Chapter 3

Embedding and NP-Completeness - Study of Some Parameters

In this chapter, we establish the possibility of embedding a graph as an induced subgraph in a harmonious graph, a felicitous graph, an elegant graph, a cordial graph, an odd-graceful graph, a polychrome graph, a strongly c-harmonious graph, a prime graph, a topologically set-graceful graph and a set-sequential graph each with a given property, leading to prove the NP-completeness of some parameters like chromatic number, clique number, domination number and independence number of these graphs. Also we prove that every complete bipartite graph is odd graceful.

3.1 Introduction

A wide variety of commonly encountered combinatorial problems, such as the problem of the determination of the clique number of a graph, are now known to be NP-complete. Many graph theory problems have been shown to be NP-complete by Graham & Sloane [GS80]. The collection of such problems continues to grow almost daily. Indeed, the NP-completeness problems are now so pervasive that it is important for any one concerned with the computational aspects of these fields to
be familiar with the meaning and implications of this concept. Theorems on embedding an arbitrary graph in a graph admitting a specific type of labellings can indeed be used to prove the NP-completeness of the determination of the parameters like the chromatic number $\chi$, the clique number $\omega$, the domination number $\gamma$ and the independence number $\beta_0$ of graphs. For instance, Acharya et al. [AAR06] have proved that the problems of determining the chromatic number, the clique number, the domination number and the independence number of a graceful graph are NP-complete.

In this chapter, we establish the embedding theorems on harmonious graphs, felicitous graphs, elegant graphs, cordial graphs, odd-graceful graphs, polychrome graphs, strongly c-harmonious graphs, prime graphs, topologically set-graceful graphs and set-sequential graphs. Nonstandard definitions of concern to us along with pertinent remarks and elementary observations are given below.

**Definition 3.1.1.** [GS80] A graph $G$ with $q$ edges, $q \geq 1$, is harmonious if there is an injection $f$ from the vertices of $G$ to the group of integers modulo $q$ such that when each edge $xy$ is assigned the label $(f(x) + f(y))(\text{mod } q)$, the resulting edge labels are all distinct; $f$ is then a harmonious labelling of $G$.

Liu and Zhang [LZ93] proved that every graph is a subgraph of a harmonious graph. Determining whether a graph has a harmonious labelling was shown to be NP-complete by Krishna et al. (2001)[23].

**Definition 3.1.2.** [SLS91] An injective function $f$ from the vertices
of a graph $G$ with $q$ edges, $q \geq 1$ to the set $\{0, 1, \ldots, q\}$ is felicitous if the edge labels induced by $(f(x) + f(y))(\mod q)$ for each edge $xy$ are all distinct.

Balakrishnan & Kumar [BS94] proved that every graph is a subgraph of a felicitous graph.

**Observation 3.1.3.** Every harmonious labelling of a harmonious graph is felicitous too; hence, every harmonious graph is felicitous.

**Definition 3.1.4.** [HCR81] An elegant labelling $f$ of a graph $G$ with $q$ edges is an injective function from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that when each edge $xy$ is assigned the label $(f(x) + f(y))(\mod q + 1)$ the resulting edge labels are all distinct positive integers; if $G$ admits such a labelling, then $G$ is an elegant graph.

Chang et al. (1981) [HCR81] proved that every graph is a subgraph of an elegant graph.

**Definition 3.1.5.** [Cah87] Let $f$ be a function from the vertices of $G$ to the set $\{0, 1\}$ and for each edge $xy$ assign the label $|f(x) - f(y)|$. Then $f$ is a cordial labelling of $G$ and $G$ is a cordial graph if the number of vertices labelled 0 and the number of vertices labelled 1 differ by at most 1, as also the number of edges labelled 0 and the number of edges labelled 1 differ at most by 1.

Cairnie & Edwards [CE00] proved that deciding whether a graph admits a cordial labelling is NP-complete.
Definition 3.1.6. [Gna91] A graph $G$ with $q$ edges is odd graceful if there is an injection $f$ from $V(G)$ into $\{0, 1, \ldots, 2q - 1\}$ such that, when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting set of edge labels is $\{1, 3, \ldots, 2q - 1\}$.

Definition 3.1.7. [Val99] For a graph $G = (V, E)$ and an abelian group $H$ a polychrome labelling of $G$ by $H$ is a bijection $f$ from $V$ to $H$ such that the edge labels induced by $f^+(uv) = f(u) + f(v)$, $uv \in E$, are all distinct.

Definition 3.1.8. [HCR81] An injective labelling $f$ of a graph $G$ with $q$ vertices is defined to be strongly $c$-harmonious if the vertex labels are from $\{0, 1, \ldots, q - 1\}$ and the edge labels induced by $f(x) + f(y)$ for each edge $xy$ are $c, c + 1, \ldots, c + q - 1$. Grace [Gra83] called such a labelling sequential.

Definition 3.1.9. A graph with vertex set $V$ is said to have a prime labelling if its vertices are labelled with distinct integers $1, 2, \ldots, |V|$ such that for each edge $xy$ the labels assigned to $x$ and $y$ are relatively prime.

A vertex-coloring of a graph $G = (V, E)$ is a function $\Phi : V \rightarrow C$ from the set of vertices to a set $C$ of colors. The coloring $\Phi$ is proper if no two adjacent vertices are assigned the same color. A graph is $k$-colorable if it admits a proper vertex coloring with at most $k$ colors. The chromatic number of the graph $G$, denoted by $\chi(G)$, is the smallest non-negative integer $k$ such that $G$ is $k$-colorable.
The \textit{clique number} of a graph $G$, denoted by $\omega(G)$, is the number of vertices in a maximal complete subgraph of $G$.

A set $S \subset V$ is a dominating set of a graph $G = (V, E)$ if each vertex in $V$ is either in $S$ or is adjacent to a vertex in $S$. The \textit{domination number} $\gamma(G)$ is the cardinality of a minimal dominating set of $G$.

For a graph $G = (V, E)$, a set $X \subseteq V$ is independent if there are no edges between vertices in $X$. The cardinality of the largest independent set in $G$ is called the \textit{independence number} of $G$ and is denoted by $\beta_0(G)$.

Here onwards the term \textit{embedding} shall mean a mapping $\zeta$ of the vertices of $G$ into the set of vertices of a graph $H$ such that the subgraph induced by the set $\{\zeta(u) : u \in V(G)\}$ is isomorphic to $G$; for all practical purposes, we shall assume then that $G$ is indeed a subgraph of $H$.

\section{3.2 Embedding and NP-Completeness of Parameters on Number Labelled Graphs}

We shall now embark on a study of embedding an arbitrary graph with a given property $\mathcal{P}$ in a graph admitting a specific type of labelling and having the property $\mathcal{P}$.

We now study the embedding of a graph into a harmonious graph and study NP-completeness of the problem of the determination of the clique number, the independence number, the domination number and the chromatic number of a connected harmonious graph.
It is well known that the determination of clique number, independence number, domination number and chromatic number of a graph is NP-complete if they are greater than or equal to 3, [GJ79]

**Theorem 3.2.1.** Every $(p, q)$ graph $G$ can be embedded in a connected Harmonious graph $H$ with $2^{p+1}$ edges and $2^{p+1} - q$ vertices.

**Proof.** Let $G$ be a graph with $V(G) = \{u_1, u_2, ..., u_p\}$. We will embed the graph $G$ in a graph $H$ with $|V(H)| = 2^{p+1} - q$ and $|E(H)| = 2^{p+1}$. Consider the set $\mathbb{Z}_{2^{p+1}}$ of integers addition modulo $2^{p+1}$. Label the vertices $u_i$ by $2^{i-1}$ where $1 \leq i \leq p$. Then the edge labels of $G$ are $2^{i-1} + 2^{j-1}$ where $1 \leq i, j \leq p$, whenever $u_i u_j$ is an edge. Here all edge labels are distinct. Now introduce $p + 2$ vertices $v_1, v_2, ..., v_{p+2}$ and label them by $0, 2^p, 2^p + 2^0, 2^p + 2^1, ..., 2^p + 2^{p-1}$ respectively. Join $v_1$ to $v_k$ where $2 \leq k \leq p + 2$.

If $G$ is connected join $v_1$ to $u_1$ and $v_2$ to all $v_k, 4 \leq k \leq p + 2$.

If $G$ is disconnected and having $t$ components, say $C_1, C_2, ..., C_t$, join $v_1$ to exactly one vertex of $C_i$, say $u_{i_1}, 1 \leq i \leq t$, join $v_2$ to all $v_k, 4 \leq k \leq p + 2, k \neq i_1 + 2, 1 \leq i \leq t$.

Then the new edge labels are $2^0, 2^1, ..., 2^{p-1}, 2^p, 2^p + 2^0, 2^p + 2^1, ..., 2^p + 2^{p-1}$, all are distinct and are distinct from edge labels of $G$. The elements of $\mathbb{Z}_{2^{p+1}}$ which are not edge labels so far are all numbers not of the form $2^i + 2^j$ where $0 \leq i < j \leq p$, and all numbers of the form $2^k + 2^l$ where $u_{k+1}$ and $u_{l+1}, 0 \leq k < l < p$ are not adjacent in $G$. Clearly these numbers are not vertex labels so far and they are $2^{p+1} - q - 2p - 1$ in number. Note that $2^p - 1$ is not of the form $2^i + 2^j$ where $0 \leq i < j \leq p$. Now introduce $S = 2^{p+1} - q - 2p - 2$ new
vertices, say \( w_1, w_2, \ldots, w_s \), label them by the elements of \( Z_{2p+1} \) which are not edge labels so far except 0, and join them to the vertex \( v_1 \) whose label is 0. Fix the vertex \( w_1 \), with the label \( 2^p - 1 \) and join it to the vertex \( v_3 \) whose label is \( 2^p + 2^0 \). Then the edge label of \( v_3w_1 \) is 0 in \( Z_{2p+1} \). Here the new graph \( H \) with \( 2^{p+1} \) edges and \( 2^{p+1} - q \) vertices is harmonious and \( G \) is an induced subgraph of \( H \).

When the given graph is connected the following figure represents the embedding of the graph into a connected harmonious graph.

![Figure 3.1](image)

When the given graph is disconnected the figure 3.2 represents the embedding of the graph into a connected harmonious graph.
Corollary 3.2.2. If $G$ is planar, so is the harmonious graph $H$.

Corollary 3.2.3. If the chromatic number, clique number and independence number of $G$ is $\geq 3$, then the chromatic number $\chi(H) = \chi(G)$, the clique number $\omega(H) = \omega(G)$ and the independence number $\beta_0(H) = \beta_0(G) + 2^{p+1} - q - p - 3$. Therefore the problems of the determination of the chromatic number, the clique number and the independence number of a connected harmonious graph are NP-complete.

The figure 3.3 illustrates the theorem when the given graph is connected with an example having $p = 5$. 
The figure 3.4 illustrates the theorem when the given graph is disconnected with an example for $p = 5$.

To study the NP-Completeness of the determination of the domination number of a harmonious graph we consider the embedding of the given graph into a disconnected harmonious graph, which is the
following corollary.

**Corollary 3.2.4.** In the theorem for a connected given graph, if we avoid the edge between $v_1$ and $u_1$ and join $v_2$ to $v_3$, we get a disconnected harmonious graph $H^*$ for which $G$ is an induced subgraph. Here if $\gamma(G) \geq 3$, then the domination number $\gamma(H^*) = \gamma(G) + 1$. Similar argument is valid for a given disconnected graph. Therefore the problem of the determination of the domination number of a harmonious graph is NP-complete.

The following figure represents an embedding into a disconnected harmonious graph.

![Diagram](image-url)
The following figure illustrates the corollary with an example for $p = 5$.

![Figure 3.6:](image)

By the observation 3.1.3 every harmonious graph is felicitous, then the following corollary is immediate.

**Corollary 3.2.5.** Every $(p,q)$-graph $G$ can be embedded in a connected felicitous graph $L$ with $2^{p+1}$ edges and $2^{p+1} - q$ vertices.

**Corollary 3.2.6.** If $G$ is planar, so is the felicitous graph $L$.

**Corollary 3.2.7.** If the chromatic number, clique number and independence number of $G$ is $\geq 3$, then the chromatic number $\chi(H) = \chi(G)$, the clique number $\omega(H) = \omega(G)$, and the independence number $\beta_0(H) = \beta_0(G) + 2^{p+1} - q - p - 3$. Therefore the problems of the determination of the chromatic number, the clique number and the independence number of a connected felicitous graph are NP-complete.
Corollary 3.2.8. Proceeding as corollary 3.2.12 we get a disconnected felicitous graph \( L^* \) for which \( G \) is an induced subgraph. If \( \gamma(G) \geq 3 \), then domination number \( \gamma(L^*) = \gamma(G) + 1 \). Therefore the problem of the determination of the domination number of a felicitous graph is NP-complete.

We now study the embedding of a graph into an elegant graph and study NP-completeness of the problems of the determination of the clique number, the independence number, the domination number and the chromatic number of a elegant graph.

Theorem 3.2.9. Every \((p, q)\) graph \( G \) can be embedded in a connected elegant graph \( H \) with \( 2^{p+1} \) edges and \( 2^{p+1} - q + 1 \) vertices.

Proof. Let \( G \) be a graph with \( V(G) = \{u_1, u_2, \ldots, u_p\} \). We will embed the graph \( G \) in a graph \( H \) with \( |V(H)| = 2^{p+1} - q + 1 \) and \( |E(H)| = 2^{p+1} \). Consider the set \( Z_{2^{p+1}} \) of integers addition modulo \( 2^{p+1} \). Label the vertices \( u_i \) by \( 2^{i-1} \) where \( 1 \leq i \leq p \). Then the edge labels of \( G \) are \( 2^{i-1} + 2^{j-1} \) where \( 1 \leq i, j \leq p \), whenever \( u_i u_j \) is an edge. Here all edge labels are distinct. Now introduce \( p + 2 \) vertices \( v_1, v_2, \ldots, v_{p+2} \) and label them by \( 0, 2^p, 2^{p+1} + 2^0, 2^{p+1} + 2^1, \ldots, 2^{p+1} + 2^{p-1} \) respectively. Join \( v_1 \) to \( v_k \) where \( 2 \leq k \leq p + 2 \).

If \( G \) is connected join \( v_1 \) to \( u_1 \) and \( v_2 \) to all \( v_k \), \( 4 \leq k \leq p + 2 \).

If \( G \) is disconnected and having \( t \) components, say \( C_1, C_2, \ldots, C_t \), join \( v_1 \) to exactly one vertex of \( C_i \), say \( u_{i_1}, 1 \leq i \leq t \), join \( v_2 \) to all \( v_k \), \( 4 \leq k \leq p + 2 \), \( k \neq i_1 + 2, 1 \leq i \leq t \).

Then the new edge labels are \( 2^0, 2^1, \ldots, 2^{p-1}, 2^p, 2^p + 2^0, 2^p + 2^1, \ldots, 2^p + \ldots \)
$2^{p-1}$, all are distinct and are distinct from edge labels of $G$. The elements of $Z_{2^{p+1}}$ which are not edge labels so far are all numbers not of the form $2^i + 2^j$ where $0 \leq i < j \leq p$, and all numbers of the form $2^k + 2^l$ where $u_{k+1}$ and $u_{l+1}$, $0 \leq k < l < p$ are not adjacent in $G$. Clearly these numbers are not vertex labels so far and they are $2^{p+1} - q - 2p - 1$ in number. Now introduce $S = 2^{p+1} - q - 2p - 1$ new vertices, say $w_1, w_2, \ldots, w_s$. Fix $w_1$ and label it by $2^{p+1}$ and label $w_i, 2 \leq i \leq s$ by the elements of $Z_{2^{p+1}}$, except 0, which are not edge labels so far and in an injective manner, and join them to the vertex $v_1$ whose label is 0. Here the new graph $H$ with $2^{p+1}$ edges and $2^{p+1} - q + 1$ vertices is elegant and $G$ is an induced subgraph of $H$.

When the given graph is connected the following figure represents the embedding of the graph into a connected elegant graph.

Figure 3.7:
When the given graph is disconnected the figure 3.8 represents the embedding of the graph into a connected elegant graph.

**Figure 3.8:**

**Corollary 3.2.10.** If $G$ is planar, so is the elegant graph $H$.

**Corollary 3.2.11.** If the chromatic number, clique number and independence number of $G$ is $\geq 3$, then the chromatic number $\chi(H) = \chi(G)$, the clique number $\omega(H) = \omega(G)$ and the independence number $\beta_0(H) = \beta_0(G) + 2^{p+1} - q - p - 1$. Therefore the problems of the determination of the chromatic number, the clique number and the independence number of a connected elegant graph are NP-complete.

The figure 3.9 illustrates the theorem when the given graph is connected with an example for $p = 5$. 
The figure 3.10 illustrates the theorem when the given graph is disconnected with an example for $p = 5$.

To study the NP-Completeness of the determination of the domination number of an elegant graph we consider the embedding of the given graph into a disconnected elegant graph, which is the following
corollary.

**Corollary 3.2.12.** In the theorem for a connected given graph, if we avoid the edge between \( v_1 \) and \( u_1 \) and join \( v_2 \) to \( v_3 \), we get a disconnected elegant graph \( H^* \) for which \( G \) is an induced subgraph. Here if \( \gamma(G) \geq 3 \), then the domination number \( \gamma(H^*) = \gamma(G) + 1 \). Similar argument is valid for a given disconnected graph. Therefore the problem of the determination of the domination number of a elegant graph is NP-complete.

The following figure represents an embedding into a disconnected elegant graph.

![Figure 3.11](image_url)
The following figure illustrates the corollary with an example. Let $p = 5$.  

![Figure 3.12](image)

The following flow chart gives an algorithm for the embedding of a given graph into a elegant graph.
Start

\[ G=(p,q), V(G)=\{u_1,u_2,...,u_p\} \]

Is \( G \) is Elegant

Yes

No

Is \( G \) is connected

No

Add \( p+2 \) vertices \( v_i \). Join \( v_1 \) to \( u_1 \). Join \( v_1 \) to \( j \leq j \leq p+2, j \neq i+2 \)

Add \( 2^{p-1}-q-2p+1 \) vertices \( w_k \) and join them to \( v_1 \).

Define \( f(u_i) = 2^{i-1}, f(v_1) = 0, f(v_2) = 2^p, f(v_j) = 2^{p+2j-3}+1, f(w_i) = 2^{p+1} \)

Label \( w_k \) by non-zero elements of \( \mathbb{Z}_2^{p+1} - \{f(u_i)\} \cup \{f(v_j)\} \) \( \cup \{2^{p+1}\} \) in an injective manner

Stop
We now study two different embeddings of a graph into a cordial graph and study NP-completeness of the problems of the determination of the clique number, the independence number, the domination number and the chromatic number of a cordial graph.

**Theorem 3.2.13.** Every connected graph $G$ with $p$ vertices can be embedded into a connected cordial graph.

**Proof.**

Case 1: $q > p$.

Let $G$ be a $(p, q)$-graph with $V(G) = \{u_1, u_2, \ldots, u_p\}$. Label $u_i$, $i$, $1 \leq i \leq p$ by 0. Add $2q - p$ new vertices $v_1, v_2, \ldots, v_q, w_1, w_2, \ldots, w_{q-p}$. Label $v_j$, $1 \leq j \leq q$ by 1 and $w_l$, $1 \leq l \leq q - p$ by 0. Fix one vertex of $G$ say $u_1$. Join $u_1$ to $v_j$, for $1 \leq j \leq p$ and to $w_l$, $1 \leq l \leq q - p$. Join $w_l$, $1 \leq l \leq q - l$ to the vertex $v_1$. The new graph $H$ is a cordial graph having $q$ vertices with label 0, $q$ vertices with label 1, $2q - p$ edges with label 0 and $2q - p$ edges with label 1.

Case 2: $q \leq p$.

Let $G$ be a $(p, q)$-graph with $V(G) = \{u_1, u_2, \ldots, u_p\}$. label $u_i$, $i$, $1 \leq i \leq p$ by 0. Add $p$ new vertices $v_1, v_2, \ldots, v_p$ and label them by 1. Fix one vertex of $G$ say $u_1$. Join $u_1$ to $v_j$, for $1 \leq j \leq p$. Join $v_1$ to $v_k$, $q + 1 \leq k \leq p$. The new graph $H$ is a cordial graph with number of vertices with label 0, number of vertices with label 1, number of edges with label 0 and number of edges with label 1 are all equal to $p$. Clearly $G$ is an induced subgraph of $H$. Hence the theorem.
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The following figure represents the embedding of the given connected \((p, q)\)-graph into a cordial graph when \(q > p\).

![Figure 3.13:](image)

The figure 3.14 represents the embedding of the given connected \((p, q)\)-graph into a cordial graph when \(q < p\).

![Figure 3.14:](image)
The figure 3.15 represents the embedding of the given connected \((p, q)\)-graph into a cordial graph when \(q = p\)

Corollary 3.2.14. If \(G\) is planar, so is the cordial graph \(H\).

Corollary 3.2.15. If the chromatic number and clique number of \(G\) is \(\geq 3\), then the chromatic number \(\chi(H) = \chi(G)\) and the clique number \(\omega(H) = \omega(G)\). Therefore the problems of the determination of the chromatic number and the clique number of a connected cordial graph are \(NP\)-complete.
The figure 3.16 illustrates the theorem for the given (5, 7) graph $G$.

![Figure 3.16: The figure 3.17 illustrates the theorem for the given (5, 4) graph $G$.](image)

The figure 3.17 illustrates the theorem for the given (5, 4) graph $G$. 

![Figure 3.17:](image)
The figure 3.18 illustrates the theorem for the given (5,5)-graph.

Figure 3.18:

To prove that the determination of some more parameters of cordial graphs are NP-complete, we will give another embedding as follows.

**Theorem 3.2.16.** Every graph $G$ with $p$ vertices can be embedded into a cordial graph.

**Proof.**

Case 1: $q > p$.

Let $G$ be a $(p,q)$-graph with $V(G) = \{u_1, u_2, \ldots, u_p\}$. Label $u_i$ by 0, $i, 1 \leq i \leq p$. Add $2q - p$ new vertices $v_1, v_2, \ldots, v_q, w_1, w_2, \ldots, w_{q-p}$. Label $v_j, 1 \leq j \leq q$ by 1 and $w_l, 1 \leq l \leq q - p$ by 0. Join the vertex $w_1$ to $v_j$, for $1 \leq j \leq q$. Then the new graph $H$ is a cordial graph having $q$ vertices with label 0, $q$ vertices with label 1, $q$ edges with label 0 and $q$ edges with label 1. Note that the number of isolated vertices is $q - p - 1$. 
The figure 3.19 represents the embedding of the given connected $(p, q)$-graph into a cordial graph when $q > p$

Case 2: $q \leq p$.

Let $G$ be a $(p, q)$-graph with $V(G) = \{u_1, u_2, \ldots, u_p\}$. Label $u_i$, for some $i$, $1 \leq i \leq p$ by 0. Add $p + 2$ new vertices $v_1, v_2, \ldots, v_{p+2}$. Label $v_1$ by 0 and $v_i$, $2 \leq i \leq p + 2$ by 1. Join $v_1$ to $v_i$, $2 \leq i \leq q + 1$. The new graph $H$ is a cordial graph with the number of vertices with label 0 and the number of vertices with label 1 both equal to $p + 1$. Number of edges with label 0 and number of edges with label 1 are both equal to $q$. Note that the number of isolated vertices is $p - q + 1$. Clearly, $G$ is an induced subgraph of $H$. Hence the theorem. ♣

The figure 3.20 represents the embedding of the given connected
$(p, q)$-graph into a cordial graph when $q < p$

**Corollary 3.2.17.** If $G$ is a triangle free graph, then so is the cordial graph $H$.

**Corollary 3.2.18.** If the domination number and independence number of $G$ are $\geq 3$, then

if $q > p$ the domination number $\gamma(H) = \gamma(G) + q - p$ and the independence number $\beta_0(H) = \beta_0(G) + 2q - p - 1$

if $q \leq p$ the $\gamma(H) = \gamma(G) + p - q + 2$ and $\beta_0(H) = \beta_0(G) + p + 1$.

Therefore, the problems of determining the domination number and the independence number of a cordial graph are NP-complete.

The figure 3.21, illustrates the theorem for the given connected $(5, 7)$-graph.
The figure 3.21 illustrates the theorem for the connected \((5, 4)\)-graph.

The figure 3.22 illustrates the theorem for the connected \((5, 4)\)-graph.

The following flow chart gives an algorithm for the embedding of the graph into a cordial graph.
Start

G=(p,q), V(G)={u_1, u_2, ..., u_p}

Yes

Is q > p

No

Is q < p

Yes

Is the new graph H is cordial

No

Add 2q-p vertices v_j, 1 ≤ j ≤ q and w_k, 1 ≤ k ≤ q-p

Join u_i to v_j, 1 ≤ j ≤ p
Join u_i to w_k, 1 ≤ k ≤ q-p

f(u_i)=0, f(v_j)=1, f(w_k)=0

Yes

Stop

No

Add p vertices v_j

Join u_i to v_j

f(u_i)=0, f(v_j)=1

Is the new graph H is cordial

Yes

No

Add p vertices v_j, 1 ≤ j ≤ p

Join u_i to v_j, 1 ≤ j ≤ p,
Join v_i to v_j, q+1 ≤ k ≤ p

f(u_i)=0, 1 ≤ i ≤ p
f(v_j)=1, 1 ≤ i ≤ p

Is the new graph H is cordial

Yes

No
It is easy to see that every odd graceful graph is bipartite.

**Theorem 3.2.19.** Every complete bipartite graph is odd graceful.

**Proof.** Let $K_{m,n}$ be a complete bipartite graph with two parts $A = \{x_1, x_2, ..., x_m\}$ and $B = \{y_1, y_2, ..., y_n\}$. Define the labelling $f$, which label $x_i$ by $(i-1)2n$ and $y_j$ by $2nm - 2j + 1$. Then the edge labels induced by $|f(x) - f(y)|$ for each edge $xy$ are distinct and takes the values from $\{1, 3, ..., 2mn-1\}$. According to this labelling $K_{m,n}$ is odd graceful. ✷

The figure 3.23, illustrates the theorem.

![Figure 3.23](image)

Figure 3.23:

Further, Gallian [Gal05] quotes Barrientos’s conjecture that every bipartite graph is odd graceful. In the absence of a proof of this conjecture, a study of embedding an arbitrary bipartite graph into an odd graceful graph might be of some value as it does lead to decide about
the NP-completeness of the problem of determining the independence number of an odd graceful graph.

**Theorem 3.2.20.** Every bipartite graph can be embedded into an odd graceful graph.

**Proof.** Let $G = (V_1, V_2)$ be a bipartite graph with partition $V_1$ and $V_2$ where $|V_1| = m$ and $|V_2| = n$. Without loss of generality, assume that $m \leq n$. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{w_1, w_2, \ldots, w_n\}$. Label the vertex $u_i$, $1 \leq i \leq m$ by $4^i + 4^{i-1} + \cdots + 4^0$ and $w_l, 1 \leq l \leq n$ by $4^l$. Here, all edge labels of $G$ are distinct. Define the set $S$ as the set of all even integers less that $4^m + 4^{m-1} + \cdots + 4^0$ and not equal to $4^l$, $1 \leq l \leq n$. Then $|S| = \frac{4^m + 4^{m-1} + \cdots + 4^0}{2} - n = \frac{4(4^m-1)}{6} - n$. Now, add $|S| + 2$ new vertices $v_1, v_2, \ldots, v_{|S|+2}$. Label $v_1$ by 1 and $v_2$ by 0 and all other $v_k$, $3 \leq k \leq |S| + 2$ by elements from $S$. Join $v_1$ to all $v_r$, $3 \leq k \leq |S| + 2$ subject to the condition that if $x$ is an edge label in $G$ then we do not join the vertex labelled by $x + 1$ to $v_1$. To get edges with label $4^l - 1$, join the vertex corresponding to the label $4^l + 4^{l-1} + \cdots + 4^1$ to the vertex $u_{l-1}$ which was labelled by $4^{l-1} + 4^{l-2} + \cdots + 4^0$. If 1 is not an edge label of $G$, join $v_1$ to $v_2$. The new graph $H$ is an odd graceful graph. ♣

**Corollary 3.2.21.** Every tree can be embedded in an odd - graceful tree.

**Corollary 3.2.22.** If the independence number of $G$ is $\geq 3$, then the independence number $\beta_0(H) = \beta_0(G) + |S| + 1$. Therefore, the
problem of the determination of the independence number of an odd graceful graph is NP-complete.

The figure 3.24, illustrates the theorem.

We now study the embedding of a graph into a polychrome graph with the abelian group $H$ as the set of all integers addition modulo $2^{p+1}$ and study NP-completeness of the problems of the determination of the clique number, the independence number, the domination number
and the chromatic number of a polychrome graph.

**Theorem 3.2.23.** Every \((p, q)\)-graph can be embedded in a connected polychrome graph with respect to the abelian group, set of all integers addition modulo \(2^{p+1} + 1\).

**Proof.** Let \(G\) be a \((p, q)\) graph. Consider the group \(H\) as the set of all integers addition modulo \(2^{p+1} + 1\). Let \(V(G) = \{u_1, u_2, \ldots, u_p\}\). Label \(u_i\) by \(2^{i-1}\), \(1 \leq i \leq p\). Let \(S_1 = \{\text{elements of } H \text{ in the form } 2^i, 0 \leq i \leq p\}\), \(S_2 = \{\text{elements of } H \text{ in the form } 2p + 2^j, 0 \leq j \leq p-1\}\), \(S_3 = \{\text{elements of } H \text{ in the form } 2^l + 2^k, 2p + 2^l + 2^k\}\) where \(u_i u_{i+1}\) is an edge of \(G\). Then \(|S_1| = p+1, \ |S_2| = p, \ |S_3| = 2q\).

Define \(S = Z_{2^{p+1}} - \{0\} - \{S_1 \cup S_2 \cup S_3\}\). Then \(|S| = 2^{p+1} - 2p - 2q - 2\). Now, introduce \(2^{p+1} - p\) new vertices \(v_1, v_2, \ldots, v_{2^{p+1}-p}\). Label \(v_1\) by \(0\), \(v_2\) by \(2^p\), \(v_t\), \(3 \leq t \leq p+2\) by elements of \(S_2\), \(v_r, p+3 \leq r \leq p+2+2q\) by elements of \(S_3\) and \(v_m, p+2q+3 \leq m \leq 2^{p+1} - p\) by elements of \(S\). Join \(v_1\) to \(u_i, 1 \leq i \leq p - 1\). Join \(v_1\) to \(v_i\), for \(2 \leq i \leq p + 2\) and for \(p + 2q + 3 \leq i \leq 2^{p+1} - p\). Join \(v_2\) to \(v_i, p + 3 \leq i \leq p + 2q + 2\). The new graph \(M\) has \(2^{p+1}\) vertices and \(2^{p+1} + q - p - 1\) edges. Here \(V(M) = Z_{2^{p+1}}\) and edges of \(M\) are all distinct. \(\Diamond\)
The following figure represents the embedding of the given \((p,q)\)-graph into a connected polychrome graph with respect to the set of addition modulo \(\mathbb{Z}_{2p+1}\)

Figure 3.25:

Corollary 3.2.24. If the chromatic number, the clique number and the independence number of \(G\) are \(\geq 3\), then the chromatic number \(\chi(H) = \chi(G) + 1\), the clique number \(\omega(M) = \omega(G) + 1\), and the independence number \(\beta_0(M) = \beta_0(G) + 2^{p+1} - p - 2\). Therefore, the problems of determining the chromatic number, the clique number and
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the independence number of a connected polychrome graph are NP-complete.

The figure 3.26, illustrates the theorem.

We now study the embedding of a graph into a strongly $c$-harmonious graph and study NP-completeness of the problems of the determination of the independence number of a strongly $c$-harmonious graph.

**Theorem 3.2.25.** Every connected $(p, q)$-graph can be embedded in a connected strongly $c$-harmonious graph, for all $c$.

**Proof.** Let $G$ be a $(p, q)$-graph with $V(G) = \{u_1, u_2, \ldots, u_p\}$. Let
\[ d = 2^i + 1. \] Label \( u_1 \) by \( 2^0 \), \( u_k \) by \( 2^{i+j-2} \), \( 2 \leq j \leq p \). Now, we will embed \( G \) into a graph \( H \) with \( 2^{i+p-1} - 1 \) edges. Define 
\[ S_1 = \{ 2^k, 1 \leq k \leq i + p - 2 \} - \{ 2^k, i \leq k \leq i + p - 2 \}, \]
\[ S_2 = \{ 2^{k_1} + 2^{k_2}, 0 \leq k_2 < k_1 \leq i + p - 2 \} - \{ \text{the edge labels of } G \}, \]
\[ S_l = \{ 2^{k_1} + 2^{k_2} + \cdots + 2^{k_l}, 0 \leq k_l < k_{l-1} < \cdots < k_1 \leq i + p - 2 \}. \]
Define \( S = \bigcup_{n=1}^{i+p-1} S_n \). Here, \( |S_1| = i - 1 \), \( |S_2| = (i + p - 1)C_2 - q \), \( |S_3| = (i + p - 1)C_3, \ldots \), \( |S_{i+p-1}| = (i + p - 1)C_{i+p-1} \). Then \( |S| = 2^{i+p-1} - q - p - 1 \). Let \( \theta = \{ \text{all elements of } S \text{ which is } \geq d \} \).
Now, introduce \( |S| + 1 \) new vertices, \( v_1, w_1, w_2, \ldots, w_{|S-\theta|} \), \( x_1, x_2, \ldots, x_{|\theta|} \). Label \( v_1 \) by 0, label \( x_i \) by elements from \( \theta \) in an increasing and injective manner and label \( w_j \) by elements from \( S - \theta \) in an increasing and injective manner. Join \( v_1 \) to \( x_i \), \( 1 \leq i \leq |\theta| \). Join \( w_j \), \( 1 \leq j \leq |S - \theta| \) to \( x_{|\theta|} \).
When \( c < 2^i + 1 \), Join \( v_1 \) to \( u_j \), \( 2 \leq j \leq p \) and join \( u_1 \) to \( x_{|\theta|} \).
when \( c = 2^i + 1 \), Join \( v_1 \) to \( u_j \), \( 3 \leq j \leq p \) and Join \( u_1, u_2 \) to \( x_{|\theta|} \).
All vertex labels are distinct and belongs to \( \{0, 1, 2, \ldots, 2^{i+p-1} - 1\} \) and the new graph \( H \) is a polychrome graph with \( 2^{i+p-1} - q \) vertices and \( 2^{i+p-1} - 1 \) edges, with edge values \( c, c + 1, \ldots, c + 2^{i+p-1} - 2 \). Clearly, \( G \) is an induced subgraph of \( H \). \( \clubsuit \)
Figure 3.27 represents the embedding of a \((p, q)\)-graph into a strongly \(c\)-harmonious graph when \(c < 2^i + 1\), where \(i \in \mathbb{N}\).
Figure 3.28 represents the embedding of a \((p, q)\)-graph into a strongly \(c\)-harmonious graph when \(c = 2^i + 1\), where \(i \in N\).

![Diagram of embedding](image)

**Figure 3.28:**

**Corollary 3.2.26.** If the independence number of \(G\) is \(\geq 3\), then the independence number \(\beta(H) = \beta_0(G) + 2^{i+p+1} - q - p - 2\). Therefore, the problem of the determination of the independence number of a connected strongly \(c\)-harmonious graph is NP-complete.
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The figure 3.29, illustrates the theorem with an example where $c = 2^2$.

The notion of prime labelling originated with Entringer. Around 1980, Entringer conjectured that all trees have a prime labelling. So far, there has been little progress towards proving this conjecture. Seoud and Youssef [SY02] have proved that every spanning subgraph
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of a prime graph is prime and every graph is a subgraph of a prime graph.

**Theorem 3.2.27.** Every graph can be embedded into a connected prime graph.

**Proof.** Let $G$ be a graph of order $n$ with $V(G) = \{u_1, u_2, \ldots, u_n\}$. Let $S = \{p_1, p_2, \ldots\}$, where $p_l < p_m$, if $l < m$ be the infinite set of primes. Label $u_i$ by $p_i$. Introduce $p_{n+1} - n - 1$ new vertices $v_j, 1 \leq j \leq p_{n+1} - n - 1$. Label $v_1$ by $p_{n+1}$ and all $v_j, 1 \leq j \leq p_{n+1} - n - 1$ by natural number $x_j$ where $2 < x_j < p_{n+1}$ and $x_j \notin S$. Join $v_1$ to all $u_i$ and $v_j, 1 \leq j \leq p_{n+1} - n - 1$. Then the new obtained graph $H$ is a prime connected graph with $G$ as an induced subgraph.

**Corollary 3.2.28.** If the chromatic number, clique number, domination number and the independence number of $G$ are $\geq 3$, then the chromatic number $\chi(H) = \chi(G) + 1$, the clique number $\omega(H) = \omega(G) + 1$, the domination number $\gamma(H) = \gamma(G) + 1$ and the independence number $\beta_0(H) = \beta_0(G) + p_{n+1} - n - 2$. Therefore, the problems of the determination of the chromatic number, the clique number, the domination number and the independence number of a connected prime graph are NP-complete.
The figure 3.30 illustrates the theorem.

3.3 Embedding and NP-Completeness of Parameters on Set-Labelled Graphs

Definition 3.3.1. [Ach83] A graph $G = (V, E)$ is said to be set-graceful if there exists a set $X$ and a set-indexer $f : V(G) \to 2^X$ such
that $f^\oplus(E(G)) = 2^X - \{\emptyset\}$, where $\oplus$ denotes the symmetric difference of sets.

**Definition 3.3.2.** [Ach83] A graph $G$ is said to be *set-sequential* if there exists a nonempty set $X$ and a bijective set-valued function $f : V(G) \cup E(G) \to 2^X - \{\emptyset\}$ such that $f^\oplus(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$, where $\oplus$ denotes the symmetric difference of sets.

**Definition 3.3.3.** [Ach83] A set-graceful graph $G = (V, E)$ is *topologically set-graceful* with respect to a set $X$ if $f(V(G)) = \{f(v) : v \in V(G)\}$ is a topology on $X$, where $f : G \to 2^X$ is the corresponding set-indexer.

Now we establish an embedding of a graph into a topologically set-graceful graph and study the NP-completeness of the problems of determination of the clique number, the independence number and the chromatic number of a connected topologically set-graceful graph.

**Theorem 3.3.4.** Every finite graph $G$ can be embedded into a connected topologically set-graceful graph $H$.

**Proof.** Let $G$ be a finite graph with $p$ vertices $u_1, u_2, \ldots, u_p$. Label these vertices by $\{1, 2\}, \{1, 3\}, \ldots, \{1, p+1\}$. Let $X = \{1, 2, \ldots, p+1\}$. Let $\tau = \{\text{All subsets of } X \text{ containing } 1\} \cup \{\emptyset\}$. Then, $\tau$ is a topology on $X$ which contains $2^p + 1$ subsets of $X$. Now, add $2^p - p + 1$ new vertices $v_i, 1 \leq i \leq 2^p - p + 1$, and label $v_1$ by $\emptyset$, $v_2$ by $\{1\}$ and $v_i, i \geq 3$ by elements of $\tau$ with cardinality $\geq 3$, in an arbitrary but injective manner. Join $v_1$ to all other $2^p$ vertices. Join $v_2$ to every
\[ u_i, \ 1 \leq i \leq p \] and to all \( v_j \) which are labelled by subsets of cardinality > 3. If \( u_i \) and \( u_j \) are not adjacent in \( G \) join \( v_2 \) to the vertex with label \( \{1, i + 1, j + 1\} \). The new graph \( H \) so obtained is topologically set-graceful and \( G \) is an induced subgraph of \( H \) as shown below. If \( \{i_1, i_2, \ldots, i_k\}, \ i_1 < i_2 < \cdots < i_k, \) with \( k \geq 3 \), is a nonempty subset of \( \{1, 2, \ldots, p + 1\} \) then,

- for \( i_1 = 1 \), the edge \( v_1w \) where the label of \( w \) is \( \{i_1, i_2, \ldots, i_k\} \) has the label \( \{i_1, i_2, \ldots, i_k\} \);
- for \( i_1 > 1 \), the edge \( v_2w^* \) where the label of \( w^* \) is \( \{1, i_1, i_2, \ldots, i_k\} \) has the label \( \{i_1, i_2, \ldots, i_k\} \);
- for \( k = 2 \) and \( i_1 = 1 \), the edge \( v_1u_{i_{k-1}} \) has the label \( \{i_1, i_k\} \);
- for \( k = 2 \) and \( i_1 > 1 \), if \( u_{i_1+1}u_{i_k+1} \) is an edge of \( G \), its label is \( \{i_1, i_k\} \);
- if \( u_{i_1+1}u_{i_k+1} \) is not an edge of \( G \), then the label of \( v_2w^{**} \), where the label of \( w^{**} \) is \( \{1, i_1, i_2\} \), is \( \{i_1, i_2\} \).

- for \( k = 1 \), then the labels of the edges \( v_1v_2, v_2u_i, \ 1 \leq i \leq p \) are respectively \( \{1\}, \{2\}, \ldots, \{p + 1\} \). Hence the theorem. ♣

**Corollary 3.3.5.** If the chromatic number, the clique number and the independence number of \( G \) are \( \geq 3 \), then the chromatic number \( \chi(H) = \chi(G) + 2 \), the clique number \( \omega(H) = \omega(G) + 2 \) and the independence number \( \beta_0(H) = \beta_0(G) + 2p - p - 1 \). Therefore, the problems of determining the chromatic number, the clique number and the independence number of a connected topologically set graceful graph are NP-complete.
The figure 3.31 illustrates the theorem.

As a topologically set-graceful graph is set-graceful, we have

**Corollary 3.3.6.** Every graph can be embedded into a connected set-graceful graph and the problems of determining the clique number, the independence number and the chromatic number of a set-graceful graph are NP-complete.
Hegde [Heg] proved embedding theorems on connected set-graceful graphs and also on connected set-sequential graphs. But the following embedding regarding set-sequential graph gives the NP-completeness of determining the clique number and the chromatic number of a connected set-sequential graph.

**Theorem 3.3.7.** *Every* \((p, q)\)-graph of order \(p \geq 5\) *can be embedded into a connected set-sequential graph.*

**Proof.** Let \(G\) be a graph with \(V(G) = \{u_1, u_2, \ldots, u_p\}\). Label the vertices of \(G\) by the sets \(\{1, 2\}, \{1, 3\}, \ldots, \{1, p+1\}\). Then, the vertex and edge labels of \(G\) are all distinct. Let \(X = \{1, 2, \ldots, p+1\}\). Define the families of sets \(S_k, 3 \leq k \leq p+1\) as follows.

\(S_3 = \{\text{subsets of } X \text{ of the form } \{1, i, j\} \text{ whenever } u_{i-1} \text{ and } u_{j-1} \text{ are not adjacent}\}\).

\(S_4 = \{\text{subsets of } X \text{ of cardinality 4 containing 1}\}\).

\(S_5 = \{\text{subsets of } X \text{ of cardinality 5}\}\).

\(S_k = \{\text{subsets of } X \text{ of cardinality } k, \text{ which does not contain the element } 1\}, k \geq 6\).

\[1 + \sum_{k=3}^{p} |S_k| = 1 + pC_2 + pC_3 + (p+1)C_5 + pC_6 + \cdots + pC_p - q = 2^p - p - q.\]

Now, introduce \(2^p - p - q\) new vertices \(v_i\).
Fix one vertex $v_1$ and label it by $\{1\}$ and the other vertices by the elements of $S_k$, $3 \leq k \leq p$, in an injective manner. Join $v_1$ to all the other vertices. If $u_{i-1}u_{j-1}$ is an edge in $G$ with $i < j$, join the vertex labelled by $\{i, j, j + 1, j + 2, j + 3\}$ to $\{1, j + 1, j + 2, j + 3\}$, $2 \leq i < j \leq p + 1$, with the convention that $p + 2, p + 3, p + 4$ are treated as $2, 3, 4$ respectively. Let the resulting graph be $H$. Here, all edge labels and vertex labels of $H$ are distinct. The new graph $H$ is a connected set-sequential graph with $2^p - q$ vertices, $2^p + q - 1$ edges and $G$ an induced subgraph of $H$. ♣

Note that the induced subgraph of $H$, on vertices other than those of $G$ and $v_1$, is a bipartite graph since the edges in this induced subgraph are only between $S_4$ and $S_5$.

**Corollary 3.3.8.** If the chromatic number and the clique number of $G$ are $\geq 3$, then the chromatic number $\chi(H) = \chi(G) + 1$ and the clique number $\omega(H) = \omega(G) + 1$. Therefore, the problems of determining the chromatic number and the clique number of a connected set-sequential graph are NP-complete.
The figure 3.32 illustrates the theorem for \((5, 4)\)-graph.

Figure 3.32:
References


