Chapter 5

Perfect square sum and strongly square sum graphs

A \((p, q)\)-graph \(G = (V, E)\) is said to be a perfect square sum graph if there exists a bijection \(f : V(G) \rightarrow \{0, 1, \ldots, p - 1\}\) such that \(f^*(uv) = [f(u)]^2 + [f(v)]^2\) for every \(uv \in E(G)\) are all distinct and \(f^*(E(G)) = \{n^2 : 1 \leq n \leq q\}\) and \(f\) is called a perfect square sum labelling. The graph \(G\) the is said to be a strongly square sum graph if the bijection \(f : V(G) \rightarrow \{0, 1, \ldots, p - 1\}\) such that \(f^*(E(G))\) consists the first \(q\) consecutive numbers of the form \(a^2 + b^2, a \leq p - 1, b \leq p - 1, a \neq b\) and the corresponding labelling is said to be a strongly square sum labelling.

5.1 Introduction

We know that it is possible to write a square as a sum of two squares and set of three integers \(x, y, z\) satisfying the equation \(x^2 + y^2 = n\) is called Pythagorian triple [Tel96]. The Pythagorian equation \(x^2 + y^2 = z^2\) was completely solved in ancient times. This chapter deals with graphs whose edge labelings are perfect squares. Such graphs are called perfect square sum graphs. Also we give another square sum
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graph which is called strongly square sum graphs.

**Definition 5.1.1.** A \((p, q)\)-graph \(G = (V, E)\) is said to be a perfect square sum graph if there exists a bijection \(f : V(G) \rightarrow \{0, 1, ..., p-1\}\) such that \(f^*(uv) = [f(u)]^2 + [f(v)]^2\) for every \(uv \in E(G)\) are all distinct and \(f^*(E(G)) = \{n^2 : 1 \leq n \leq q\}\) and \(f\) is called a perfect square sum labelling.

Figure 5.2 gives two graphs which are perfect square sum and figure ?? gives two graphs which are not perfect square sum respectively.

![Figure 5.1](image1)

![Figure 5.2](image2)

**Definition 5.1.2.** A \((p, q)\)-graph \(G = (V, E)\) is said to be a strongly square sum graph if there exists a bijection \(f : V(G) \rightarrow \{0, 1, ..., p-1\}\) such that \(f^*(uv) = [f(u)]^2 + [f(v)]^2\) for every \(uv \in E(G)\) are all distinct.
distinct and $f^*(E(G))$ consists the first $q$ consecutive numbers of the form $a^2 + b^2$, $a \leq p - 1, b \leq p - 1, a \neq b$, then $f$ is said to be a strongly square sum labeling.

Figure 5.3 gives two graphs which are strongly square sum and figure 5.4 gives two graphs which are not strongly square sum respectively.
5.2 perfect square sum graphs

Theorem 5.2.1.  Stars are perfect square sum.

Proof. For a star $K_{1,p-1}$, there exists a perfect square sum labeling which assigns 0 to the central vertex and the numbers $1, 2, ..., p-1$ to the vertices of unit degree.

Observation 5.2.2.  Complete graph $K_n$ is perfect square sum if and only if $n \leq 2$

Theorem 5.2.3.  Every connected triangle free perfect square sum graph is a star.

Proof. Let $G = (V, E)$ be a connected triangle free $(p, q)$ graph which is not a star having a perfect square sum labeling $f$. Since $f(V(G)) = \{0, 1, 2, ..., p-1\}$ then there exists a vertex $u \in V(G)$ such that $f(u) = 0$. Since $f$ is a perfect square sum labeling $1 \in f^*(E(G))$, so that there exists a vertex $v_1$ in the neighbourhood $N(u)$ of $u$ with $f(v_1) = 1$. Similarly $4 \in f^*(E(G))$ and since $f$ is injective there exists a vertex $v_2 \in N(u)$ such that $f(v_2) = 4$. Let $t^2$ be the largest integer in $f^*(E(G))$ such that $1, 4, 9, ..., (t-1)^2 \in f^*(E_u(G))$. Let $f^*(xy) = t^2$ for some $xy \in E(G)$. Since $t^2 \notin f^*(E_u(G))$, we must have $f(x) \neq 0$ and $f(y) \neq 0$. So that $f(x) \neq t$ and $f(y) \neq t$. Then $t^2 = f^*(xy) = [f(x)]^2 + [f(y)]^2$, where $0 < f(x) < t$ and $0 < f(y) < t$. Hence the vertices $x$ and $y$ should be adjacent to the vertex $u$. It follows that $(u, x, y, u)$ is a triangle in $G$, a contradiction.
Corollary 5.2.4. If $G$ is a perfect square sum graph with a triangle, then any perfect square sum labeling of $G$ must assign 0 to one of the vertex of the triangle in $G$.

Corollary 5.2.5. If a unicyclic graph $G$ is perfect square sum then its unique cycle must be a triangle.

Corollary 5.2.6. The cycles $C_n$ for $n \geq 3$ are not perfect square sum.

Our next attempt is to embed the cycles into a perfect square sum graph. In the case of $C_3$ the embedding is a connected for $C_4$ and $C_5$ the embedding is not connected.

Theorem 5.2.7. The cycle $C_3$ can be embedded as an induced subgraph of a perfect square sum graph.

Proof. Define $V(C_3) = \{v_1, v_2, v_3, v_1\}$. Let $f(v_1) = 0$. Assign $f(v_2) = x$ and $f(v_3) = y$ so that $x, y, x^2 + y^2$ forms a Pythagorian triple. If $x > y$, attach $\sqrt{x^2 + y^2} - 3$ pendant vertices to the vertex $v_1$. Label them by all the consecutive numbers $1, 2, ... p - 1$ less than $\sqrt{x^2 + y^2}$ and not equal to $x$ and $y$. Then it is easy to verify that the induced edge values are $1^2, 2^2, ..., x^2 + y^2$. 

The following figures 5.5 and 5.6 give the embeddings of the cycles $C_4$ and $C_5$ respectively. For the figure 5.5 there is one isolated vertex and that of figure 5.6 there are two isolated vertices.
From the above results we summarize the following.

**Conjecture 5.2.8.** The cycle $C_n, n \geq 3$ can be embedded as an induced subgraph of a perfect square sum graph with $n - 3$ isolated vertices.

**Conjecture 5.2.9.** Every triangle free graph has a perfect square sum labeling.

**Theorem 5.2.10.** If a perfect square sum graph contain more than one triangle, all the triangles should have a common vertex which is
labeled as 0.

Proof. Let us suppose that there exists a triangle \((uvwu)\) in \(G\) having non-zero vertex labeling. Let \(f\) be the perfect square sum labelling of \(G\) and \(a, b, c\) be the three positive integers such that \(f(u) = a, f(v) = b, f(w) = c\). Since \(f\) is a perfect square sum labeling, we have \(a^2 + b^2, b^2 + c^2, a^2 + c^2\) are perfect squares. Let \(a^2 + b^2 = x^2\) and \(a^2 + c^2 = y^2\). Then \(b^2 + c^2 = x^2 + y^2 - 2a^2\). That is \(b^2 + c^2 = (x + y)^2 - 2(a^2 + xy)\). In order to get \(b^2 + c^2\) a perfect square we have either \(a^2 = xy\) or \(a^2 = -xy\) and in both cases \(a^4 = x^2y^2\). But \((a^2 + b^2)(a^2 + c^2) = x^2y^2 = a^4\). That is \(a^4 + b^2(a^2 + c^2) + a^2c^2 = a^4\). By cancelling like terms on both sides and simplifying we get \(b^2y^2 + a^2c^2 = 0\). This is not possible since the sum of two positive numbers never equal to zero. Hence either one of \(a\) or \(b\) or \(c\) must be zero. That is if a perfect square sum graph contain any number of triangle, all the triangles should have a common vertex which is labelled as 0.

**Theorem 5.2.11.** In any connected perfect square sum graph \(G\) there can be atmost three triangles.

Proof. The following figures 5.7 5.8 5.9 gives graphs with one, two and three triangles respectively. In figure 5.9 we should assign the values 1, 2, ..., 799 except the values 80, 224, 351, 720, 768 and 798 to the vertices adjacent to the vertex which is labelled as 0.

Next let us assume that \(G\) be a graph which contains four triangles. Since \(G\) is perfect square sum, all the four triangles should have a common vertex \(u\) such that \(f(u) = 0\). Also \(f^*(E(G)) = \{1^2, 2^2, ..., q^2\} \)
and since $G$ is connected, we have the highest four values of $f^*(E(G))$ are square of four consecutive numbers each of which can be represented as sum of two non-zero integer squares. Let one of such number be $a^2$ and write $a^2 = x^2 + y^2$. But we have the non-zero integer solutions of the above Pythagorian equation are of the form $a = k^2 + l^2, x = 2kl, y = k^2 - l^2$. That implies, the number $a$ can also be represented as the sum of two squares. Since the choice of $a$ is arbitrary we get a conclusion that any four consecutive integers can be represented as the sum of two squares. This is not possible and it contradicts our supposition that $G$ contains four triangles. Hence there can be atmost three triangles in any perfect square sum graph.
Theorem 5.2.12. Let $G$ be a perfect square sum graph. Then $d(u)$ cannot exceed the number of distinct positive solutions of the equation $x^2 - y^2 = [f(u)]^2$ for all $u \in E(G)$ except the vertex $u$ with $f(u) = 0$.

Proof. Let $d(u) = n$ and $u_1, u_2, ..., u_n$ be the vertices adjacent to the vertex $u$. Since $G$ is a perfect square sum graph we have $[f(u)]^2 + [f(u_i)]^2 = a_i^2$ for $1 \leq i \leq n$. That is, $a_i^2 - [f(u_i)]^2 = [f(u)]^2$. Hence the pairs $(a_i, f(u_i))$ are nothing but the solution of the equation $x^2 - y^2 = [f(u)]^2$. 
5.3 strongly square sum graphs

Theorem 5.3.1. The complete graph $K_n$ is strongly square sum if and only if $n \leq 5$.

Proof. Strongly square sum labeling of $K_1, K_2, K_3, K_4, K_5$ are given in figure ???. In $K_6$ the induced edge labeling $f^*$ is not injective since $0^2 + 5^2 = 3^2 + 4^2$. See figure ??.

Theorem 5.3.2. Every strongly square sum graph except $K_1, K_2$ and $K_{1,2}$ contain at least one triangle.

Proof. Clearly $K_1, K_2$ and $K_{1,2}$ are triangle free strongly square sum graphs. For a strongly square sum graph with three edges, the possible edge values are 1, 4 and 5. To obtain the edge value 5, the vertices which label 1 and 2 should be adjacent. In this case the graph should contains a triangle $(uvwu)$ with $f(u) = 0, f(v) = 1$ and $f(w) = 2$.

Corollary 5.3.3. The cycles $C_n, n \geq 4$ are not strongly square sum.

Corollary 5.3.4. Complete bipartite graph $K_{a,b}$ is strongly square sum if and only if $a = 1, b \leq 2$.

Theorem 5.3.5. The cycles $C_4$ and $C_5$ can be embedded as an induced subgraph of a strongly square sum graph.

Proof. We can embed $C_4$ into a $(p, q)$-graph $G$ which is strongly square sum. Let $f(G) = \{0, 1, ..., 10\}$, where $f(C_4) = \{0, 3, 4, 10\}$. Make the vertices adjacent so as to get the edge values $1, 4, 5, ..., 181$, which are the first $q$ consecutive numbers of the form $a^2 + b^2, a \leq$
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$p - 1, b \leq p - 1, a \neq b$, without making forbidden edges. The two forbidden edges have labeling 25 and 100 which is obtained by joining the vertices with labeling 0 and 5 and 6 and 8 respectively. In a similar way, we can also embed the cycle $C_5$. Note that in $C_4$, we have $0^2 + 10^2 = 100 = 6^2 + 8^2$ and $3^2 + 4^2 = 25 = 0^2 + 5^2$. Also in $C_5$, we have $0^2 + 25^2 = 625 = 20^2 + 15^2$, $25^2 + 10^2 = 725 = 26^2 + 7^2$, $5^2 + 10^2 = 125 = 11^2 + 2^2$, $5^2 + 15^2 = 250 = 13^2 + 9^2$, $0^2 + 15^2 = 225 = 12^2 + 9^2$.

Figure 5.10:

Figure 5.11:
Conjecture 5.3.6. The cycles $C_n, n \geq 4$ can be embedded as an induced subgraph of a strongly square sum graph.

Theorem 5.3.7. A unicyclic graph is square sum if and only if it is either $C_3$ or $C_3$ with one pendant edge.

Proof. Clearly $C_3$ and $C_3$ with one pendant edge are strongly square sum. See figure 5.12. Conversely suppose $G$ is a unicyclic graph which is strongly square sum. Then $f^*(E(G)) = \{1, 4, 5, 9, 10, \ldots\}$.

If $f^*(E(G)) = \{1, 4, 5\}$, then $G \cong C_3$

If $f^*(E(G)) = \{1, 4, 5, 9\}$, then $G$ is isomorphic to $C_3$ along with one pendant edge.

If $f^*(E(G)) = \{1, 4, 5, 9, 10\}$, then to obtain the edge value 10, the vertices with labels 3 and 1 should be adjacent. In this case one more triangle is formed. Hence we cannot construct a unicyclic graph $G$ with $f^*(E(G)) = \{1, 4, 5, 9, 10, \ldots\}$. Therefore the unicyclic graphs which admits a strongly square sum labelling are either $C_3$ or $C_3$ with one pendant edge.

Corollary 5.3.8. There exists exactly one strongly square sum graph for each $q$ with $6 \leq q \leq 9$. (See figure 5.13)
Corollary 5.3.9. There exists exactly two strongly square sum graphs with $q = 10$, one with $p = 5$ and the other with $p = 6$. (See figures 5.14 and 5.15)

Problem 5.3.10. Find the maximum and minimum number of edges of a strongly square sum graph with $p$ vertices.
References


