) d.f.

**3.4.20** If \( X \sim \psi^2_{(n)} \), then \( \frac{X}{\pi} \sim G\left(\frac{n}{2}\right). \)
3.4.21 If $X_i \sim M (\mu, \delta)$ ($i = 1,2,\ldots,n$) are independent variates then
\[
\begin{bmatrix}
\overline{X} - \mu \\
\frac{\delta}{\sqrt{n}}
\end{bmatrix}^2
\]
is a psi-square variate with 1 d.f.

3.4.22 PSI ($\psi$) DISTRIBUTION: -

(i) DERIVATION OF $\psi$ DISTRIBUTION: -

We know that if $X \sim \chi^2_{(n)}$ then $\sqrt{X} \sim \chi_{(n)}$. Similarly, if $X \sim \psi^2_{(n)}$ then we want to derive the distribution of $\sqrt{X}$ which is as follows.

If $X \sim \psi^2_{(n)}$ then
\[
f(x) = \frac{1}{\pi^{n/2} \Gamma(n/2)} e^{-\frac{x}{2}} \left(\frac{n}{2}\right)^{n/2-1} ; 0 \leq x < \infty.
\]

Let $Y = \sqrt{X}$ i.e. $Y^2 = X$.

\[
\therefore f(y) = \left| \frac{dx}{dy} \right| f(x) = 2y f(y^2)
\]
\[
= 2y \frac{1}{\pi^{n/2} \Gamma(n/2)} e^{-\frac{1}{2} y^2} \left(\frac{n}{2}\right)^{n/2-1} ; 0 \leq y < \infty
\]
\[
= \frac{2}{\pi^{n/2} \Gamma(n/2)} e^{-\frac{1}{n} y^2} y^{n-1} ; 0 \leq y < \infty
\]

Then variate $Y$ follows psi distribution with $n$ d.f.

In notation this may be written as $X \sim \psi_{(n)}$. 

(ii) $\mu_r = \pi^2 \frac{r^{(n+r)/2}}{\Gamma(n/2)}$

i.e. $E(\psi^r) = \pi^2 \frac{r^{(n+r)/2}}{\Gamma(n/2)}$

NOTE: - (i) $E(\psi) = \sqrt{n\pi} \frac{r^{(n+1)/2}}{\Gamma(n/2)}$

Using Appendix - C(10) for large $n$,

$E(\psi) = \sqrt{n\pi} \frac{r^{1/2}}{\Gamma(n/2)} = \sqrt{\frac{nr}{2}}$.

Also, $E(\psi^2) = \pi \frac{r^{(n+2)/2}}{\Gamma(n/2)} = \frac{n\pi}{2}$.

Hence, for large $n$, $E(\psi^2) \approx [E(\psi)]^2$, i.e. $\text{Var}(\psi) = 0$.

(ii) If $X_i (i = 1, 2, \ldots, n)$ are $n$ independent moderate variates with means $\mu_i$ and mean deviations $\delta_i (i = 1, 2, \ldots, n)$, then

$\psi^2 = \sum_{i=1}^{n} \left( \frac{X_i - \mu_i}{\delta_i} \right)^2$, is a psi-square variate with $n$ d.f. Therefore,

$\psi = \sqrt{\sum_{i=1}^{n} \left( \frac{X_i - \mu_i}{\delta_i} \right)^2}$ follows psi distribution with $n$ d.f.

(iii) It may be noted that for $X \sim \chi_{(n)}$, the p.d.f is defined as

$$f(x) = \frac{1}{2^{(n/2)-1} \Gamma(n/2)} e^{-\frac{1}{2} x^2} x^{n-1}; 0 \leq x < \infty.$$
3.4.23 Exponential distribution with mean $\pi$ is a particular case of psi-square distribution when $n = 2$.

3.4.24 If $X_i (i = 1, 2, \ldots, n)$ are i.i.d. exponential variates with parameters $\theta$ then $\pi \theta \sum_{i=1}^{n} X_i$ is a psi-square variate with $2n$ d.f.

3.4.25 If $X \sim M(\mu, \delta)$ then $E \left[ \left| \frac{X-\mu}{\delta} \right|^{2r} \right]$ is $r^{th}$ raw moment of psi-square variate with 1 d.f.

3.4.26 If $X_1$ and $X_2$ are independently distributed each as psi-square variate with 2 d.f. then characteristic function of each of them is $(1 - it\pi)^{-1}$ and the density function of $Y = \frac{1}{\pi} (X_1 - X_2)$ is standard Laplace distribution.

3.4.27 For psi-square variate $X$ with $n$ d.f., the m.g.f. of $Z = \log X$ is

$$\frac{\pi^t \Gamma \left( \frac{n}{2} + t \right)}{\Gamma \left( \frac{n}{2} \right)}.$$

**Proof:**

Let $Z = \log X \Rightarrow X = e^Z \quad \therefore \quad dx = e^Z dz$

The probability differential of $X$ is given by:

$$dP(x) = \frac{1}{\frac{n}{2} \Gamma \left( \frac{n}{2} \right)} e^{-\frac{x}{\pi x} \left( \frac{n}{2} \right)} - 1 \quad dx; \quad 0 \leq x < \infty.$$
\[
\therefore \text{d}G(z) = \frac{1}{\pi^2 \Gamma\left(\frac{n}{2}\right)} e^{-\frac{z^2}{\pi}} \left(e^{z^2}\right)^{\left(\frac{n}{2}-1\right)} e^{z}dz; \quad -\infty \leq z < \infty.
\]

\[\therefore \text{The m.g.f. of Z is}
\]
\[M_Z(t) = \frac{1}{\pi^2 \Gamma\left(\frac{n}{2}\right)} \int_{-\infty}^{\infty} \frac{e^{-\frac{z^2}{\pi}} + nz + tz}{e^{\pi t/2}} dz.
\]

Let \(Y = \frac{e^z}{\pi}\).

\[\therefore M_Y(t) = \frac{1}{\pi^2 \Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} e^{-y}(\pi y)^{\left(\frac{n}{2}+t\right)} y^{-1} dy = \frac{\pi^t \Gamma\left(\frac{n}{2}+t\right)}{\Gamma\left(\frac{n}{2}\right)}.
\]

**3.4.28** If \(X\) and \(Y\) are independent psi-square variates, each with \(n\) d.f. and \(U = \frac{X}{Y}\), then for \(r > 0\), 
\[E(U^r) = \frac{\Gamma\left(\frac{n}{2}+r\right)\Gamma\left(\frac{n}{2}-r\right)}{\Gamma\left(\frac{n}{2}\right)^2}, \quad n > 2r.
\]

**Proof:** -

\[E(U^r) = E\left[\left(\frac{X}{Y}\right)^r\right] = E\left[e^{\left(\frac{X}{Y}\right)^r}\right] = E\left(e^{r \log x - r \log y}\right)
\]

\[= E\left(e^{r \log x}\right)E\left(e^{-r \log y}\right) \quad (\because X \text{ and } Y \text{ are independent.})
\]

\[= M_{\log x}(r) \cdot M_{\log y}(-r).
\]

Using 3.4.27, we can write
\[
E(U^r) = \frac{\pi^r \Gamma\left(\frac{n}{2} + r\right)}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\pi^{-r} \Gamma\left(\frac{n-r}{2}\right)}{\Gamma\left(\frac{n-r}{2}\right)} = \frac{\Gamma\left(\frac{n}{2} + r\right) \Gamma\left(\frac{n}{2} - r\right)}{\left[\Gamma\left(\frac{n}{2}\right)\right]^2}, \quad n > 2r.
\]

**NOTE:**

(i) \(E(U) = E\left(\frac{X}{Y}\right) = \frac{n}{n-2}, \quad n > 2.\)

(ii) \(E\left(U^2\right) = E\left(\frac{X^2}{Y^2}\right) = \frac{n(n+2)}{(n-2)(n-4)}, \quad n > 4.\)

(iii) \(\text{Var}(U) = V\left(\frac{X}{Y}\right) = \frac{4n(n-1)}{(n-2)^2(n-4)}, \quad n > 4.\)

3.4.29 If \(x_i (i = 1, 2, \ldots, n)\) is a random sample from \(M(0,1)\) then

distribution of \(\bar{m}x_m + (n-m)\bar{x}_{n-m}\) is \(\psi_2(2).\)

**Proof:**

Since \(x_i (i = 1, 2, \ldots, n)\) is a random sample from \(M(0,1),\)

\(\bar{x}_m \sim M\left(0, \frac{1}{\sqrt{m}}\right)\) and \(\bar{x}_{n-m} \sim M\left(0, \frac{1}{\sqrt{n-m}}\right).\)

i.e. \(\bar{m}x_m \sim \psi_1(1)\) and \((n-m)\bar{x}_{n-m} \sim \psi_1(1).\)

Since \(\bar{x}_m\) and \(\bar{x}_{n-m}\) are independent, by additive property of psi-square distribution, \(\bar{m}x_m + (n-m)\bar{x}_{n-m}\) is \(\psi_2(2).\)

3.4.30 **Theorem:** If \(x_i (i = 1, 2, \ldots, n)\) are independent and identically distributed (i.i.d.) as \(M(\mu, \delta)\) then

\[\sum_{i=1}^{n} \frac{|x_i - \bar{x}|^2}{\delta^2} = \frac{ns^2}{\delta^2} = \frac{(n-1)s^2}{\delta^2}\]

is a psi-
square variate with \( n-1 \) d.f. where \( s^2 = \frac{1}{n} \sum_{i=1}^{n} |x_i - \bar{x}|^2 \) and
\[
S^2 = \frac{1}{n-1} \sum_{i=1}^{n} |x_i - \bar{x}|^2 .
\]

**Proof:**

Since \( \bar{x} \) and \( x_i - \bar{x}, i = 1, 2, 3 \ldots n. \) are independently distributed, \( \bar{x} \) and \( s^2 = \frac{1}{n} \sum_{i=1}^{n} |x_i - \bar{x}|^2 \) are also independently distributed.

Now,
\[
\sum_{i=1}^{n} (x_i - \mu)^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2 .
\]

\[
\therefore \sum_{i=1}^{n} \left( \frac{x_i - \mu}{\delta} \right)^2 = \sum_{i=1}^{n} \left( \frac{x_i - \bar{x}}{\delta} \right)^2 + n \left( \frac{\bar{x} - \mu}{\delta} \right)^2 \Rightarrow U = V + W
\]

Here \( U \) is a psi-square variate with \( n \) d.f and \( W \) is also a psi-square variate with 1 d.f.

Hence,
\[
M_U(t) = (1 - \pi t)^{-\frac{n}{2}}, \quad |t| < \frac{1}{\pi} .
\]

and
\[
M_W(t) = (1 - \pi t)^{-\frac{1}{2}}, \quad |t| < \frac{1}{\pi} .
\]

Further, since \( \bar{x} \) and \( s^2 \) are independent, \( V \) and \( W \) are also independently distributed.
\[
M_U(t) = M_{V + W}(t) = M_V(t)M_W(t)
\]

\[
M_V(t) = (1 - \pi t)^{-(n-1)/2}, \quad |t| < \frac{1}{\pi},
\]

which is m.g.f. of psi-square variate with \(n-1\) d.f.

Hence by uniqueness theorem of m.g.f.'s,

\[
V = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\delta^2} = \sum_{i=1}^{n} \left| x_i - \bar{x} \right|^2 = \frac{ns^2}{\delta^2} = \frac{(n-1)s^2}{\delta^2}
\]

is a psi-square variate with \(n-1\) d.f.

**NOTE:**
(i) If \(x_1, x_2, \ldots, x_n\) is a random sample from a standard moderate distribution then \(\sum_{i=1}^{n} \left( x_i - \bar{x} \right)^2 \) is a psi-square variate with \(n-1\) d.f.

(ii) The probability distribution of \(S^2\) is

\[
f(S^2) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \left( \frac{n-1}{\pi \delta^2} \right)^{\frac{n-1}{2}} e^{-\left(\frac{(n-1)s^2}{\pi \delta^2}\right)} (S^2)^{\frac{n-3}{2}} ; 0 < S^2 < \infty.
\]

(iii) The probability distribution of \(S\) is

\[
f(S) = \frac{2}{\Gamma\left(\frac{n-1}{2}\right)} \left( \frac{n-1}{\pi \delta^2} \right)^{\frac{n-1}{2}} e^{-\left(\frac{(n-1)s^2}{\pi \delta^2}\right)} S^{n-2} ; -\infty < S < \infty.
\]

(iv) The probability distribution of \(s^2\) is

\[
f(s^2) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \left( \frac{n}{\pi \delta^2} \right)^{\frac{n-1}{2}} e^{-\frac{ns^2}{\pi \delta^2}} (s^2)^{\frac{n-3}{2}} ; 0 < s^2 < \infty.
\]

(v) The probability distribution of \(s\) is
\[
f(s) = \frac{2}{\Gamma\left(\frac{n-1}{2}\right)} \left[ \frac{n}{\pi \delta^2} \right]^{\frac{n-1}{2}} e^{-\frac{n s^2}{\pi \delta^2}} s^{-2} ; -\infty < s < \infty.
\]

(vi) The joint probability distribution of \( \bar{x} \) and \( S^2 \) is

\[
f(\bar{x}, S^2) \, d\bar{x} \, dS^2 = \frac{\sqrt{n} (n-1) \frac{n-1}{2}}{\pi \delta^n \Gamma\left(\frac{n-1}{2}\right)} e^{-\left[ \frac{n(\bar{x}-\mu)^2 + (n-1)S^2}{\pi \delta^2} \right]} (S^2)^{\frac{n-1}{2}} \, d\bar{x} \, dS^2.
\]

(vii) The joint probability distribution of \( \bar{x} \) and \( S \) is

\[
f(\bar{x}, S) \, d\bar{x} \, dS = \frac{2\sqrt{n} (n-1) \frac{n-1}{2}}{\pi \delta^n \Gamma\left(\frac{n-1}{2}\right)} e^{-\left[ \frac{n(\bar{x}-\mu)^2 + (n-1)S^2}{\pi \delta^2} \right]} S^{n-2} \, d\bar{x} \, dS.
\]

3.4.31 MEANS AND VARIANCES OF \( S, S^2, s \) and \( s^2 \):

**METHOD:**

We know that the p.d.f. of \( S^2 \) is

\[
f(S^2) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \left[ \frac{n-1}{\pi \delta^2} \right]^{\frac{n-1}{2}} e^{-\left[ \frac{(n-1)S^2}{\pi \delta^2} \right]} (S^2)^{\frac{n-3}{2}} ; 0 < S^2 < \infty.
\]

\[
E(S^r) = \left[ \frac{n-1}{\pi \delta^2} \right]^{\frac{n-1}{2}} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty S^r e^{-\left[ \frac{n-1}{\pi \delta^2} \right] S^2} (S^2)^{\frac{n-3}{2}} \, dS^2.
\]

Using Appendix - C(14),
\[ E(S^r) = \left( \frac{n-1}{\pi \delta^2} \right)^{\frac{n-1}{2}} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \left( \frac{n+r-1}{\pi \delta^2} \right)^{\frac{n+r-1}{2}}. \]

\[ \therefore E(S^r) = \frac{\Gamma\left(\frac{n+r-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \left[ \pi n \right]^r \delta^r. \quad (3.4.31.1) \]

For \( r = 1, \)

\[ E(S) = \sqrt{\frac{\pi}{n-1}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \delta = C_4'(n) \delta, \quad (3.4.31.2) \]

Where \( C_4'(n) = \sqrt{\frac{\pi}{n-1}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}. \)

For \( r = 2, \)

\[ E(S^2) = \frac{\pi}{2} \delta^2 \quad (3.4.31.3) \]

For \( r = 4, \)

\[ E(S^4) = \frac{\pi^2}{4} \left( \frac{n+1}{n-1} \right) \delta^4 \quad (3.4.31.4) \]

Hence,

\[ \text{Var}(S) = \left( \frac{\pi}{2} - C_4'(n) \right) \delta^2 = \left( 1 - C_4^2(n) \right) \frac{\pi}{2} \delta^2 \quad (3.4.31.5) \]

where \( C_4(n) = \sqrt{\frac{2}{n-1}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}, \ n > 1 \) and

\[ C_4'(n) = \sqrt{\frac{\pi}{2}} C_4(n). \]
Also,
\[ \text{Var}(S^2) = \frac{\pi^2}{2(n-1)} \delta^4. \]  

(3.4.31.6)

We know that \( S^2 = \frac{n}{(n-1)} s^2 \). i.e. \( S = \sqrt{\frac{n}{n-1}} s \) or \( s = \sqrt{\frac{n-1}{n}} S \).

Using (3.4.31.1), we can write,
\[ \text{E}(s^r) = \frac{\Gamma\left(\frac{n+r-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \left(\frac{\pi}{n}\right)^{\frac{r}{2}} \delta^r. \]  

(3.4.31.7)

For \( r = 1 \),
\[ \text{E}(s) = \sqrt{\frac{\pi}{n}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \delta = C''''n \delta \]  

(3.4.31.8)

Where \( C''''n = \sqrt{\frac{\pi}{n}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \).

For \( r = 2 \),
\[ \text{E}(s^2) = \left(1 - \frac{1}{n}\right) \frac{\pi}{2} \delta^2. \]  

(3.4.31.9)

For \( r = 4 \),
\[ \text{E}(s^4) = \left(1 - \frac{1}{n^2}\right) \frac{\pi^2}{4} \delta^4. \]  

(3.4.31.10)

\[ \therefore \text{Var}(s) = \left(1 - \frac{1}{n}\right) \left(1 - C^2_{4(n)}\right) \frac{\pi}{2} \delta^2 \]  

(3.4.31.11)

and

\[ \text{Var}(s^2) = \frac{1}{n} \left(1 - \frac{1}{n}\right) \frac{\pi^2}{2} \delta^4. \]  

(3.4.31.12)
METHOD: -2

We know that \( \frac{(n-1)S^2}{\sigma^2} \) is a psi-square variate with \( n-1 \) d.f.

\[
\therefore E \left[ \frac{(n-1)S^2}{\sigma^2} \right] = \frac{(n-1)\pi}{2}.
\]

\[
\therefore E(S^2) = \frac{\pi}{2} \sigma^2.
\]

Also,

\[
\text{Var} \left[ \frac{(n-1)S^2}{\sigma^2} \right] = (n-1)^2 \frac{\pi^2}{2}.
\]

\[
\therefore \text{Var}(S^2) = \frac{\pi^2}{2(n-1)} \sigma^4.
\]

Now,

\[
S = \sqrt{\frac{n}{n-1}} s \Rightarrow E(s) = \left( \sqrt{\frac{n-1}{n}} S \right).
\]

\[
\therefore E(s) = \sqrt{\frac{\pi}{n}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \sigma = C_4(n) \sigma.
\]

Similarly, we can find \( \text{Var}(s) \) and \( \text{Var}(s^2) \).

METHOD: -3 (Using M.G.F.)

The m.g.f. of \( S^2 \) is

\[
M_{S^2}(t) = \left[ \frac{n-1}{\pi \sigma^2} \right]^{\frac{n-1}{2}} \int_0^\infty \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} e^{-\frac{n-1}{\pi \sigma^2} t} S^2 \left( \sigma^2 \right)^{-\frac{n-3}{2}} dS^2.
\]

\[
= \left[ 1 - \frac{\pi \sigma^2}{n-1} t \right]^{\frac{n-1}{2}}.
\]

\[
\therefore M_{S^2}(t) = 1 + \frac{\pi}{2} \sigma^2 t + \left( \frac{n+1}{n-1} \right) \frac{\pi^2}{4} \sigma^4 + ...
\]
\[ \therefore \ E(S^2) = \frac{\pi^2}{2} \delta^2, \quad E(S^4) = \frac{\pi^2}{4} \left( \frac{n+1}{n-1} \right) \delta^4. \]

**NOTE:** - Using \( \frac{n-1}{n} \left( 1 - C^2_{4(n)} \right) = n \left( n - 1 \right) - \frac{2 \left( \Gamma \left( \frac{n}{2} \right) \right)^2}{\left( \Gamma \left( \frac{n-1}{2} \right) \right)^2} = \frac{1}{2n} - \frac{1}{8n^2} - \frac{3}{16n^3} - ... \)

[See: J.F. Kenney and E.S. keeping(1951), p.p.171]

and

\[ 1 - C^2_{4(n)} = \left( 1 - \frac{1}{n} \right)^{-1} \left[ 1 - \frac{1}{2n} - \frac{1}{8n^2} - ... \right] = 1 + \frac{1}{2n} - \frac{5}{8n^2} - ... \]

For large \( n \), neglecting terms of higher than order \( n^{-1} \), from (3.4.31.11) we get

(i) \[ \text{Var}(s) = \frac{\pi^2}{2} \frac{\delta^2}{2n}. \]  
(3.4.31.13)

\[ \therefore \text{S.E.}(s) = \sqrt{\frac{n}{2} \frac{\delta^2}{2n}}. \]

**REMARK:** - For normal distribution, \( \text{Var}(s) = \frac{\sigma^2}{2n} \).

(ii) \[ \text{Var}(s^2) = \frac{\pi^2}{2} \frac{\delta^4}{n}. \]  
(3.4.31.14)

\[ \therefore \text{S.E.}(s^2) = \frac{\pi}{2} \sqrt{n} \delta^2. \]

(iii) For large \( n \), neglecting terms of more than order \( n^{-1} \), from (3.4.31.5) we can write

\[ \text{Var}(S) = \left( 1 + \frac{1}{2n} \right) \frac{\pi}{2} \delta^2. \]  
(3.4.31.15)
3.4.32 For psi-square distribution with n d.f.

\[ \mu'_r = \pi \left[ \frac{n}{2} + r - 1 \right] \mu'_{r-1} \]

which is recurrence relation between \( r^{th} \) and \( r-1^{th} \) raw moments.

3.4.33 For psi-square distribution with n d.f.

\[ \mu_{r+1} = \frac{\pi r}{2} \left[ 2\mu_r + n\pi \mu_{r-1} \right], \quad r \geq 1 \]

which is recurrence relation between the central moments.

**Proof:** -

If \( X \) is a psi-square variate with n d.f. then

\[ M_X(t) = (1 - \pi t)^{\frac{n}{2}}. \]

\[ \therefore \] the m.g.f. about mean is

\[ M(t) = M_{X-\mu}(t) = e^{-\frac{n\pi t}{2}} (1 - \pi t)^{\frac{n}{2}}. \]

Taking logarithms of both sides and differentiating with respect to ‘\( t \)’, we get

\[ 2(1 - \pi t) \frac{d}{dt} M(t) = n\pi^2 t M(t). \]

Differentiating \( r \) times with respect to ‘\( t \)’, by Leibniz’s theorem

[See: Appendix-C(12)(II)], we get,

\[ 2(1 - \pi t) \frac{d^{r+1}}{dt^{r+1}} M(t) - 2\pi t \frac{d^r}{dt^r} M(t) = n\pi^2 t \frac{d^r}{dt^r} M(t) + n\pi^2 r \frac{d^{r-1}}{dt^{r-1}} M(t). \]

Putting \( t = 0 \) and using the relation \( \mu_r = \left[ \frac{d^r}{dt^r} M(t) \right]_{t = 0} \), we get
\[ \mu_{r+1} = \frac{\pi r}{2} \left[ 2\mu_r + n\pi\mu_{r-1} \right]. \]

**NOTE:** - Taking \( r = 1, 2, 3 \), we get the different values \( \mu_2, \mu_3 \) and \( \mu_4 \).

3.4.34 The variables \( X_1, X_2, \ldots, X_n \) are independently distributed in the rectangular form:

\[ dF = dx, 0 \leq x \leq 1. \]

If \( P = X_1X_2 \ldots X_n \) then \( -\pi \log_e P \) has psi-square distribution with \( 2n \) d.f.

**Proof:**

We have

\[ -\pi \log_e P = \sum_{i=1}^{n} (-\pi) \log(X_i) = \sum_{i=1}^{n} Y_i, \quad \text{where} \quad Y_i = (-\pi) \log X_i. \]

The probability function of \( Y_i \) is given by

\[ g(y_i) = \left| \frac{dx_i}{dy_i} \right| f(x_i). \]

Here

\[ \left| \frac{dx_i}{dy_i} \right| = -\frac{y_i}{\pi} \quad \text{and} \]

\[ f(x) = 1, \quad 0 \leq x \leq 1 \]

\[ = 0, \quad \text{otherwise}. \]

\[ \therefore g(y_i) = \frac{1}{\pi} e^{-\frac{y_i}{\pi}}, \quad 0 \leq y_i \leq \infty, \]

which is the probability function of psi-square distribution with 2 d.f. i.e.
$Y_i$ ( $i = 1,2,3,\ldots,n$ ) are independent psi-square variates each with 2 d.f.

Hence $-\pi \log_e P = \sum_{i=1}^{n} Y_i$ is a psi-square variate with $2n$ d.f.

3.4.35 If $n$ is even then for psi-square variate

$$P = \frac{2}{\pi} \int_{-\infty}^{\infty} e^{-\frac{y^2}{\pi}} y^{n-1} dy = e^{-\frac{\psi^2}{\pi}} \left[ 1 + \frac{\psi^2}{\pi^2} + \frac{\psi^4}{\pi^4} + \ldots + \frac{\psi^{2n-2}}{\pi^{2n-2}} \Gamma\left(\frac{n}{2}\right) \right]$$

and the values of $P$ for a given psi-square variate can be derived from tables of Poisson’s exponential limit.

Proof:-

Let us consider the incomplete gamma integral.

$$Q(r + 1, \beta) = I_r = \frac{1}{r!} \int_{\beta}^{\infty} e^{-t} t^r dt, \quad \text{where } r \text{ is a positive integer.}$$

[See: Appendix- C(13)]

Integrating by parts, we get

$$I_r = \frac{e^{-\beta} \beta^r}{r!} + I_{r-1}.$$  

Repeated application of this gives

$$I_r = \frac{e^{-\beta} \beta^r}{r!} + \frac{e^{-\beta} \beta^{r-1}}{(r-1)!} + \ldots + \frac{e^{-\beta} \beta}{1!} + I_0.$$  

Here

$$I_0 = \frac{1}{0!} \int_{\beta}^{\infty} e^{-t} t^0 dt = e^{-\beta}.$$
\[ I_r = \frac{1}{r!} \int_0^{\infty} e^{-\beta t} t^r dt = e^{-\beta \left( 1 + \frac{\beta^2}{2!} + \ldots + \frac{\beta^n}{r!} \right)}. \]

Putting \( \beta = \frac{\psi^2}{\pi} \) and \( r = \frac{1}{2} (n - 2) = \frac{n}{2} - 1 \),

(Since \( r \) is an integer, \( n = 2r + 2 \) must be even)

We get,

\[ \frac{1}{(n/2 - 1)!} \int_0^{\psi^2/\pi} e^{-t \cdot \frac{n/2 - 1}{\pi}} dt = e^{-\frac{\psi^2}{\pi}} \left( 1 + \frac{\psi^4}{\pi^2 2!} + \ldots + \frac{\psi^{n-2}}{\pi^{n-1} (n/2 - 1)!} \right). \]

Taking \( t = \frac{\psi^2}{\pi} \) in the integral on the L.H.S., we get

\[ \text{L.H.S.} = \frac{2}{\pi^{n/2} \Gamma(n/2)} \int_0^{\psi} e^{-y^2/\pi} \left( \frac{y^2}{\pi} \right)^{n/2 - 1} \frac{y}{\pi} dy \]

\[ = \frac{2}{\pi^{n/2} \Gamma(n/2)} \int_0^{\psi} e^{-y^2/\pi} y^{n-1} dy. \]

Let the given value of \( \psi^2 \) be \( \psi_0^2 \), then

\[ P = P(\psi^2 > \psi_0^2) = \frac{2}{\pi^{n/2} \Gamma(n/2)} \int_0^{\psi_0} e^{-y^2/\pi} y^{n-1} dy \]

\[ = e^{-\frac{\psi_0^2}{\pi}} \left( 1 + \frac{\psi_0^4}{\pi^2 2!} + \ldots + \frac{\psi_0^{n-2}}{\pi^{n-1} (n/2 - 1)!} \right). \]
Let \( \lambda = \frac{\psi_0^2}{\pi} \).

\[
\therefore P = e^{-\lambda} \left( 1 + \lambda + \frac{\lambda^2}{2!} + \ldots + \frac{\lambda^{n-1}}{(n-1)!} \right).
\]

The terms on R.H.S. viz, \( e^{-\lambda}, \lambda e^{-\lambda}, \frac{\lambda^2}{2!} e^{-\lambda} \ldots \) etc. are the successive terms of the Poisson distribution with parameter \( \lambda = \frac{\psi_0^2}{\pi} \).

**NOTE:** - (i) We have

\[
P = \frac{2}{\pi^2 \Gamma \left( \frac{n}{2} \right)} \int_0^{\infty} e^{-\frac{y^2}{\pi}} y^{n-1} dy
\]

\[
= \frac{\psi_0^2}{\pi^2 \Gamma \left( \frac{n}{2} \right)} \int_0^{\infty} e^{-\lambda t} t^{n-1} dt
\]

\[
\therefore P = 1 - P(\psi_0^2)
\]

Where \( P(\psi_0^2) = \frac{\psi_0^2}{\pi^2 \Gamma \left( \frac{n}{2} \right)} \int_0^{\infty} e^{-\lambda t} t^{n-1} dt \).

(ii) \( P = e^{-\frac{\psi_0^2}{\pi}} \left[ 1 + \frac{\psi_0^2}{\pi} + \frac{\psi_0^4}{\pi^2 2!} + \ldots + \frac{\psi_0^{n-2}}{\pi^2 \frac{n}{2}!} \right] \)

\[
= \sum_{j=0}^{k-1} \frac{e^{-\lambda \cdot j}}{j!} \quad \text{where} \quad K = \frac{n}{2}, \lambda = \frac{\psi_0^2}{\pi}.
\]
\[
\therefore P = 1 - P(\psi^2_{(n)}) = \sum_{j=0}^{k-1} \frac{e^{-\lambda} \lambda^j}{j!}.
\]

Hence, we can derive value of P using cumulative Poisson distribution for even value of n and also for different values of \( \psi^2 \) [See: Appendix-D].

(iii) Using cumulative sums of Poisson distribution table [See: Appendix-D], for different even values of n and given psi-square probabilities, we get corresponding different values \( \alpha \). e.g, for \( n = 14 \) and \( \psi^2 = 6.4 \) we get \( \alpha = 0.995 \).

Note that In Psi-square distribution table [See: Appendix-B], for different values of n and \( \alpha \), we get corresponding different psi-square probabilities. e.g, for \( n = 14 \) and \( \alpha = 0.995 \) we get \( \psi^2 = 6.4 \).

3.4.36 For psi-square variate with n d.f.

\[
\mu_r = E(\psi^{2r}) = \pi^r \frac{\Gamma\left(\frac{n+r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.
\]

3.4.37 If the random variables X and Y are independent and follow \( \psi^2 \) distribution with n d.f. then \( \sqrt{n}(\overline{X}-\overline{Y}) \) follows student’s t distribution with n d.f., independently of \( X + Y \).

**Proof:**

Since X and Y are independent \( \psi^2 \) variates, each with n d.f., their joint p.d.f. is given by:
\[ f(x,y) = \frac{1}{\pi^n \left( \Gamma \left( \frac{n}{2} \right) \right)^2} e^{-\frac{1}{\pi}(x+y)} (xy)^{\frac{n-1}{2}}, 0 \leq x \leq \infty, 0 \leq y \leq \infty. \]

Let \( u = \frac{\sqrt{n}(x-y)}{2\sqrt{xy}} \) and \( v = x + y \) \( \Rightarrow \) \( x = \frac{v}{2} \left[ 1 + \frac{1}{\sqrt{1+\frac{n}{u^2}}} \right] \) and

\[ y = \frac{v}{2} \left[ 1 - \frac{1}{\sqrt{1+\frac{n}{u^2}}} \right]. \]

Also,

\[ |J| = \frac{v}{2\sqrt{n} \left( 1 + \frac{u^2}{n} \right)^{\frac{3}{2}}}. \]

\[ \therefore \] the joint p.d.f. of \( U \) and \( V \) becomes:

\[ g(u,v) = \frac{1}{\pi^n \left( \Gamma \left( \frac{n}{2} \right) \right)^2} 2^{n-1} \sqrt{n} e^{-\frac{1}{\pi}v} \frac{v^{n-1}}{\left( 1 + \frac{u^2}{n} \right)^{\frac{n+1}{2}}}, -\infty < u < \infty, 0 \leq v < \infty. \]

Using Legendre’s duplication formula, [ see: Appendix-C(15)]

\[ \Gamma^n = \frac{2^{n-1} \Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi}}. \]

we get

\[ \pi^n \left( \Gamma \left( \frac{n}{2} \right) \right)^2 2^{n-1} \sqrt{n} = \pi^n \sqrt{n} \Gamma \left( \frac{1}{2}, \frac{n}{2} \right). \]

Hence,
\[ g(u, v) = \frac{1}{\pi^n \Gamma_n} e^{-\frac{1}{2} u^2} \frac{1}{\sqrt{n}} B\left(\frac{1}{2}, \frac{n}{2}\right) \left(1 + \frac{u^2}{n}\right)^{-\frac{n+1}{2}}, -\infty < u < \infty, 0 \leq v < \infty. \]

\[ = g_1(u)g_2(v) \]

where \( g_1(u) = \frac{1}{\sqrt{n}} B\left(\frac{1}{2}, \frac{n}{2}\right) \left(1 + \frac{u^2}{n}\right)^{-\frac{n+1}{2}}, -\infty < u < \infty \)

and \( g_2(v) = \frac{1}{\pi^n \Gamma_n} e^{-\frac{1}{2} v^2} v^{n-1}, 0 \leq v < \infty. \)

(a) \( u = \frac{\sqrt{n}(x-y)}{2\sqrt{xy}} \) and \( v = x + y \) are independently distributed.

(b) \( u = \frac{\sqrt{n}(x-y)}{2\sqrt{xy}} \) follows student’s t distribution with \( n \) d.f..

(c) \( v = x + y \) follows \( G\left(\frac{1}{\pi}, n\right) \).

3.4.38 **Lemma**: For large value of \( n \), standardized psi-square variate has approximately moderate distribution with mean 0 and mean deviation 2.

**Proof**: -

The m.g.f. of \( \psi^2 \) is

\[ M_{\psi^2}(t) = (1 - \pi t)^{-\frac{n}{2}}. \]

For large \( n \), consider standardized psi-square variate defined as

\[ U = \frac{\psi^2 - E(\psi^2)}{\text{M.D.}(\psi^2)} = \frac{\psi^2 - \frac{n\pi}{2}}{\frac{1}{2\sqrt{n\pi}}} \]

(See: 3.4.17).
\[ \therefore M_U(t) = e^{-\frac{n \pi t}{\sqrt{n \pi}}}.M_{\psi}^2(\frac{2t}{\sqrt{n \pi}}) = e^{-\sqrt{n \pi t}}.(1-\frac{2t}{\sqrt{n \pi}})^{-\frac{n}{2}}. \]

Taking logarithm both the sides,

\[
\log M_U(t) = -\sqrt{n \pi t} - \frac{n}{2} \log\left(1-2\sqrt{\frac{\pi}{n} t}\right)
\]

\[
= -\sqrt{n \pi t} + \frac{n}{2} \left(2\sqrt{\frac{\pi}{n} t} + \frac{1}{2} \left(2\sqrt{\frac{\pi}{n} t}\right)^2 + \ldots \right)
\]

\[
= -\sqrt{n \pi t} + \sqrt{n \pi t} + \pi t^2 + \frac{4}{3} \frac{\pi^3}{n} t^3 + \ldots
\]

\[
\log M_U(t) = \frac{\pi}{4} (2)^2 t^2 + \frac{4}{3} \frac{\pi^3}{n} t^3 + \ldots
\]

For large value of \(n\),

\[
\log M_U(t) = \frac{\pi}{4} (2)^2 t^2.
\]

i.e. \(\therefore M_U(t) = e^{\frac{\pi}{4} (2)^2 t^2}\)

which is m.g.f. of \(M(0,2)\).

Hence,

\[ U \sim M(0,2). \]

**NOTE:** - \[ W = \frac{U}{2} \sim M(0,1). \]
3.5 APPROXIMATING (OR TRANSFORMING) PSI-SQUARE VARIATE USING MODERATE VARIATE: -

For the d.f. \( n > 30 \), the \( \chi^2 \) distribution is approximated by normal distribution. There are at least three such approximations. In the same manner, for \( n > 30 \), the \( \psi^2 \) distribution may also be approximated by a moderate distribution. In this section three such approximations are discussed.

Firstly for \( X \sim \psi^2_{(n)} \), we will considered a transformation which can construct using standardized variable

\[
Z = \frac{X - \text{E}(X)}{\text{M.D.}(X)} = \frac{X - \frac{n\pi}{2}}{\sqrt{\frac{n\pi}{2}}} = \frac{2X - n\pi}{\sqrt{n\pi}} \quad \text{(for large \( n \))}
\]

which would tend to moderateness i.e. \( M (0,1) \) as \( n \) increases. This transformation may be called standardized \( \psi^2 \) transformation.

Secondly we will consider square root transformation, which is similar to \( \chi^2 \) approximation (due to R.A.Fisher), defined using standardized variable

\[
Z = \sqrt{2\psi^2} - \sqrt{\frac{n}{2}(2n - 1)}
\]
which would also approaches a standard moderate distribution. This transformation may be called $\sqrt{2\psi^2}$ (square root) standardized transformation.

Thirdly we can consider cube root transformation, which is similar to $\chi^2$ transformation (due to Wilson and Hilferty’s), defined using standardized variable

$$Z = \left[3\sqrt{\frac{\psi^2}{n}} - \left(\frac{1}{2}\right)\left(1 - \frac{2}{9n}\right)\right]3\sqrt{n}\left(\frac{\sqrt{\pi}}{4}\right)^{\frac{1}{3}}$$

so that for large n, $Z$ approaches M (0,1). This transformation may be called $\sqrt[3]{\frac{\psi^2}{n}}$ (cube root) standardized transformation.

In this section we will compare the precision of transformation by computing the various values of $\psi^2(\alpha)$, for $0 < \alpha < 1$ and $n > 30$. We will study the relative speed of the three transformation to attain moderateness by comparing the values of their skewness and kurtosis which are the function of n.

It may be noted for $\psi^2$ from (3.4.11) and (3.4.12), we have

$$\beta_1 = \frac{8}{n}, \quad \beta_2 = \frac{12}{n} + 3 \quad \text{and} \quad \gamma_1 = \frac{8}{n}, \quad \gamma_2 = \frac{12}{n}.$$
As we know that skewness $\beta_1$ and kurtosis $\beta_2$ of a variable and its standardized form are same, the values of $\beta_1$ and $\beta_2$ of standardized $\psi^2$ transformation remain same i.e. $\beta_1 = \frac{8}{n}$, $\beta_2 = \frac{12}{n} + 3$.

3.5.1 SQUARE ROOT TRANSFORMATION:

**LEMMA:** For large $n$, $\sqrt{2\psi^2}$ follows moderate distribution with mean $\sqrt{\frac{\pi}{2}(2n-1)}$ and mean deviation 1 i.e. $Z = \sqrt{2\psi^2} - \sqrt{\frac{\pi}{2}(2n-1)} \sim M(0,1)$.

**Proof:**

From (3.4.36), $r^{th}$ moment of $\sqrt{\psi^2}$ is

$$
\mu_r' = \pi^2 \cdot \frac{\Gamma\left(\frac{n+r}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.
$$

For $r = 1$,

$$
\mu_1' = E(\sqrt{\psi^2}) = \pi^2 \cdot \frac{1}{2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.
$$

∴ $\mu_1' = \sqrt{\frac{\pi}{2}} \left(1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{5}{128n^3} + o(n^{-3})\right)$  \hspace{1cm} (3.5.1.1)


For $r = 2$, $\mu_2' = \frac{n\pi}{2}$.

For $r = 3$, $\mu_3' = \frac{\pi(n+1)}{2} \cdot \mu_1'$.

For $r = 4$, $\mu_4' = \frac{\pi^2 n(n+2)}{4}$.
Now, using (3.5.1.1) we get,

$$\mu_1^2 = \frac{n \pi}{2} \left( 1 - \frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{16n^3} + o(n^{-3}) \right).$$

We will find central moments using above raw moments.

$$\mu_2 = \frac{\pi}{4} \left( 1 - \frac{1}{4n} - \frac{1}{8n^2} + o(n^{-2}) \right) \quad (3.5.1.2)$$

Further,

$$\mu_3 = \frac{\pi^3}{64 \sqrt{2n}} \left( 1 - \frac{1}{4n} - \frac{3}{32n^2} + \frac{1}{64n^3} + o(n^{-3}) \right) \quad (3.5.1.3)$$

$$\mu_4 = \frac{3\pi^2}{16} \left( 1 - \frac{1}{2n} - \frac{1}{16n^2} + o(n^{-2}) \right) \quad (3.5.1.4)$$

For the moments of $\sqrt{\psi^2}$, using (3.5.1.1), we get,

$$\mu_1' = E(\sqrt{\psi^2}) = \sqrt{n \pi} \left( 1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{5}{128n^3} + o(n^{-3}) \right).$$

$$\therefore \mu_1' = \sqrt{\frac{\pi}{2}} \sqrt{2n} \left( 1 - \frac{1}{4n} + \ldots \right)$$

$$\therefore \mu_1' = \sqrt{\frac{\pi}{2}} \left( 2n - 1 \right)^{\frac{1}{2}} + o(n^{\frac{1}{2}}).$$

(See: Stuart and Ord (1987), p.p. 511)

On neglecting $o(n^{\frac{1}{2}})$, we get,

$$\mu_1' = E(\sqrt{\psi^2}) = \sqrt{\frac{\pi}{2}} \left( 2n - 1 \right)^{\frac{1}{2}}. \quad (3.5.1.5)$$
Using (3.5.1.2),

\[ \mu_2 = V(\sqrt{2\psi^2}) = \frac{\pi}{2} \left(1 - \frac{1}{4n} - \frac{1}{8n^2} + o(n^{-2})\right) \]

\[ = \frac{\pi}{2} \left(1 + O(n^{-1})\right). \]

On neglecting \(O(n^{-1})\), we get,

\[ \mu_2 = V(\sqrt{2\psi^2}) = \frac{\pi}{2} \quad (3.5.1.6) \]

\[ \therefore \text{For large } n, \text{ using (3.5.1.5) and (3.5.1.6)} \]

\[ \sqrt{2\psi^2} \sim N\left(\frac{\pi}{2}(2n-1), \frac{\pi}{2}\right) \text{ approximately.} \]

i.e. For large \(n\),

\[ \sqrt{2\psi^2} \sim M\left(\frac{\pi}{2}(2n-1), 1\right) \text{ approximately.} \quad (3.5.1.7) \]

i.e. \(Z = \sqrt{2\psi^2} - \sqrt{\frac{\pi}{2}(2n-1)} \sim M(0,1). \)

Now to find the measures of skewness and kurtosis of \(\sqrt{2\psi^2}\) we have to find its \(\mu_3\) and \(\mu_4\).

From (3.5.1.3), it is clear that

\[ \mu_3 = E[(\sqrt{2\psi^2})^3] = \frac{\pi^3}{8} \sqrt{2n} \left(1 - \frac{1}{4n} - \frac{3}{32n^2} + \frac{1}{64n^3} + o(n^{-3})\right). \]

Also from (3.5.1.4), we get,

\[ \mu_4 = E[(\sqrt{2\psi^2})^4] = \frac{3\pi^2}{4} \left(1 - \frac{1}{2n} - \frac{1}{16n^2} + o(n^{-2})\right). \]
\[
\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\pi^3}{8} \left( \frac{1}{2n} - \frac{1}{8n^2} + o(n^{-2}) \right) = \frac{1}{2n} + o(n^{-1}) \quad (3.5.1.8)
\]

\[
\gamma_1 = \frac{1}{\sqrt{2n}} + o(n^{-\frac{1}{2}}). \quad (3.5.1.9)
\]

Also,

\[
\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\pi^2}{4} \left( \frac{1}{2n} - \frac{1}{16n^2} + o(n^{-2}) \right) = 3 + \frac{3}{4n^2} + o(n^{-2}) \quad (3.5.1.10)
\]

\[
\gamma_2 = \frac{3}{4n^2} + o(n^{-2}). \quad (3.5.1.11)
\]

A comparison of skewness and kurtosis of \(\psi^2\) with those of \(\sqrt{2\psi^2}\) (obtained in 3.5.1.8 to 3.5.1.11) indicates that \(\sqrt{2\psi^2}\) tends to moderate distribution more rapidly than \(\psi^2\) does. This can be noticed more clearly in the table 3.5.2.

**NOTE**: (i) **Another proof of lemma**: -

The above lemma can also be proved using the relationship between \(\chi^2\) and \(\psi^2\) distribution as follows.

Fisher’s has shown that \(\sqrt{2\chi^2}\) is approximately normally distributed with mean \(\sqrt{(2n-1)}\) and unit variance [See: Stuart, A. and Ord, J.K. (1987), p.p.508].
i.e. $\sqrt{2\chi^2} \sim N(\sqrt{2n-1},1)$ for large n.

$\therefore \sqrt{2\chi^2} \sim M\left(\sqrt{2n-1},\sqrt{\frac{2}{\pi}}\right)$ ($\because \delta = \sqrt{\frac{2}{\pi}\sigma}$).

$\therefore \sqrt{2(\frac{\pi}{2}\chi^2)} \sim M\left(\sqrt{\frac{\pi}{2}(2n-1)},1\right)$.

$\therefore \sqrt{2\psi^2} \sim M\left(\sqrt{\frac{\pi}{2}(2n-1)},1\right)$ ($\because \psi^2 = \frac{\pi}{2}\chi^2$). 

Hence $\sqrt{2\psi^2}$ is approximately moderately distributed with mean $\sqrt{\frac{\pi}{2}(2n-1)}$ and mean deviation 1 which is shown in (3.5.1.7).

(ii) Computing the value of $\psi^2(\alpha)$ using the value of $Z_\alpha$ for given $0 < \alpha < 1$.

For large n, if we want to calculate value of $\psi^2(\alpha)$ such that $P(\psi^2 > \psi^2(\alpha)) = \alpha$ using moderate distribution then it can be calculated from (3.5.1.7) as follows.

From (3.5.1.7) we can define a standardized variable

$$Z = \sqrt{2\psi^2} - \sqrt{\frac{\pi}{2}(2n-1)}.$$ 

$$\Rightarrow \psi^2 = \frac{1}{2}\left(Z + \sqrt{\frac{\pi}{2}(2n-1)}\right)^2. \quad (3.5.1.12)$$

Hence, for given $\alpha$ and $n(>30)$ we can calculate

$$\psi^2(\alpha) \text{ as } \frac{1}{2}\left(Z_\alpha + \sqrt{\frac{\pi}{2}(2n-1)}\right)^2.$$
e.g. for \( \alpha = 0.05 \) and \( n = 40 \), \( \psi^2_{(0.05)} \) can be calculated as

\[
\frac{1}{2} \left( Z_{0.05} + \sqrt{\frac{\pi}{2}} (2n - 1) \right)^2 = \frac{1}{2} \left( 2.061518115 + \sqrt{\frac{\pi}{2} \times 79} \right)^2 = 87.13608
\]

which is approximately same as actual value of \( \psi^2_{(0.05,40)} = 87.58521 \) from psi-square table given Appendix-B.

Thus, we can calculate different values of \( \psi^2 \) using moderate distribution for large \( n \).

### 3.5.2 CUBE ROOT TRANSFORMATION:

**LEMMA:** For large \( n \), \( \left( \frac{\psi^2}{n} \right)^{\frac{1}{3}} \) is follows moderate distribution with

mean \( \left( \frac{\pi}{2} \right) \left( 1 - \frac{2}{9n} \right) \) and mean deviation \( \frac{1}{3\sqrt{n}} \left( \frac{4}{\sqrt{\pi}} \right)^{\frac{1}{3}} \).

i.e. \( Z = \left[ \frac{3\psi^2}{n} \right] \left( \frac{4}{\sqrt{\pi}} \right)^{\frac{1}{3}} \sim M (0,1) \).

**Proof:**

We know that the mean of psi-square distribution is \( \frac{n\pi}{2} \).

Let \( Y = \left( \frac{\psi^2}{n\pi/2} \right)^h = \left( \frac{2\psi^2}{n\pi} \right)^h \), where \( h \) is undetermined.

Let \( w = \psi^2 - \frac{n\pi}{2} \). Then \( \frac{w}{n\pi/2} = \frac{\psi^2}{n\pi/2} - 1 \).
\[ \therefore \left( \frac{w}{n\pi} + 1 \right)^h = y. \]

\[ \therefore \left( \frac{w}{n\pi} + 1 \right)^{rh} = y^r. \]

\[ \therefore y^r = \sum_{j=0}^{rh} \left( \frac{rh}{n\pi} \right)^j. \]

Taking expectation, we get

\[ \mu'_r(y) = E(y^r) = \sum_{j=0}^{rh} \left( \frac{rh}{n\pi} \right)^j E(w^j) = \sum_{j=0}^{rh} \left( \frac{2}{n\pi} \right)^j \mu_j(\psi^2). \]

For \( r = 1, \)

\[ \mu'_1(y) = 1 + \left( \frac{h}{2} \right) \frac{2}{n} + \left( \frac{h}{3} \right) \frac{8}{n^2} + \left( \frac{h}{4} \right) \left[ \frac{48}{n^3} + \frac{12}{n^2} \right] + \left( \frac{h}{5} \right) \left[ \frac{160}{n^3} + \frac{284}{n^4} \right] \]

\[ + \left( \frac{h}{6} \right) \left[ \frac{120}{n^3} + \frac{2080}{n^4} + \frac{3840}{n^5} \right] + \ldots \]

\[ \therefore \mu'_1(y) = 1 + \frac{h(h-1)}{n} + \frac{h(h-1)(h-2)(3h-1)}{6n^2} + \frac{h^2(h-1)^2(h-2)(h-3)}{6n^3} + O(n^{-3}). \]

\[ \Rightarrow \text{Put } h = \frac{1}{3}, \text{ we get} \]

\[ \mu'_1(y) = E \left[ \left( \frac{2\psi^2}{n\pi} \right)^{\frac{1}{3}} \right] = 1 - \frac{2}{9n} + \frac{80}{3\cdot n^3} + O(n^{-3}) \]

\[ = 1 - \frac{2}{9n} + O(n^{-3}). \]

On neglecting \( O(n^{-3}) \), we get

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\[ \mu_1(y) = E \left[ \left( \frac{2\psi^2}{n\pi} \right)^{\frac{1}{3}} \right] = 1 - \frac{2}{9n} \] (3.5.2.1)

\[ \mu_2(y) = 1 + \left( \frac{2h}{2n} \right) + \left( \frac{2h}{3n} \right)^2 + \left( \frac{2h}{4n} \right)^3 + \left( \frac{2h}{5n} \right)^4 + \ldots \]

\[ \therefore \mu_2(y) = 1 + \frac{2h(2h-1)}{n} + \frac{2h(2h-1)(h-1)(6h-1)}{3n^2} + \frac{4h^2(2h-1)^2(h-1)(2h-3)}{3n^3} + o(n^{-3}). \]

\[ \Rightarrow \text{Put } h = \frac{1}{3}, \text{ we get} \]

\[ \mu_2(y) = 1 - \frac{2}{9n} + \frac{4}{3^4n^2} + \frac{56}{3^7n^3} + o(n^{-3}). \]

\[ \mu_3(y) = 1 + \left( \frac{3h}{2n} \right) + \left( \frac{3h}{3n} \right)^2 + \left( \frac{3h}{4n} \right)^3 + \left( \frac{3h}{5n} \right)^4 + \ldots \]

\[ \therefore \mu_3(y) = 1 + \text{ terms containing factor } (3h-1). \]

\[ \Rightarrow \text{Put } h = \frac{1}{3}, \text{ we get} \]

\[ \mu_3(y) = 1. \]

\[ \therefore \text{For } r = 4, \]

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\[ \mu'_4(y) = 1 + \left(\frac{4h}{2}\right)^2 + \left(\frac{4h}{3}\right)\frac{8}{n^2} + \left(\frac{4h}{4}\right)\left(\frac{48}{n^3} + \frac{12}{n^2}\right) + \left(\frac{4h}{5}\right)\left(\frac{160}{n^3} + \frac{384}{n^4}\right) \]
\[ + \left(\frac{4h}{6}\right)\left(\frac{120}{n^3} + \frac{2080}{n^4} + \frac{3840}{n^5}\right) + \ldots \]
\[ \therefore \mu_4(y) = 1 + \frac{4h(4h-1)}{n} + \frac{4h(4h-1)(2h-1)(12h-1)}{3n^2} \]
\[ + \frac{16h^2(4h-1)^2(2h-1)(4h-3)}{3n^3} + o(n^{-3}). \]

\[ \Rightarrow \text{Put } h = \frac{1}{3}, \text{ we get} \]
\[ \mu'_4(y) = 1 + \frac{4}{9n} - \frac{4}{3n^2} + \frac{80}{3^7 n^3} + o(n^{-3}) \]

Now, for central moments,
\[ \mu_2(y) = \mu'_2(y) - \mu'_1(y) = V \left[ \left(\frac{2\psi^2}{n\pi}\right)^{\frac{1}{3}} \right] = \frac{2}{9n} - \frac{104}{3^7 n^3} + o(n^{-3}). \]
\[ = \frac{2}{9n} + O(n^{-3}) \]

On neglecting \( O(n^{-3}) \), we get
\[ \mu_2(y) = V \left[ \left(\frac{2\psi^2}{n\pi}\right)^{\frac{1}{3}} \right] = \frac{2}{9n} . \quad (3.5.2.2) \]
\[ \therefore \text{For large } n, \text{ using (3.5.2.1) and (3.5.2.2), we get} \]
\[ 3\sqrt{\frac{2\psi^2}{n\pi}} \sim N\left(1 - \frac{2}{9n}, \frac{2}{9n} \right) \]

which also means that for large \( n \), we get
\[ \sqrt[3]{\frac{2\psi^2}{n\pi}} \sim M \left( 1 - \frac{2}{9n}, \frac{2}{3} \cdot \frac{1}{\sqrt{n\pi}} \right). \]

i.e.
\[ \sqrt[3]{\frac{2\psi^2}{n}} \sim M \left( \frac{\pi}{2}, \frac{1}{3} \left( 1 - \frac{2}{9n} \right) \cdot \frac{1}{3} \sqrt{n} \left( \frac{4}{\sqrt{\pi}} \right)^{\frac{1}{3}} \right). \] (3.5.2.3)

i.e. \[ Z = \left[ \sqrt[3]{\frac{2\psi^2}{n}} - \left( \frac{\pi}{2} \right)^{\frac{1}{3}} \left( 1 - \frac{2}{9n} \right) \right] \sqrt{n} \left( \frac{\sqrt{\pi}}{4} \right)^{\frac{1}{3}} \sim M(0,1). \]

Now to find the measures of skewness and kurtosis of \[ \sqrt[3]{\frac{2\psi^2}{n\pi}} \] (which is same as for \[ \sqrt[3]{\frac{2\psi^2}{n}} \]), we have to find its \( \mu_3 \) and \( \mu_4 \).

Using above raw moments,
\[ \mu_3(y) = \mu_3(y) - 3\mu_2(y)\mu_1(y) + 2\mu_1^3 = \frac{32}{3^6 n^3} + o(n^{-3}). \]
\[ \mu_4(y) = \mu_4(y) - 4\mu_3(y)\mu_1(y) + 6\mu_2(y)\mu_1^2(y) - 3\mu_1^4(y). \]
\[ = \frac{4}{3^3 n^2} - \frac{16}{3^6 n^3} + o(n^{-3}). \]

Also,
\[ \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{2^{10}}{3^{12} n^6} + o(n^{-6}) = \frac{2^7}{3^6 n^3} + o(n^{-3}). \] (3.5.2.4)

\[ \gamma_1 = \frac{7}{3^2 n^2} + o(n^{-\frac{3}{2}}). \] (3.5.2.5)
\[ \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\frac{4}{3^3 n^2} - \frac{16}{3^6 n^3} + o(n^{-3})}{\frac{4}{3^4 n^2} - \frac{13}{3^9 n^4} + \frac{13^2}{3^{14} n^6} + o(n^{-6})} = 3 - \frac{4}{3^2 n} + o(n^{-1}). \quad (3.5.2.6) \]

\[ \therefore \gamma_2 = -\frac{4}{3^2 n} + o(n^{-1}). \quad (3.5.2.7) \]

A comparison of skewness and kurtosis of \( \psi^2 \) with \( \sqrt[3]{\frac{\psi^2}{n}} \) (3.5.2.4 to 3.5.2.7) indicates that \( \sqrt[3]{\frac{\psi^2}{n}} \) tends to approximately moderate distribution more rapidly than \( \psi^2 \) and \( \sqrt{2\psi^2} \), with mean \( \left( \frac{\pi}{2} \right)^{\frac{1}{3}} \left( 1 - \frac{2}{9n} \right) \) and mean deviation \( \frac{1}{3 \sqrt{n}} \left( \frac{4}{\sqrt{\pi}} \right)^{\frac{1}{3}} \).

**NOTE:** (i) Another proof of lemma: -

The above lemma can also be proved using the relationship between \( \chi^2 \) and \( \psi^2 \) distribution as follows.

Wilson and Hilferty’s (1931) have been that \( \left( \frac{\chi^2}{n} \right)^{\frac{1}{3}} \) is approximately normally distributed with mean \( 1 - \frac{2}{9n} \) and variance \( \frac{2}{9n} \).

i.e. \( \left( \frac{\chi^2}{n} \right)^{\frac{1}{3}} \sim N \left( 1 - \frac{2}{9n}, \frac{2}{9n} \right) \) for large n.

\[ \therefore \left( \frac{\chi^2}{n} \right)^{\frac{1}{3}} \sim M \left( 1 - \frac{2}{9n}, \frac{2}{3\sqrt{n}\pi} \right) \left( \therefore \delta = \sqrt{\frac{2}{\pi}} \sigma \right). \]
\[ \therefore \left( \frac{\pi \chi^2}{n} \right)^{\frac{1}{3}} \sim M \left( \frac{1}{3} \left( 1 - \frac{2}{9n} \right), \frac{1}{3\sqrt{n}} \frac{4}{\sqrt{\pi}} \right)^{\frac{1}{3}}. \]

\[ \therefore 3 \sqrt{\frac{\psi^2}{n}} \sim M \left( \frac{1}{3} \left( 1 - \frac{2}{9n} \right), \frac{1}{3\sqrt{n}} \frac{4}{\sqrt{\pi}} \right)^{\frac{1}{3}} \text{ (}: \psi^2 = \frac{\pi \chi^2}{2} \). \]

Hence, \( 3 \sqrt{\frac{\psi^2}{n}} \) is approximately moderately distributed with mean 
\[ \left( \frac{\pi}{2} \right)^{\frac{1}{3}} \left( 1 - \frac{2}{9n} \right) \text{ and mean deviation } \frac{1}{3\sqrt{n}} \left( \frac{4}{\sqrt{\pi}} \right)^{\frac{1}{3}} \text{ which is shown in (3.5.2.7).} \]

(ii) Computing the value of \( \psi^2_{(\alpha)} \) using the value of \( Z_\alpha \) for given \( 0 < \alpha < 1 \).

For large \( n \), if we want to calculate value of \( \psi^2_{(\alpha)} \) such that \( P(\psi^2 > \psi^2_{(\alpha)}) = \alpha \) using moderate distribution then it can be calculated from (3.5.2.3) as follows.

From (3.5.2.3) we can write a standardized variable
\[ Z = \left[ \sqrt{\frac{\psi^2}{n}} - \left( \frac{\pi}{2} \right)^{\frac{1}{3}} \left( 1 - \frac{2}{9n} \right) \right] \frac{1}{3\sqrt{n}} \left( \frac{\sqrt{\pi}}{4} \right)^{\frac{1}{3}}. \]

\[ \Rightarrow \psi^2 = n \left[ \sqrt{\frac{\psi^2}{2}} \left( 1 - \frac{2}{9n} \right) + Z \frac{1}{3\sqrt{n}} \left( \frac{4}{\sqrt{\pi}} \right)^{\frac{1}{3}} \right]^3. \quad (3.5.2.8) \]

Hence, for given \( \alpha \) and \( n(>30) \) we can calculate \( \psi^2_{(\alpha)} \) as
\[ n \left( \frac{3\pi}{2} \left( 1 - \frac{2}{9n} \right) + z_{\alpha} \frac{1}{\sqrt{\pi}} \left( \frac{4}{\sqrt{n}} \right)^{1/3} \right)^3. \]

e.g. for \( \alpha = 0.05 \) and \( n = 40 \), \( \psi^2_{(0.05)} \) can be calculated as

\[
\psi^2_{(0.05)} = 40 \left( \frac{3\pi}{2} \left( 1 - \frac{2}{360} \right) + 2.061518115 \frac{1}{3\sqrt{40}} \left( \frac{4}{\sqrt{\pi}} \right)^{1/3} \right)^3
\]

\[= 87.57727 \]

which is approximately same as \( \psi^2_{(0.05,40)} = 87.58521 \) obtained from Psi-square table given Appendix-B.

Thus, we can calculate different values of \( \psi^2 \) using moderate distribution for large \( n \).

It should be noted that this transformation is more accurate as compared to previous one (i.e. 3.5.1.12).

**COMPARISON:** - (i) From the values of \( \gamma_1 \) and \( \gamma_2 \) of \( \psi^2, \sqrt{2\psi^2} \) and \( \frac{3\psi^2}{\sqrt{n}} \), summarized in the table 3.5.2., it can be seen that \( \frac{3\psi^2}{\sqrt{n}} \) tends to symmetry, as measured by \( \gamma_1 \), more rapidly than either \( \psi^2 \) or \( \sqrt{2\psi^2} \).
Conclusion: -The second approximation is probably the most well known but the latter is approaching moderately even faster.

(ii) Garwood (1936) has made some comparisons of the square-root and cube-root approximations with the exact values of chi-square distribution. Similarly, we can compare of the square-root and cube-root approximations with the exact values of psi-square distribution. The exact values of $Z_{\alpha}$ for $\alpha = 0.99, 0.95, 0.05$ and $0.01$ are given in table (3.5.1). Using these values of $Z_{\alpha}$ and for some d.f. $n > 30$ (i) the exact values of $\frac{1}{\pi} \psi_{(\alpha,n)}^2$ are obtained using Psi-square table given Appendix-B and (ii) the approximate values of $\frac{1}{\pi} \psi_{(\alpha,n)}^2$ are obtained using the following two formulae given in 3.5.2.9 and 3.5.2.10 for different values of $Z_{\alpha}$ corresponding to different values $\alpha$ using table (3.5.1). They are presented in table 3.5.3 for comparison with the exact values.

<table>
<thead>
<tr>
<th>Variate</th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi_1$</td>
<td>$\sqrt{\frac{8}{n}}$</td>
<td>$\frac{12}{n}$</td>
</tr>
<tr>
<td>$\sqrt{2}\psi_1^2$</td>
<td>$\frac{1}{\sqrt{2n}} + o(n^{-\frac{1}{2}})$</td>
<td>$\frac{3}{4n^2} + o(n^{-2})$</td>
</tr>
<tr>
<td>$3\sqrt{\frac{\psi_1^2}{n}}$</td>
<td>$\frac{7}{3\sqrt{n}} + o(n^{-\frac{3}{2}})$</td>
<td>$-\frac{4}{9n} + o(n^{-1})$</td>
</tr>
</tbody>
</table>
The two formulae are

\[
\frac{1}{\pi} \psi^2 = \frac{1}{2\pi} \left( Z + \frac{\pi}{\sqrt{2}} (2n-1) \right)^2
\]  
(3.5.2.9)

[See - 3.5.1.12]

and

\[
\frac{1}{\pi} \psi^2 = \frac{n}{\pi} \left( \frac{3\pi}{2} \left( 1 - \frac{2}{9n} \right) + Z - \frac{1}{3n} \left( \frac{4}{\sqrt{n}} \right)^3 \right)
\]  
(3.5.2.10)

[See - 3.5.2.8].

Tab.3.5.2 Some values of 'Z' corresponding to different values of \( \alpha \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>0.99</th>
<th>0.95</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
<td>-2.9156</td>
<td>-2.0615</td>
<td>2.0615</td>
<td>2.9156</td>
</tr>
</tbody>
</table>

In the table (3.5.3), the column (A) provides the exact values of \( \frac{1}{\pi} \psi^2_{(\alpha, n)} \). The column (B) and column (D) provides the approximate values of \( \frac{1}{\pi} \psi^2_{(\alpha, n)} \) obtained using the formulae (3.5.2.9) and (3.5.2.10) respectively.
Tab: 3.5.3 Comparison of approximations to the Psi-square d.f.

<table>
<thead>
<tr>
<th>Prob.(α)</th>
<th>n</th>
<th>Exact (A)</th>
<th>Square-root (B)</th>
<th>(A) - (B) (C)</th>
<th>Cube-root (D)</th>
<th>(A) - (D) (E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99</td>
<td>30</td>
<td>7.47672831</td>
<td>7.16845529</td>
<td>0.30827302</td>
<td>7.46240538</td>
<td>0.01432292</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>11.0821307</td>
<td>10.7644455</td>
<td>0.31765819</td>
<td>11.0697123</td>
<td>0.01241837</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>14.8534144</td>
<td>14.5295251</td>
<td>0.32381631</td>
<td>14.8422455</td>
<td>0.01109598</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>18.742426</td>
<td>18.4142283</td>
<td>0.3281977</td>
<td>18.7323131</td>
<td>0.01011296</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>26.770039</td>
<td>26.4358896</td>
<td>0.33414942</td>
<td>26.7613104</td>
<td>0.00872855</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>35.032478</td>
<td>34.6934645</td>
<td>0.3380833</td>
<td>35.0246638</td>
<td>0.00778401</td>
</tr>
<tr>
<td>0.95</td>
<td>30</td>
<td>9.2463306</td>
<td>9.10920457</td>
<td>0.13712603</td>
<td>9.24527643</td>
<td>0.00105418</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>13.2564518</td>
<td>13.1164952</td>
<td>0.13815668</td>
<td>13.2539932</td>
<td>0.00065868</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>17.382126</td>
<td>17.2433409</td>
<td>0.13878505</td>
<td>17.3816936</td>
<td>0.0004324</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>21.593796</td>
<td>21.4547669</td>
<td>0.13921271</td>
<td>21.5936905</td>
<td>0.00028911</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>30.1957398</td>
<td>30.0559764</td>
<td>0.13976342</td>
<td>30.195618</td>
<td>0.00012183</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>38.9647333</td>
<td>38.824657</td>
<td>0.1410754</td>
<td>38.9647027</td>
<td>3.0542E-05</td>
</tr>
<tr>
<td>0.05</td>
<td>30</td>
<td>21.8864859</td>
<td>21.7435677</td>
<td>0.14291815</td>
<td>21.8833187</td>
<td>0.0031672</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>27.872397</td>
<td>27.7362772</td>
<td>0.14296251</td>
<td>27.8767128</td>
<td>0.00252683</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>33.7524033</td>
<td>33.6094314</td>
<td>0.14297186</td>
<td>33.7502783</td>
<td>0.00214293</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>39.509722</td>
<td>39.3580054</td>
<td>0.14296676</td>
<td>39.5009125</td>
<td>0.0018472</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>50.939737</td>
<td>50.7967959</td>
<td>0.14294111</td>
<td>50.9382516</td>
<td>0.00148539</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>62.1710569</td>
<td>62.0281466</td>
<td>0.14291029</td>
<td>62.1697992</td>
<td>0.00125768</td>
</tr>
<tr>
<td>0.01</td>
<td>30</td>
<td>25.469097</td>
<td>25.3075004</td>
<td>0.04859205</td>
<td>25.4570695</td>
<td>-0.0169788</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>31.8453699</td>
<td>31.4415102</td>
<td>0.04385968</td>
<td>31.8552147</td>
<td>-0.0088448</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>38.0769456</td>
<td>37.6764306</td>
<td>0.04051501</td>
<td>38.0859521</td>
<td>-0.009065</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>44.1897095</td>
<td>43.7917274</td>
<td>0.0379206</td>
<td>44.1908646</td>
<td>-0.0083551</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>56.1643963</td>
<td>55.7700662</td>
<td>0.0343015</td>
<td>56.171792</td>
<td>-0.0073956</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>67.9033616</td>
<td>67.515913</td>
<td>0.0397703</td>
<td>67.9107333</td>
<td>-0.0067117</td>
</tr>
</tbody>
</table>

where \((A) = \frac{1}{\pi} \psi^2_{\alpha,n}\), \((B) = \frac{1}{2\pi} \left( Z_{\alpha} + \frac{\pi}{\sqrt{2n-1}} \right)^2\) and \((D) = \frac{n}{\pi} \left( \frac{3}{\sqrt{n}} \left( 1 - \frac{3}{2n^2} \right) + Z_{\alpha} \frac{1}{\sqrt{n}} \left( \frac{4}{\sqrt{n}} \right) \right)^3\)

**Conclusion:** - As noted by Garwood (1936) in case of \(\chi^2\) distribution, here too it is evident from column (C) and column (D) that the Cube root approximation is evidently very good and the Square-root approximation is fair.

**3.6 APPLICATIONS OF PSI-SQUARE:** -

**Theorem:** - For a large random sample, \(\psi^2 = \frac{\pi}{2} \sum_{i=1}^{m} \left( \frac{(n_i - np_i)^2}{np_i} \right)\) follows
Psi-square distribution with \((m-1)\) d.f., where \(n_i\) and \(npi\) are the observed and expected frequencies of the \(i^{th}\) cell \((i=1,2,3,\ldots,n)\) and \(\sum_{i=1}^{m} n_i = n\).

**Proof:**

Suppose the units of a large random sample of size \(n\) are divided into \(m\) cells and suppose that the units are distributed at random in these cells. If the probabilities for different cells are \(p_1, p_2, \ldots, p_m\) then the probability \(p\) of there being \(n_1\) units in the 1st cell, \(n_2\) in the 2nd, \ldots, and \(n_m\) in the \(m^{th}\) cell is the term \((p_1^{n_1} p_2^{n_2} \ldots p_m^{n_m})\) in the multinomial expansion of \((p_1 + p_2 + \ldots + p_m)^n\) so that

\[
P = \frac{n!}{n_1! n_2! \ldots n_m!} p_1^{n_1} p_2^{n_2} \ldots p_m^{n_m}\]

where \(\sum_{i=1}^{m} n_i = n, \sum_{i=1}^{m} p_i = 1\).

If \(n\) is sufficiently large, so that \(n_1, n_2, \ldots, n_m\) are not small, then using stirling’s formula [See: Appendix-C(7)], we have

\[
P \approx \frac{\sqrt{2\pi e^{-n}} n^{n+\frac{1}{2}}}{\prod_{i=1}^{m} \sqrt{2\pi e^{-n_i} n_i^{n_i+\frac{1}{2}}}} p_1^{n_1} p_2^{n_2} \ldots p_m^{n_m}\]

\[
\therefore P \approx C \prod_{i=1}^{m} \left( \frac{np_i}{n_i} \right)^{n_i + \frac{1}{2}}\]

where \(C = \frac{1}{(2\pi)^{\frac{m-1}{2}} n^{\frac{1}{2}} (p_1 p_2 \ldots p_m)^{\frac{1}{2}}}\) is a constant, independent of \(n_1, n_2, \ldots, n_m\).

Taking logarithm of both sides,
\[ \log P \approx \log C + \sum_{i=1}^{m} \left( n_i + \frac{1}{2} \right) \log \left( \frac{np_i}{n_i} \right) \]

\[
\therefore \log \left( \frac{p}{C} \right) \approx \sum_{i=1}^{m} \left( n_i + \frac{1}{2} \right) \log \left( \frac{\lambda_i}{n_i} \right) \tag{3.6.1}
\]

where \( \lambda_i = np_i \) is the expected frequency for the \( i \)th cell.

Let us define \( \theta_i = \frac{n_i - \lambda_i}{\sqrt{\pi \lambda_i}} \).

\[
\therefore n_i = \lambda_i + \theta_i \sqrt{\frac{2}{\pi \lambda_i}}.
\]

Substituting in (3.6.1), we get

\[
\therefore \log \left( \frac{p}{C} \right) \approx \sum_{i=1}^{m} \left( \lambda_i + \theta_i \sqrt{\frac{2}{\pi \lambda_i}} + \frac{1}{2} \right) \log \left( \frac{\lambda_i}{\lambda_i + \theta_i \sqrt{\frac{2}{\pi \lambda_i}}} \right)
\]

\[
= - \sum_{i=1}^{m} \left( \lambda_i + \theta_i \sqrt{\frac{2}{\pi \lambda_i}} + \frac{1}{2} \right) \log \left( 1 + \theta_i \sqrt{\frac{2}{\pi \lambda_i}} \right).
\]

We assume that \( \theta_i \) is small compared with \( \lambda_i \).

\[
\therefore \log \left( \frac{p}{C} \right) \approx - \sum_{i=1}^{m} \left( \lambda_i + \theta_i \sqrt{\frac{2}{\pi \lambda_i}} + \frac{1}{2} \right) \left( \theta_i \sqrt{\frac{2}{\pi \lambda_i}} - \frac{1}{2} \theta_i^2 \frac{2}{\pi \lambda_i} + O\left( \lambda_i^{-\frac{3}{2}} \right) \right)
\]

Since \( \sum_{i=1}^{m} \theta_i \sqrt{\frac{2}{\pi \lambda_i}} = \sum_{i=1}^{m} (n_i - \lambda_i) = 0 \) and \( n \) is large, i.e. \( \lambda_i = np_i \) is large, \( O(\lambda_i^{-\frac{1}{2}}) \) approaches to zero,

\[
\log \left( \frac{p}{C} \right) \approx - \sum_{i=1}^{m} \frac{1}{\pi} \theta_i^2.
\]
which shows that $\theta_i (i = 1, 2, \ldots, m)$ are independent variates with $M(0,1)$.

Hence, \[
\sum_{i=1}^{m} \theta_i^2 = \pi \sum_{i=1}^{m} \left( \frac{(n_i - np_i)^2}{np_i} \right) \sim \psi^2_{(m-1)}.
\]

**NOTE:** - (i) Let $O_i$ and $E_i$ ($i = 1, 2, \ldots, m$) be a set of observed and expected frequencies then

\[
\psi^2 = \pi \sum_{i=1}^{m} \frac{(O_i - E_i)^2}{E_i} = \pi \sum_{i=1}^{m} \left( \frac{O_i^2}{E_i} \right) - N
\]

follows $\psi^2_{(m-1)}$; where $\sum_{i=1}^{m} O_i = \sum_{i=1}^{m} E_i = N = \text{Total frequency}$.

Hence, like $\chi^2$, $\psi^2$ can also be used for testing the independence of two attributes.

(ii) If we have $2 \times 2$ contingency table

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>d</td>
</tr>
</tbody>
</table>

then the Psi-square test of independence gives

\[
\psi^2 = \frac{\pi}{2} \frac{N(ad - bc)^2}{(a+b)(c+d)(a+c)(b+d)}, \text{ where } N = a + b + c + d. \quad (3.6.2)
\]

(iii) If we have $n \times 2$ contingency table

<table>
<thead>
<tr>
<th>a_1</th>
<th>b_1</th>
<th>T_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_2</td>
<td>b_2</td>
<td>T_2</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

\[
T_a = \sum_{i=1}^{n} a_i, \quad T_b = \sum_{i=1}^{n} b_i, \quad T = T_a + T_b
\]

then the Psi-square test of independence gives
\[ \psi^2 = \frac{\pi}{2} \frac{T_a T_b}{T} \left[ T \sum_{i=1}^{n} \frac{a_i^2}{T_i} - T_a^2 \right] \]


If we put \(n = 2\) in above formula of \(\psi^2\) then we get (3.6.2).

(iv) we have \(2 \times n\) contingency table

\[
\begin{array}{c|cccc|c}
\phantom{1} & a_1 & \ldots & a_i & \ldots & a_n \\
\hline
b_1 & b_2 & \ldots & b_i & \ldots & b_n \\
T_1 & T_2 & \ldots & T_i & \ldots & T_n \\
\end{array}
\]

then the Psi-square test of independence gives

\[ \psi^2 = \frac{\pi}{2} \frac{T_a T_b}{a_i + b_i} \left[ \frac{a_i T_a}{T} - \frac{b_i T_b}{T} \right]^2 \]


If we put \(n = 2\) in above formula of \(\psi^2\) then we get (3.6.2).

(v) **PSI-SQUARE TEST FOR TESTING** \(H_0: \delta = \delta_0\): -

Just as the chi-square statistic for testing \(\sigma = \sigma_0\) is defined as

\[ \chi^2 = \frac{\sum(x_i - \bar{x})^2}{\sigma_0^2} = \frac{ns^2}{\sigma_0^2} = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{(n-1)} \]

to test the hypothesis that the population mean deviation \(\delta = \delta_0\); the psi-square test statistic

\[ \psi^2 = \frac{\sum(x_i - \bar{x})^2}{\delta_0^2} = \frac{ns^2}{\delta_0^2} = \frac{(n-1)S^2}{\delta_0^2} \sim \psi^2_{(n-1)} \]

can be used.
3.7 **NON CENTRAL PSI - SQUARE DISTRIBUTION**: -

3.7.1 **INTRODUCTION**: -

We know that non central chi-square distribution is generalization of (central) chi-square distribution. Similarly, we can also derive non central psi-square distribution, which is a generalized case of (central) psi-square distribution.

The distribution of the sum of the square of independent moderate variates each having mean deviation but with non zero means is known as non central psi-square distribution. In symbols it may be denoted by $\psi^2$.

3.7.2 **DERIVATION AND DEFINITION**: -

If $X_i \sim M(\mu_i, \delta_i)$ for $i=1,2,3,...,n$ are ‘n’ independent $M(\mu_i, \delta_i)$ random variables then

$$\psi^2 = \sum_{i=1}^{n} \frac{X_i^2}{\delta_i^2}$$

has the non central psi-square distribution with $n$ d.f.

Here we will derive non central psi-square distribution using m.g.f. in the following way:

If $X_i \sim M(\mu_i, \delta_i)$, for $(i = 1,2,3,...,n)$.

Let $Y_i = \frac{X_i^2}{\delta_i^2}$.  

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\[
\therefore M_{Y_1}(t) = M_{X_i} \left( \frac{t}{\delta_i^2} \right) = \frac{1}{\pi \delta_i} \int_{-\infty}^{\infty} e^{\frac{t}{\delta_i^2} x_i^2} \cdot e^{-\frac{1}{\pi} \left( \frac{x_i - \mu_i}{\delta_i} \right)^2} \, dx_i
\]

\[
= \frac{1}{\pi \delta_i} \int_{-\infty}^{\infty} e^{\frac{t}{\delta_i^2} x_i^2 - \frac{1}{\pi} \left( \frac{x_i - \mu_i}{\delta_i} \right)^2} \, dx_i
\]

\[
= \frac{1}{\pi \delta_i} \int_{-\infty}^{\infty} e^{\left( \frac{1-\pi t}{\pi \delta_i^2} \right) x_i^2 + 2 \frac{\mu_i}{\pi \delta_i^2} x_i - \frac{\mu_i^2}{\pi \delta_i^2}} \, dx_i
\]

\[
= \frac{1}{\pi \delta_i} \int_{-\infty}^{\infty} e^{\left( \frac{1-\pi t}{\pi \delta_i^2} \right) x_i^2} \cdot \left( \frac{\mu_i}{\pi \delta_i^2} \right)^2 \, dx_i
\]

\[
= (1 - \pi t)^{\frac{1}{2}} \cdot e^{\left( \frac{\mu_i^2}{\delta_i^2} \right)} \cdot \frac{t}{1 - \pi t} ; \quad t < \frac{1}{\pi}.
\]

Now,

\[
M_{\psi}^{(2)}(t) = M_{\sum Y_i}(t) = \prod_{i=1}^{n} M_{Y_i}(t)
\]

\[
= \prod_{i=1}^{n} \left( 1 - \pi t \right)^{-\frac{1}{2}} e^{\left( \frac{\mu_i^2}{\delta_i^2} \right) \cdot \frac{t}{1 - \pi t}}
\]

\[
= \prod_{i=1}^{n} \left( 1 - \pi t \right)^{-\frac{1}{2}} e^{\frac{\lambda^{'}}{1 - \pi t}} \text{ where } \lambda' = \sum_{i=1}^{n} \left( \frac{\mu_i}{\delta_i} \right)^2 \quad \text{and} \quad t < \frac{1}{\pi}.
\]
Hence,

\[ M_{\psi^2}(t) = (1 - \pi t)^{-\frac{n}{2}} e^{\frac{\lambda'}{\pi}} e^{1-\pi t}. \]

\[ \therefore M_{\psi^2}(t) = \sum_{r=0}^{\infty} \frac{\left(\frac{\lambda'}{\pi}\right)^{r}}{r!} (1 - \pi t)^{-\left(\frac{n+2r}{2}\right)}; \quad t < \frac{1}{\pi}. \]

Hence, by uniqueness theorem, the p.d.f. of non central \( \psi^2 \) distribution with \( n \) d.f. and with non centrality parameter \( \lambda' \) is given by:

\[ f(\psi^2(n)) = \sum_{r=0}^{\infty} \frac{\left(\frac{\lambda'}{\pi}\right)^{r}}{r!} \cdot P(\psi^2(n+2r)) \]

where \( P(\psi^2(n+2r)) = \frac{1}{\pi^{\frac{n+2r}{2}} \cdot \Gamma\left(\frac{n+2r}{2}\right)} \cdot e^{-\frac{1}{\pi} \psi^2} \cdot (\psi^2)^{\frac{n+2r-1}{2}}, 0 \leq \psi^2 < \infty \)

with \( (n+2r) \) d.f.

**NOTE:** -(i) This distribution has two parameters: \( n \) which specifies the number of d.f. and \( \lambda' \) which is non centrality parameter and it is positive.

(ii) If we take \( \lambda' = 0 \Rightarrow \mu_i = 0 \quad \forall \ i = 1,2,3,...n \) the m.g.f. of non central \( \psi^2 \) distribution reduces to \( (1 - \pi t)^{-\frac{n}{2}} \) i.e. the m.g.f. of central \( \psi^2 \) distribution.
3.7.3 ADDITIVE PROPERTY: -

If \( X_i \) (\( i = 1, 2, 3, \ldots, n \)) are independent non central psi-square distribution with \( n_i \) d.f. and non centrality element \( \lambda_i \) then \( \sum Y_i \) is also non central psi-square distribution with \( \sum n_i \) d.f. and non centrality element \( \sum \lambda_i \).

3.7.4 CUMULANTS: -

If \( X \sim \psi^2 \) then

\[
M_X(t) = (1 - \pi t) \frac{n}{2} e^{\frac{\lambda t}{1 - \pi t}}.
\]

\[
\therefore K_X(t) = -\frac{n}{2} \log(1 - \pi t) + \frac{\lambda t}{1 - \pi t}.
\]

After simplification, we get

\[
K_r = \frac{1}{2} \pi^{r-1} (r - 1)! (n \pi + 2r \lambda').
\]

Hence,

\[
K_1 = \mu_1 = \frac{n \pi}{2} + \lambda'.
\]

\[
K_2 = \mu_2 = \frac{n \pi^2}{2} + 2\pi \lambda'.
\]

\[
K_3 = \mu_3 = n \pi^3 + 6\pi^2 \lambda'.
\]

\[
K_4 = 3n \pi^4 + 24\pi^3 \lambda'.
\]

3.7.5

\[
\beta_1 = \frac{8(n + 6 \frac{\lambda'}{\pi})^2}{(n + 4 \frac{\lambda'}{\pi})^3}.
\]
3.7.6 \[ \beta_2 = 3 + \frac{12}{n+4\left(\frac{\lambda}{\pi}\right)} \].

**NOTE:** - If we take \( \lambda' = 0 \), we get

\[ \beta_1 = \frac{8}{n} \] and \( \beta_2 = 3 + \frac{12}{n} \).

Also, \( n \to \infty \), \( \beta_1 \to 0 \) and \( \beta_2 \to 3 \).