Chapter 5

Distance Compatible Sets and Hypergraphs

This chapter contains some foundational definitions and results on hypergraphs. We extend the concept of 1-uniform dcsl graphs to hypergraphs. We define a hypergraph corresponding to a dcsl $f$ of a graph $G$, denoted by $H^f_G$, and study some of the characteristics of these hypergraphs. We find that the hypergraph corresponding to 1-uniform dcsl paths and stars satisfies the coloured-edge property. We also prove that the hypergraph and its dual are isomorphic. We establish that the stability number of a hypergraph corresponding to a $k$-uniform dcsl-labeling is the order of $G$ if and only if $k = 2$ and $G \cong K_n$.

5.1 Introduction

To begin with, Hypergraph theory is something different and much more generalized concept of graph theory. Given a set $V$ of vertices, an edge of a simple graph on $V$ is a set of two vertices, while an edge of a hypergraph on $V$ is any subset of $V$. The theory of hypergraphs
popularized and enriched by many contributions of Berge [5], [6], is the extension of theorems about graphs to hypergraphs. The problem is to find a suitable formulation of the theorems for hypergraphs in such a way that they contain the graph as a special case.

Precisely, C. Berge [5], defined a hypergraph on a finite set $X = \{x_1, x_2, x_3, \ldots, x_n\}$ as a family $H = (E_1, E_2, E_3, \ldots, E_m)$ of subsets of $X$ such that

$$E_i \neq \emptyset, \quad (1 \leq i \leq m)$$

$$\bigcup_{i=1}^{m} E_i = X$$

The elements $x_1, x_2, x_3, \ldots, x_n$ of $X$ are called vertices, and the sets $E_1, E_2, E_3, \ldots, E_m$ are the edges of the hypergraph. The order of $H$, denoted by $n(H)$, is the number of vertices and the number of edges is denoted by $m(H)$.

**Definition 5.1.1.** [5] A hypergraph $H = (X; E_1, E_2, \ldots, E_m)$ is said to be simple, if the edges $E_i$ are all distinct.

**Definition 5.1.2.** [5] Given a hypergraph $H = (X, \mathcal{E})$ and nonempty subset $S \subseteq X$, the hypergraph $H_S = (S, \{E \in \mathcal{E} : E \subseteq S\})$ is called the subhypergraph induced by $S$ in $H$. 
Definition 5.1.3. [5] The dual of a hypergraph $H = (E_1, E_2, \ldots, E_m)$ on $X$ is a hypergraph $H^* = (X_1, X_2, \ldots, X_n)$ whose vertices corresponds to the edges of $H$, and with the edges $X_i = \{ e_j : x_i \in E_j \in H \}$.

Definition 5.1.4. [5] Given an integer $k > 0$, the $k$-section of hypergraph $H$ is defined to be the couple $H_{(k)} = (X, \mathcal{E}_{(k)})$ formed by the set $\mathcal{E}_{(k)} = \{ F : F \subset X : 1 \leq |F| \leq k, F \subset E_i \text{ for some } E_i \in \mathcal{E} \}$. The $2$-section $H_{(2)}$ of $H$ is a symmetric graph with the same vertices as $H$ and with a loop attached to each vertex and $[H]_2$ denotes this two section without loops.

Definition 5.1.5. [5] A simple hypergraph $H = (E_1, E_2, \ldots, E_m)$ has the Helly property if every intersecting family of $H$ is a star. That is, for $J \subset \{1, 2, 3, \ldots, m\}$

$$E_j \cap E_k \neq \emptyset; \quad j, k \in J$$

(5.1.3)
implies

$$\bigcap_{j \in J} E_j \neq \emptyset.$$  

(5.1.4)

Definition 5.1.6. [5] The chromatic index of a hypergraph $H$ is the least number of colours necessary to colour the edges of $H$ such that two intersecting edges are always coloured differently. The chromatic index of hypergraph $H$ is denoted by $q(H)$. 
Definition 5.1.7. [6] Degree $\triangle(H)$ of a hypergraph $H$ is defined as

$$\triangle(H) = \max \{d_H(x), x \in X\},$$

where the degree $d_H(x)$ of $x$ which is the number of edges of $H(x)$, where $H(x)$ is the star with center $x$ formed by the edges containing $x$.

A Hypergraph $H$ has the coloured edge property if $q(H) = \triangle(H)$.

Definition 5.1.8. [6] A hypergraph $H = (X, \mathcal{E})$ is said to be hereditary if $A \in \mathcal{E}$, $B \subset A$ and $B \neq \emptyset \Rightarrow B \in \mathcal{E}$.

5.2 Distance compatible set-labeled graphs and hypergraphs

In this section, we try to apply the properties of distance compatible set-labeled graphs (dcsl-graphs) to hypergraphs. Let $(G, f)$ be a 1-uniform dcsl-graph, such that $f(u) \neq \emptyset$ for all $v \in V(G)$. We can construct a hypergraph, $H^f_G$, using the labels of the dcsl-graph, which are subsets of the corresponding dcsl-set. Optimal hypergraphs $H^f_G$ are those hypergraphs, for which $f$ is minimal, in the sense that the dcsl-set with respect to which the 1-uniform dcsl $f$ is minimal.

Remark 5.2.1. It is interesting to note that, using the rows of the $(0,1)$–matrix of the hypergraph $H^f_G$, where $f$ is a 1-uniform dcsl, we
can embed \( G \) into the hypercube \( Q_n \), where \( n \) is the cardinality of the dcsl-set.

**Definition 5.2.1.** \[6\] A transversal of a hypergraph \( H = (X; E_1, E_2, \ldots, E_m) \) is defined to be a set \( T \subset X \) such that \( T \cap E_i \neq \emptyset \), \( (i = 1, 2, \ldots, m) \). The transversal number is defined to be the minimum number of vertices in a transversal. The transversal number of a hypergraph \( H \) is denoted by \( \tau(H) = \min |T| \).

**Definition 5.2.2.** \[10\] Given a hypergraph \( H = (X, \mathcal{E}) \), a set \( S \subset X \) is defined to be stable if it contains no edge \( E_i \) with \( |E_i| > 1 \). The stability number of \( H \) is defined to be the maximum cardinality of a stable set of \( H \).

In the second chapter, we have proved that complete graph \( K_n \) is dispersible. Here we are considering the corresponding hypergraph \( H_{f K_n}^f \) of the dispersible dcsl-labeling \( f \). Recall that we defined the dispersible function \( f : V(K_n) \to 2^X \) where, \( X = \{1, 2, \ldots, 2^{n-1}\} \). We have for each \( v_i \in V(K_n) \), \( f(v_i) = \{1, 2, \ldots, 2^{i-1}\} \). Thus, order of \( H_{f K_n}^f \) is \( 2^{n-1} \) and \( E_i = f(v_i) = \{1, 2, \ldots, 2^{i-1}\} \). Such a graph is being shown in Figure 5.1

Now consider \( E_1 = \{1\} \). We have \( \{1\} \cap E_i \neq \emptyset \). Therefore \( \{1\} \)
Figure 5.1: The Hypergraph representation of dispersible dcsl-$K_5$

is a transversal and, it is a minimal transversal. Note that no other singleton is a transversal of $H$. Thus, the transversal number $\tau(H)$ of this hypergraph $H$ is equal to one. The stability number of this graph is also equal to one. Following is an open problem which may be dealt later.

**Problem 10.** Find the transversal number and stability number of an optimal hypergraph $H_G^{f}$.

Consider the 1-uniform dcsl of path $P_6$ as follows. Let $v_1, v_2, \ldots, v_6$ be the vertices of $P_6$. Assign the vertices of $P_6$ by the subsets $\{1\}, \{1, 2\}, \{2\}, \{2, 3\}, \{2, 3, 4\}$ and $\{2, 3, 4, 5\}$ of a dcsl-set $X = \{1, 2, 3, 4, 5\}$ taken in that order. Figure 5.2 illustrates the construction of the hypergraph
using the dcsl-labeling of path $P_6$.

![1-uniform dcsl Path $P_6$ and its corresponding hypergraph](image)

Figure 5.2: 1-uniform dcsl Path $P_6$ and its corresponding hypergraph

Also the $(0,1)$— matrix of the hypergraph illustrated in Figure 5.2 is given below.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$
Remark 5.2.2. Note that chromatic index of the hypergraph given in Figure 5.2 is four. Moreover, the chromatic index \( q(H) \) is equal to the degree, \( \Delta(H) \) of \( H \). Thus, the hypergraph \( H \) corresponding to the dcs1 path \( P_6 \) has the coloured edge property.

Figure 5.3 depicts the 1-uniform dcs1-labeling of star \( K_{1,7} \) and its corresponding hypergraph.

(0, 1)-matrix of this dcs1-labeling of star \( K_{1,7} \) is given below.
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}

Remark 5.2.3. Chromatic index of the hypergraph $H$ of dcsl-graph $K_{1,7}$ is eight, equal to the degree of $H$. Thus, we strongly believe that the hypergraphs of dcsl-graphs are graphs which satisfy the coloured edge property.

In general, a hypergraph and its dual hypergraph need not be isomorphic. But it happens in the case of 1-uniform dcsl-graphs. It is interesting to note that, if $(G, f)$ is a 1-uniform dcsl-graph then, the hypergraph $H^f_G$ and the dual hypergraph of $H^f_G$ are isomorphic. Figure 5.4 give the hypergraph corresponding to 1-uniform dcsl path $P_6$ and Figure 5.5 depicts its dual graph. Note that the hypergraphs in Figure
Figure 5.4: Hypergraph corresponding to 1-uniform dcs1 path $P_6$.

Figure 5.5: Dual graph of a hypergraph whose initial graph is 1-uniform dcs1 path $P_6$. 
and the Figure 5.5 are isomorphic.

**Remark 5.2.4.** Consider the 2-uniform dcsl-graph $K_n$ with the 2-uniform dcsl-set, $X = \{1, 2, \ldots, n\}$. In the corresponding hypergraph, the whole set is the only stable set. It is interesting to note that the transversal number of this hypergraph is also $n$.

**Theorem 5.2.1.** The stability number of a hypergraph corresponding to a $k$-uniform dcsl-labeling is the order of $G$ if and only if $k = 2$ and $G \cong K_n$.

*Proof.* Suppose $k = 2$ and $G \cong K_n$. Then, by Remark 5.2.4, stability number of the hypergraph corresponding to the 2-uniform dcsl-labeling is $n$, order of $G$.

Conversely, let $G$ be a graph such that the hypergraph corresponding to the $k$-uniform dcsl-labeling $f$ is $n$, order of $G$. Then, since $E_i \subset X$ for all $i$, we have $|E_i| = 1$, for all $i$. Also each $E_i = f(v_i)$ for some $v_i \in V(G)$.

**Case 1:** $k = 1$.

If $k = 1$ then, $|f(v_i)| > 1$ for atleast one $i$, which implies $|E_i| > 1$ for atleast one $i$. This is a contradiction to the assumption that stability number of the hypergraph corresponding to the $k$-uniform dcsl-labeling
is $n$, the order of $G$.

**Case 2:** $k > 2$.

$G$ is $k$-uniform, $k > 2$ implies, there exists atleast one vertex $v_i$ with $|f(v_i)| \geq 2$. That is, every stable set of the hypergraph corresponding to the $k$-uniform dcs-l-labeling $f$ is a proper subset of $X$, which is again a contradiction to the assumption.

**Case 3:** $k = 2$ and $G \not\cong K_n$.

Since $G$ is not complete, there exist atleast one pair of vertices, say, $(v_i, v_j)$ of $G$ such that $v_i$ and $v_j$ are non-adjacent in $G$. Then, $d(v_i, v_j) \geq 2$. Since the graph is 2-uniform, $|f(v_i) \oplus f(v_j)| \geq 4$. But this is possible only if $|f(v_i)| \geq 2$ or $|f(v_j)| \geq 2$. That is, $|E_i| \geq 2$ or $|E_j| \geq 2$. Thus, we reach a contradiction to the assumption that the stability number is $n$.

Thus, the stability number of a hypergraph corresponding to a $k$-uniform dcs-l-labeling is the order of $G$ if and only if $k = 2$ and $G \cong K_n$. 

**Definition 5.2.3.** [5] Let $H = (X; E_1, E_2, \ldots, E_n)$ be a hypergraph with $n$ edges. The representation graph of $H$ is defined to be the simple graph $L(H)$ of order $n$, whose vertices $e_1, e_2, \ldots, e_n$ respectively represent the edges $E_1, E_2, \ldots, E_n$ of $H$ and with vertices $e_i$ and $e_j$ joined by an edge
if and only if, $E_i \cap E_j \neq \emptyset$. For each simple graph $G$, there exists a hypergraph $H$ such that $G = L(H)$. The minimum order of the hypergraphs for which $G$ is a representative graph is denoted by $\Omega(G)$.

**Definition 5.2.4.** Let $G$ be a simple graph with vertices $x_1, x_2, \ldots, x_n$, none of which is isolated. Let $\overline{G}$ be a graph, each of whose vertices represents an edge of $G$, with two vertices corresponding to edges $(a, b)$ and $(x, y)$ of $G$ joined together if and only if, $\{a, b, x, y\}$ is not a clique in $G$. The minimum order $\Omega(G)$ of the hypergraphs for which $G$ is a representative graph equals the chromatic number $\gamma(\overline{G})$.

**Theorem 5.2.2.** Let $G$ be a simple graph without triangles and with $m$ edges. The minimum order of the hypergraphs $H$ for which $G = L(H)$ is $\Omega(G) = m$.

**Corollary 5.2.3.** If $G$ is a 1-uniform dcsl-graph of size $q$ then, $\Omega(G) = q$ where, $\Omega(G)$ is the minimum order of the hypergraphs $H$ for which $G = L(H)$.

**Proof.** Let $G$ be a 1-uniform dcsl-graph of size $q$. Then, by Proposition 2.5.3, $G$ is triangle free. Thus the proof follows from Theorem 5.2.2.

**Remark 5.2.5.** Let $(G, f)$ be a dcsl-graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. If $\bigcap f(v_i) \neq \emptyset$, $1 \leq i \leq n$, then the representative graph $G' = L(H_G^f)$
is a complete graph of order $n$.

Following are the problems identified for further investigation.

**Problem 11.** Characterize $dcs$-graph $G$ such that $G \cong L(H_G^f)$.

**Problem 12.** Characterize $k$-uniform $dcs$ hypergraph.

**Problem 13.** Characterize $(k, r)$-arithmetic $dcs$ hypergraphs.

**Problem 14.** Characterize $(0, 1)$- matrix of $1$-uniform $dcs$ hypergraphs.

**Problem 15.** Characterize hypergraphs of $dcs$ graphs satisfy the coloured edge property.

**Problem 16.** Characterize hypergraph representation of dispersible-$dcs$ graphs, $(k, r)$-arithmetic $dcs$-graphs, $k$-uniform $dcs$-graphs and study their various properties.
5.3 References


7. M. Buratti, G. Burosch and P.V. Ceccherint, *A characterization of*
hypergraphs which are products of a finite number of edges, Rendiconti di Matematica, Serie VII Volume 17, Roma (1997), 373 – 385.


