Chapter 4

Infinite Family of Graphs with a Given Property

Since, by their very definitions, set-graceful, topologically set-graceful and set-sequential graphs have exponential orders or sizes. It is not hard to see that most graphs do not fall under any of these classes of graphs. *Even within the classes of graphs satisfying the order or size conditions for a graph to be in any of these classes, we surmise that similar conclusion holds.* Hence, it becomes important to have infinite families of such graphs towards gaining deeper insight into the properties of these very special graphs. In this chapter, given an arbitrary graph $G$, we describe ways to generate infinite ascending chains of set-graceful graphs, topologically set-graceful graphs and set-sequential graphs with $G$ as an *initializing graph* and such that at each stage of construction the preceding graph is an induced subgraph of the succeeding ‘host’ graph. Also for a given integer $t \geq 1$, we construct an embedded family of cordial graphs with minimum degree greater than or equal to $t$.

4.1 Introduction

This chapter deals with construction of infinite family of graphs with a given property.

Definition 4.1.1. [Cah87] Let $f$ be a function from the vertices of
G to the set \{0, 1\} and for each edge \(xy\) assign the label \(|f(x) - f(y)|\). Then \(f\) is a cordial labelling of \(G\) and \(G\) is a cordial graph if the number of vertices labelled 0 and the number of vertices labelled 1 differ by at most 1, as also the number of edges labelled 0 and the number of edges labelled 1 differ at most by 1.

**Definition 4.1.2.** [Ach83] A graph \(G = (V, E)\) is said to be set-graceful if there exists a set \(X\) and a set-indexer \(f : V(G) \rightarrow 2^X\) such that \(f^\oplus(E(G)) = 2^X - \{\emptyset\}\), where the induced edge function \(f^\oplus\) is defined as the symmetric difference of sets.

**Definition 4.1.3.** [Ach83] A graph \(G\) is said to be set-sequential if there exists a non-empty set \(X\) and a bijective set-valued function \(f : V(G) \cup E(G) \rightarrow 2^X - \{\emptyset\}\) such that \(f^\oplus(uv) = f(u) \oplus f(v)\) for every \(uv \in E(G)\), where \(\oplus\) denotes the symmetric difference of sets.

**Definition 4.1.4.** [Ach83] A set-graceful graph \(G = (V, E)\) is topologically set-graceful with respect to a set \(X\) if \(f(V(G))\) is a topology on \(X\) where \(f : V(G) \rightarrow 2^X\) is the corresponding set-indexer.

### 4.2 Construction of Infinitely Many Property Loaded Families of Graphs.

By theorem 3.2.13, every graph can be embedded into a connected cordial graph. The following theorem gives an infinite family of connected cordial graphs with the given graph is an induced subgraph. In this embedding, there are vertices of small degrees in the embedded
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graph. The following theorem proves the existence of a connected cordial graph with minimum degree greater than a pre-assigned number.

4.2.1 Cordial Graphs

**Theorem 4.2.1.** For any graph $G$ and for any $\theta$, an arbitrary integer, there exists a connected cordial graph $C$ with minimum degree $\delta(C) > \theta$.

**Proof.** Let $G$ be a graph. By Theorem 3.2.13, $G$ can be embedded into a connected cordial graph $H$. We can obtain connected cordial graphs $H_k, k \geq 1$ as follows.

Put $H_1 = H + K_1$, where $V(K_1) = \{v_1\}$, which was labelled by 0; $H_2 = H_1 + K_1$, where $V(K_1) = \{v_2\}$, which was labelled by 1; $H_3' = H_2 + K_1$, where $V(K_1) = \{v_3\}$, which was labelled by 0. Now, obtain $H_3$ by removing the edge between $v_2$ and $v_3$. Let $H_4 = H_3 + K_1$, where $V(K_1) = \{v_4\}$, which was labelled by 1; $H_5' = H_4 + K_1$, where $V(K_1) = \{v_5\}$, which was labelled by 0. Now, to obtain $H_5$ remove the edge between $v_4$ and $v_5$. Proceeding like this, we get connected cordial graphs $H_k, k \geq 1$ having minimum degree 1 more than the graph $H_{k-1}$. Choose an integer $l$ so that $H_l = C$ having minimum degree $> \theta$. Hence the theorem. ♣
The figure 4.1 and 4.2 illustrates the theorem for a \((5, 7)\)-graph.

Figure 4.1:
4.2.2 Set-Graceful Graphs

Every set-graceful \((p, q)\)-graph \(G = (V, E)\) with respect to a set of cardinality \(n\) can be embedded in a set-graceful \((q + 1, q)\)-graph \(H\), where by an embedding of \(G\) means identifying an induced subgraph in \(H\) that is isomorphic to \(G\). Such a ‘host’ graph \(H\) of \(G\) together with its set-graceful labelling is considered as fully augmented. Fully augmented set-graceful graph of a set-graceful \((p, q)\)-graph \(G\) can be obtained by adding \(2^n - p\) isolated vertices with labels as those subsets.
of $X$ that are not present as vertex labels in the set-graceful labelling
given on $G$. The following results will be the means by which we shall
devise a procedure to obtain ascending chains of set-graceful graphs of
the required type with the arbitrarily given graph $G$ as the ‘initializing
graph’.

The corollary 3.3.6 on Chapter 3 give an embedding of a graph in
a connected set-graceful graph which is as follows.

**Theorem 4.2.2.** *Every graph can be embedded in a connected set-
graceful graph.*

The theorem 2.2.3 on Chapter 2 give a necessary and sufficient
condition for the graph $H + \overline{K}_m$ to be set-graceful, where $H$ is a set-
graceful graph with $n$ edges ($n \geq 1$) and $n + 1$ vertices which is given
below.

**Theorem 4.2.3.** *If $H$ is a set-graceful graph with $n$ edges ($n \geq 1$)
and $n + 1$ vertices then the join of $H + \overline{K}_m$ is set-graceful if and only
if $m = 2^{n_1} - 1$, $n_1 \in N$, where $N$ is the set of natural numbers.*

Let $H$ be a fully augmented set-graceful graph, with $G$ as an in-
duced subgraph. Then by Theorem 4.2.2, $H + \overline{K}_n$ is set-graceful for
all $n = 2^x - 1$, $x \in N$. It is interesting to see that such set-graceful
graphs could be used to generate an infinite ascending chain of fully
augmented set-graceful graphs each of which contains its predecessor
as an induced subgraph, which leads to the following theorem.

**Theorem 4.2.4.** *Given any set-graceful fully augmented graph $H$, there exists an infinite ascending chain $\mathbb{H} := (H_1 \subset H_2 \subset \ldots)$ of set-
graceful graphs $H_1, H_2, \ldots$ such that $H_i$ is an induced subgraph of $H_{i+1}$ for every non-negative integer $i$ and $H_i$ is a fully augmented graph.

**Theorem 4.2.5.** Let $G$ be a set-graceful $(p, q)$-graph and $H_1$ be its full augmentation, then there exists an infinite ascending chain $H := (H_1 \subset H_2 \subset \ldots)$ of set-graceful graphs such that $H_i$ is an induced subgraph of $H_{i+1}$ where $H_i$ is the full augmentation of $K_1 + H_{i-1}, i \geq 2$.

**Proof.** Let $G$ be $(p, q)$-graph and $H_1$ be its full augmentation, with respect to a set $X = \{1, 2, \ldots, n\}$. Then the graph $K_1 + H$ is set-graceful if we assign the set $X_1 = \{1, 2, \ldots, n, n+1\}$ to the vertex of $K_1$. Let the full augmentation of $K_1 + H$ be $H_1$. Again, the graph $K_1 + H_1$ is set-graceful by assigning the set $X_2 = \{1, 2, \ldots, n, n+1, n+2\}$ to the new vertex adjoined to $H_1$. Let the full augmentation of $K_1 + H_1$ be denoted by $H_2$. We may continue this procedure indefinitely invoking the following procedure: For any integer $i \geq 2$, define $H_i$ to be a full augmentation of the set-graceful graph $K_1 + H_{i-1}$ in which the vertex of $K_1$ is assigned the set $X_i = \{1, 2, \ldots, n, n+1, n+2, \ldots, n+i\}$. It is obvious from the construction that the sequence $H_1, H_2, \ldots$ is an infinite ascending chain of set-graceful graphs. ♣
Figure 4.3 illustrates the construction described in the proof of Theorem 4.2.4.

Given a connected set-graceful graph $G$, we have given two methods of constructing an infinite sequence $\mathcal{G} := (G = G_1, G_2, \ldots)$ of connected set-graceful graphs $G_i$ such that $G_i$ is contained in $G_{i+1}$ as an induced subgraph for $i = 1, 2, \ldots$. This is evident from the fact that, in view of Theorem 4.2.2, even if $G$ is not set-graceful, we can obtain an infinite sequence $\mathcal{H} := (H = G_1, G_2, \ldots)$ of set-graceful graphs $G_i$, where $H$ is the set-graceful host of $G$ and $G_i$ contains $G_{i-1}$ as an induced subgraph for each $i, \; i = 1, 2, \ldots$.

### 4.2.3 Topologically Set-Graceful Graphs

A set-graceful labelling of $G$ that is a topological set-indexer too is called a *topological set-graceful labelling* of $G$ and $G$ a topologically
set-graceful graph. The theorem 3.3.4 give an embedding of a graph in a connected topologically set-graceful graph which is as follows..

Theorem 4.2.6. Every graph can be embedded in a connected topologically set-graceful graph.

Theorem 4.2.7. Let $G$ be a graph. Then there exists an infinite sequence $(H = G_1, G_2, \ldots)$ of connected topologically set-graceful graphs $G_i$ where $H$ contains $G$ as an induced subgraph and $G_i$ contains $G_{i-1}$ as an induced subgraph for all integers $i \geq 2$.

Proof. Let $G$ be a graph of finite order, say $n$ with $V(G) = \{u_1, u_2, \ldots, u_n\}$. Let $X_1 = \{1, 2, \ldots, n + 1\}$. Define $S$ as the set of all subsets of $X$, containing the element 1 with cardinality $\geq 3$. Then, $|S| = nC_2 + nC_3 + \cdots + nC_n = 2^n - n - 1$. Label each $u_i$ by $\{1, i + 1\}$. Now, introduce $|S| + 2$ new vertices $\{v_1, v_2, \ldots, v_{|S|+2}\}$, label $v_1$ by $\emptyset$, $v_2$ by $\{1\}$ and $v_j, j \geq 3$ by members of $S$ in an injective manner. Join $v_1$ to all $u_i$’s and $v_j$’s, and $v_2$ to all $u_i$’s with vertices of cardinality $> 3$. If $u_i$ is not adjacent to $u_j$ then, join $v_2$ to the vertex labelled by $\{1, i + 1, j + 1\}$. Then, the new graph $G_1$ is topologically set-graceful and $G$ is an induced subgraph of $G_1$. To construct $G_2$, introduce a new vertex $w_2$, label it by $\{1, n + 2\}$, join it to all other vertices. Now, add $2^n - 1$ new vertices, label them by subsets of $X_2 = \{1, 2, \ldots, n + 2\}$ that contain the elements 1 and $n + 2$ and join them to $v_1$. Then, $G_2$ is topologically set-graceful with $G$ as an induced subgraph.

In general, to construct $G_j$ from $G_{j-1}$ introduce $w_j$, label it by
\{1, n + j\} and join it to all other vertices. Now, add \(2^{n+j-2} - 1\) new vertices, label them by the subsets of \(X_j = \{1, 2, \ldots, n + j\}\) of cardinality > 2 which contain the elements 1 and \(n + j\), and join them to \(v_1\). Then, \(H = (G_1, G_2, \ldots)\) is an infinite sequence of topologically set-graceful graphs that are in ascending orders and \(H\) contains \(G\) as an induced subgraph.

Figure 4.4 illustrates the construction described in the of above theorem.

Figure 4.4:
4.2.4 Set-Sequential Graphs

An immediate observation from the very definition is that a necessary condition for a \((p, q)\)-graph \(G = (V, E)\) to admit a set-sequential labelling is \(p + q + 1 = 2^m\) for some positive integer \(m\). By theorem 3.3.7 every graph can be embedded into a set-sequential graph. The following result describes a method of constructing an ascending chain of set-sequential graphs for an arbitrarily given graph \(G\) containing it as an induced subgraph.

**Theorem 4.2.8.** Let \(G\) be any graph. Then, there exists an infinite sequence \(H := (H = G_1, G_2, \ldots)\) of set-sequential graphs where 
\(H\) contains \(G\) as an induced subgraph and \(G_i\) contains \(G_{i-1}\) as an induced subgraph for all integers \(i \geq 2\).

**Proof.** Let \(G\) be a finite graph of order \(n\) with \(V(G) = \{u_1, u_2,\ldots, u_n\}\). Label \(u_i\) by \(\{1, i + 1\}\). Let \(S\) be the set of all subsets of \(X = \{1, 2, \ldots, n + 1\}\) which contains the element 1 and of cardinality \(\geq 3\). Then, \(|S| = 2^n - n - 1\). Now, introduce \(1 + |S|\) new vertices, say \(\{v_1, v_2, \ldots, v_{1+|S|}\}\). Label \(v_1\) by \(\{1\}\) and all other \(v_i\) by elements of \(S\) in an injective manner.
Join $v_1$ to all $u_i, 1 \leq i \leq n$ and to all vertices which were labelled by distinct sets of cardinality $\geq 4$. If $u_i$ is not adjacent to $u_j$ then join $v_1$ to $\{1, i+1, j+1\}$. The new graph $G_1$ is a set-sequential graph, with $G$ as an induced subgraph. Construct $G_2$ by introducing $2^1$ new vertices with labels $\{n+2\}, \{1, n+2\}$ and join them to all vertices $G_1$ except $v_1$.

To obtain $G_3$, introduce $2^2$ newer vertices to $G_2$, label them by subsets of $\{1, n+2, n+3\}$ that contain the element $n + 3$, and join them to all vertices $G_2$ except $v_1$.

In general, $G_j$ can be constructed from $G_{j-1}$ by introducing $2^{j-1}$ new vertices, label them by subsets of $Y = \{1, n+2, \ldots, n+j\}$ which contains the element $n + j$. Join them to all vertices of $G_{i-1}$ except $v_1$. Then, $(H = G_1, G_2, \ldots)$ is an infinite sequence of set-sequential graphs in ascending order where $H$ contains $G$ as an induced subgraph.
Figure 4.5 illustrate the construction described in the proof of Theorem 4.2.8 at the first stage.
Figure 4.6 illustrate the construction described in the proof of Theorem 4.2.8 at the second stage.


References


