Chapter 2

Weak Armendariz and Weak McCoy Rings Relative To Monoid

In this chapter, we give the concept of weak \( S\)-Armendariz ring (or weak Armendariz ring relative to monoid) and generalize some of the results of weak Armendariz ring to weak \( S\)-Armendariz ring. Further, we extend some of the results of weak \( S\)-Armendariz, weak McCoy and \( S\)-McCoy ring to weak \( S\)-McCoy ring.

2.1 Introduction

Lee and Wong [37] introduced the concept of weak Armendariz ring which is a generalization of Armendariz ring, and they extended some important results of Armendariz ring to weak Armendariz ring. Further, Jeon et al. [27] studied some other properties of Armendariz ring to weak Armendariz ring. Moreover, they also established a connection between weak Armendariz ring and other well known classes of rings. A generalization of weak Armendariz ring, weak McCoy ring, introduced by Camillo and Nielsen [8], and they investigated some properties of weak McCoy ring. In this chapter, we study some of the important properties of \( S\)-Armendariz and weak Armendariz ring proved by Liu [43], Lee and Wong [37] and Jeon et al. [27] to weak \( S\)-Armendariz ring. Afterwards, we generalize some of the results of weak \( S\)-Armendariz, weak McCoy and \( S\)-McCoy ring to weak \( S\)-McCoy ring.
This chapter is organized as follows: In section 2.2, we discuss some well known results of weak Armendariz ring which are an important part of our motivation to develop the following sections.

In section 2.3, we introduce the concept of weak $S$-Armendariz ring which is a common generalization of weak Armendariz rings and $S$-Armendariz rings. Further, we prove some results of weak Armendariz ring due to Lee and Wong [37], and Jeon et al. [27] to weak $S$-Armendariz ring, and also extend some results of $S$-Armendariz ring studied by Liu [43] to weak $S$-Armendariz ring.

In the last section, we study the concept of weak $S$-McCoy ring introduced by Hashemi [20]. The class of weak $S$-McCoy rings is a common generalization of weak $S$-Armendariz ring, $S$-Armendariz ring and $S$-McCoy ring. Further, we study some results of all above mentioned classes of the rings to weak $S$-McCoy rings.

2.2 Weak Armendariz Rings

In this section, we recall basic definitions, examples and properties of weak Armendariz ring.

Lee and Wong [37] introduced the concept of Weak Armendariz ring, a generalization of Armendariz ring, which is defined as follows.

**Definition 2.2.1.** A ring $R$ is called weak Armendariz if whenever the linear polynomials $f(x) = a_0 + a_1x, g(x) = b_0 + b_1x \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$, where $0 \leq i \leq 1, 0 \leq j \leq 1$.

Clearly, every Armendariz ring is weak Armendariz ring.

Due to Bell [6], a right (or left) ideal $I$ of a ring $R$ is said to have the insertion-of-factors-property (simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. So a ring $R$ is called IFP if the zero ideal of $R$ has the IFP, whereas, Norbonne [55] called IFP rings semicommutative.
In following Lemma, Jeon et al. [27] established a connection between weak Armendariz ring and some other rings.

**Lemma 2.2.1 ([27, Lemma 1.1]).** 1. Armendariz rings are weak Armendariz.

2. The class of (weak) Armendariz ring is closed under subrings and direct products.

3. Reduced rings are Armendariz.

4. Reduced rings are IFP.

5. IFP rings are abelian.

6. Weak Armendariz rings are abelian.

Jeon et al. [27, Example 1.2] investigated that the converse of (1), (2), (3), (4), (5) and (6) need not be true, and that the classes of (weak) Armendariz rings and IFP rings do not contains each other.

Anderson and Camillo [2] proved that for a ring $\mathcal{R}$, $\mathcal{R}[x]/(x^n)$ is Armendariz if and only if $\mathcal{R}$ is reduced, where $n \geq 2$ a natural number. In following Theorem, Lee and Wong [37] generalized above result to weak Armendariz ring.

**Theorem 2.2.1 ([37, Theorem 3.1]).** Let $\mathcal{R}$ be a ring and $n \geq 2$ a natural number. Then $\mathcal{R}[x]/(x^n)$ is weak Armendariz if and only if $\mathcal{R}$ is reduced.

Huh et al. [26] investigated that a ring $\mathcal{R}$ is Armendariz if and only if $\mathcal{Q}$ is Armendariz when $\mathcal{R}$ be a right Ore ring with classical right quotient ring $\mathcal{Q}$ of $\mathcal{R}$. The extension of this result studied by Jeon et al. [27], which is given as follows.

**Theorem 2.2.2 ([27, Theorem 2.5]).** Let $\mathcal{R}$ be a right Ore ring with classical right quotient ring $\mathcal{Q}$ of $\mathcal{R}$. Then $\mathcal{R}$ is a weak Armendariz ring if and only if $\mathcal{Q}$ is a weak Armendariz ring.
A proper ideal \( I \) of a ring \( R \) is called (weak) Armendariz if \( I \) is (weak) Armendariz as a ring without identity. It is a natural to ask whether a given ring \( R \) is a (weak) Armendariz ring when \( R/I \) and \( I \) are (weak) Armendariz for any nonzero proper ideal \( I \) of \( R \). However, Example [31, Example 14] erases the possibility. But Huh et al. [26] gave an affirmative answer when \( I \) is reduced. In next Proposition, Jeon et al. [27] studied the above results to weak Armendariz ring.

**Proposition 2.2.1** ([27, Proposition 2.10]). *Let \( R \) be a ring such that \( R/I \) is weak Armendariz for some proper ideal of \( I \) of \( R \). If \( I \) is reduced then \( R \) is weak Armendariz.*

### 2.3 Weak Armendariz Rings Relative To Monoid

This section deals with the concept of a new class of rings which is known as weak \( S \)-Armendariz ring. Weak \( S \)-Armendariz ring is a common generalization of \( S \)-Armendariz ring and weak Armendariz ring. Further, we study some of the results of \( S \)-Armendariz ring and weak Armendariz rings as investigated by Liu [43], Lee and Wong [37], and Jeon et al. [27] to weak Armendariz ring relative to monoid.

The weak \( S \)-Armendariz ring is define as follows.

**Definition 2.3.1.** Let \( S \) be a monoid ring. We say that a ring \( R \) is called weak \( S \)-Armendariz ring (weak Armendariz ring relative to monoid) if whenever the elements \( \alpha = a_1 g_1 + a_2 g_2 , \beta = b_1 h_1 + b_2 h_2 \in R[S] \) satisfy \( \alpha \beta = 0 \), then \( a_i b_j = 0 \) for all \( 1 \leq i \leq 2 \) and \( 1 \leq j \leq 2 \), where \( a_i, b_j \in R \) and \( g_i, g_j \in S \).

If \( S = \{ e \} \) then each ring is weak \( S \)-Armendariz. Let \( S = (\mathbb{N} \cup \{0\}, +) \), then \( R \) is weak \( S \)-Armendariz if and only if \( R \) is weak Armendariz. It is clear that every \( S \)-Armendariz ring is a weak \( S \)-Armendariz ring and subrings of weak \( S \)-Armendariz
rings are weak $S$-Armendariz rings. The converse may not be true, as shown by the following example.

**Example 2.3.1.** Let $R = \mathbb{Z}_3[x, y] \setminus (x^3, x^2y^2, y^3)$, where $\mathbb{Z}_3$ is a Galois field of order 3, $\mathbb{Z}_3[x, y]$ is the polynomial ring with two indeterminates $x, y$ over $\mathbb{Z}_3$, and $(x^3, x^2y^2, y^3)$ is the ideal of $\mathbb{Z}_3[x, y]$ generated by $x^3, x^2y^2, y^3$ [37, Example 3.2]. Let $S$ be a monoid and $R[S]$ be a monoid ring over $S$. Since $(\bar{e}x + \bar{g}y)^3 = (\bar{e}x + \bar{g}y)(\bar{e}x^2 e + 2\bar{e}xyg + \bar{g}y^2 g) = 0$ with $\bar{e} \neq \bar{g}$, where $e, g \in S$ with $e \neq g$, then $R$ is not $S$-Armendariz. But $R$ is weak $S$-Armendariz.

Further, there are some examples in which $R$ is not weak $S$-Armendariz; in particular, we consider the following example.

**Example 2.3.2.** Let $\mathbb{Z}_8$ be a ring of integers of modulo 8 and $R = T(\mathbb{Z}_8, \mathbb{Z}_8)$ be a trivial extension of $\mathbb{Z}_8$ [27, Example 1.2(4)]. Suppose $S$ be a monoid with $|S| \geq 2$. Take $e, g \in S$ such that $e \neq g$. Now consider $\alpha = (4, 0)e + (4, 1)g \in R[S]$. The square of $\alpha$ is zero but the product $(4, 0)(4, 1) = (0, 4) \neq 0$. Thus $R$ is not weak $S$-Armendariz.

Now, we study some properties of weak $S$-Armendariz ring.

**Proposition 2.3.1.** Let $S$ be a u.p.-monoid and $R$ a reduced ring. Then $R$ is weak $S$-Armendariz.

**Proof.** Let $S$ be u.p.-monoid and $R$ a reduced ring. So by [43, Proposition 1.1] $R$ is $S$-Armendariz and therefore $R$ is weak $S$-Armendariz.

**Lemma 2.3.1.** Let $S$ be a monoid with $|S| \geq 2$ and $R$ is weak $S$-Armendariz ring. If $a, b, c \in R$ are such that $ab = 0$ and $c^2 = 0$, then $acb = 0$.

**Proof.** Suppose $g \in S$ with $g \neq e$ and $\alpha = (ae - acg), \beta = (be + cbg) \in R[S]$. Then $(ae - acg)(be + cbg) = 0$. Since $R$ is weak $S$-Armendariz, therefore $acb = 0$. □
Proposition 2.3.2. Let \( S \) be a monoid with \(|S| \geq 2\). Then every weak \( S \)-Armendariz ring is abelian.

Proof. Suppose \( R \) be a weak \( S \)-Armendariz and \( g \in S \) with \( g \neq e \). Consider \( e_1^2 = e_1 \in R[S] \) and \( a = e_1, \ b = 1 - e_1, \ c = e_1r(1 - e_1) \). Then clearly, \( ab = 0 \) and \( c^2 = 0 \) whence \( acb = 0 \). Therefore by Lemma 2.3.1, \( e_1r(1 - e_1) = 0 \) which implies that \( e_1r = e_1re_1 \).

Now, we take \( a_1 = 1 - e_1, \ b_1 = e_1, \ c_1 = (1 - e_1)re_1 \). Then \( a_1b_1 = 0 \) and \( c_1^2 = 0 \), whence \( a_1c_1b_1 = 0 \). Thus by Lemma 2.3.1, \( (1 - e_1)re_1 = 0 \) which implies \( re_1 = e_1re_1 \).

Therefore \( e_1 \) is a central element of \( R \). Hence \( R \) is abelian.

Next Example shows that converse of Proposition 2.3.2 need not be true.

Example 2.3.3. Let \( R \) be abelian ring and

\[
R_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in R \right\}.
\]

Then \( R_4 \) is abelian by [26, Lemma 2]. Let \( S \) be a monoid with \(|S| \geq 2\). Take \( e, g \in S \) such that \( e \neq g \). Consider

\[
\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad g \in R[S]
\]

\[
\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad g \in R[S].
\]
Then $\alpha \beta = 0$, but

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\neq 0.
$$

So $R_4$ is not weak $S$-Armendariz.

Recall that Kaplansky [29] introduced the concept of Baer ring. A ring $R$ is said to be Baer if the right (left) annihilator of every nonempty subset is generated by an idempotent. Whereas, p.p.-ring is a generalization of a Baer ring. A ring $R$ is said to be a right p.p.-ring if the right annihilator of an element of $R$ is generated by an idempotent.

In following Theorem, we show that converse of Proposition 2.3.2 is true if $R$ is a right p.p.-ring.

**Theorem 2.3.1.** Let $S$ be a u.p.-monoid and $R$ is an abelian ring. If $R$ is a right p.p.-ring, then $R$ is a weak $S$-Armendariz ring.

**Proof.** For the proof see the section 3.3, Theorem 3.3.1. \qed

Huh et al. [26] showed that if $I$ is reduced ideal of $R$ such that $R/I$ is an Armendariz ring, then $R$ is Armendariz. Further, Liu [43] extended the above result and proved that if $I$ is reduced ideal of $R$ such that $R/I$ is an $S$-Armendariz ring, then $R$ is $S$-Armendariz. Afterwards, Jeon et al. [27] generalized the result of Huh et al. [26] and investigated that if $I$ is reduced ideal of $R$ such that $R/I$ is an weak Armendariz ring, then $R$ is weak Armendariz. Here, we prove these results to weak $S$-Armendariz ring which is a generalization of all these results.

**Proposition 2.3.3.** Let $S$ be a strictly totally ordered monoid and $I$ an ideal of $R$. If $I$ is reduced and $R/I$ is weak $S$-Armendariz, then $R$ is weak $S$-Armendariz.
Proof. Suppose $\alpha, \beta \in \mathcal{R}[\mathcal{S}]$ be such that $\alpha \beta = 0$. We write $\alpha = a_1g_1 + a_2g_2$ and $\beta = b_1h_1 + b_2h_2$ with $g_1 < g_2$ and $h_1 < h_2$. We will use transfinite induction on the strictly totally ordered set $(\mathcal{S}, \leq)$ to show that $a_ib_j = 0$ for all $i, j$. Note that in $(\mathcal{R}/\mathcal{I})[\mathcal{S}]$, $(\bar{a}_1g_1 + \bar{a}_2g_2)(\bar{b}_1h_1 + \bar{b}_2h_2) = 0$. Since $\mathcal{R}/\mathcal{I}$ is weak $\mathcal{S}$-Armendariz, $a_ib_j \in \mathcal{I}$ for all $i$ and $j$. If there exists $1 \leq i \leq 2$ and $1 \leq j \leq 2$ such that $g_ih_j = g_1h_1$, then $g_1 \leq g_i$ and $h_1 \leq h_j$. If $g_1 < g_i$, then $g_1h_1 < g_ih_1 \leq g_ih_j = g_1h_1$, a contradiction. Thus $g_1 = g_i$. Similarly $h_1 = h_j$. Hence $a_1b_1 = 0$ then $b_1\mathcal{I}a_1 = 0$. Similarly $a_2b_2 = 0$. Now only show that $a_1b_2 = 0 = a_2b_1$. Suppose $a_1b_2 \neq 0$. Then $(a_2b_1)(a_1b_2)^2 = (a_2b_1)(a_1b_2)(a_1b_2) = a_2(b_1a_1b_2a_1)b_2 \in a_2(b_1\mathcal{I}a_1)b_2 = 0$, from $b_1\mathcal{I}a_1 = 0$ and $a_1b_2 \in \mathcal{I}$. Thus,\[0 = (\alpha \beta)(a_1b_2)^2 = (a_1b_2g_1h_2 + a_2b_1g_2h_1)(a_1b_2)^2 = (a_1b_2)^3(g_1h_2) + (a_2b_1)(a_1b_2)^2g_2h_1 = (a_1b_2)^3(g_1h_2),\]which implies $(a_1b_2)^3 = 0$. Since $\mathcal{I}$ is reduced, so $a_1b_2 = 0$, a contradiction. Therefore $a_1b_2 = 0$. Similarly $a_2b_1 = 0$. Hence, $\mathcal{R}$ is weak $\mathcal{S}$-Armendariz. \hfill \Box

**Proposition 2.3.4.** Let $\mathcal{S}$ be a monoid and $\mathcal{T}$ a u.p.-monoid. If $\mathcal{R}$ is reduced and weak $\mathcal{S}$-Armendariz then $\mathcal{R}[\mathcal{S}]$ is a weak $\mathcal{T}$-Armendariz.

Proof. Suppose $\alpha = a_1g_1 + a_2g_2 \in \mathcal{R}[\mathcal{S}]$ such that $\alpha^2 = 0$. Then $a_ib_j = 0$ for all $1 \leq i, j \leq 2$ since $\mathcal{R}$ is weak $\mathcal{S}$-Armendariz ring. In particular $a_i = 0$ for all $1 \leq i \leq 2$ which implies $a_i = 0$ since $\mathcal{R}$ is reduced. Thus $\alpha = 0$. It follows that $\mathcal{R}[\mathcal{S}]$ is reduced. Hence, $\mathcal{R}[\mathcal{S}]$ is weak $\mathcal{T}$-Armendariz. \hfill \Box

Liu [43] showed that if $\mathcal{R}$ is reduced and $\mathcal{S}$-Armendariz, then $\mathcal{R}$ is $\mathcal{S} \times \mathcal{T}$-Armendariz, where $\mathcal{S}$ be a monoid and $\mathcal{T}$ a u.p.-monoid. In next Theorem, we prove
the above result to weak \( S \)-Armendariz ring, which is an extension of the above result of Liu [43].

**Theorem 2.3.2.** Let \( S \) be a monoid and \( T \) a u.p.-monoid. If \( R \) is a reduced and weak \( S \)-Armendariz then \( R \) is a weak \( S \times T \)-Armendariz.

**Proof.** Suppose \( \alpha = a_1(m_1, n_1) + a_2(m_2, n_2) \in R[S \times T] \). We consider that \( n_1 \neq n_2 \) and \( m_1 \neq m_2 \). So we write \( \alpha_1 = (a_1m_1)n_1 + (a_2m_2)n_2 \in R[S][T] \). Now, it is easy to see that there exists an isomorphism of a rings \( R[S \times T] \rightarrow R[S][T] \), defined by \( \alpha \rightarrow \alpha_1 \). Assume that

\[
\alpha\beta = \{a_1(m_1, n_1) + a_2(m_2, n_2)\}\{b_1(m'_1, n'_1) + b_2(m'_2, n'_2)\} = 0,
\]

where \( \alpha, \beta \in R[S \times T] \). Then from the above isomorphism \( ((a_1m_1)n_1 + (a_2m_2)n_2)((b_1m'_1)n'_1 + (b_2m'_2)n'_2) = 0 \). So, by Proposition 2.3.4, \( (a_1m_1 + a_2m_2)(b_1m'_1 + b_2m'_2) = 0 \). Therefore \( a_ib_j = 0 \) for all \( 1 \leq i, j \leq 2 \) since \( R \) is weak \( S \)-Armendariz. Hence, \( R \) is weak \( S \times T \)-Armendariz.

Liu [43] investigated that a finitely generated abelian group \( G \) is torsion-free if and only if there exists a ring \( R \) with \( |R| \geq 2 \) such that \( R \) is \( G \)-Armendariz. In Theorem 2.3.3, we extend this result to weak \( G \)-Armendariz ring. To prove this result, we need to prove following Lemmas and Propositions.

**Proposition 2.3.5.** Let \( S \) be a cancellative monoid and \( T \) be an ideal of \( S \). If \( R \) is weak \( T \)-Armendariz then \( R \) is weak \( S \)-Armendariz.

**Proof.** Let \( \alpha = a_1g_1 + a_2g_2, \beta = b_1h_1 + b_2h_2 \in R[S], \) where \( a_1, a_2, b_1, b_2 \in R \) and \( g_1, g_2, h_1, h_2 \in S \) such that \( \alpha\beta = 0 \). Take \( g \in T \), then \( gg_1, gg_2, h_1g, h_2g \in T \) and \( gg_i \neq gg_j \) and \( h_ig \neq h_jg \) when \( i \neq j \). Now, from

\[
\left( \sum_{i=1}^{2} a_i gg_i \right) \left( \sum_{j=1}^{2} b_j h_j g \right) = 0
\]

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and from the hypothesis that $\mathcal{R}$ is weak $\mathcal{T}$-Armendariz, $a_i b_j = 0$ for all $i, j$. Thus $\mathcal{R}$ is weak $\mathcal{S}$-Armendariz.

**Lemma 2.3.2.** Let $\mathcal{S}$ be a cyclic group of order $n \geq 2$ and $\mathcal{R}$ a ring with $0 \neq 1$. Then $\mathcal{R}$ is not weak $\mathcal{S}$-Armendariz.

*Proof.* Suppose that $\mathcal{S} = \{e, g\}$. Let $\alpha = 1e + 1g$ and $\beta = 1e + (-1)g \in \mathcal{R}[\mathcal{S}]$ then $\alpha \beta = 0$. Thus $\mathcal{R}$ is not weak $\mathcal{S}$-Armendariz. □

**Lemma 2.3.3.** Let $\mathcal{S}$ be a monoid and $\mathcal{T}$ a submonoid of $\mathcal{S}$. If $\mathcal{R}$ is weak $\mathcal{S}$-Armendariz then $\mathcal{R}$ is weak $\mathcal{T}$-Armendariz.

*Proof.* Proof is trivial. □

**Lemma 2.3.4.** Let $\mathcal{S}$ and $\mathcal{T}$ be u.p. monoids. Then so is the monoid $\mathcal{S} \times \mathcal{T}$.

*Proof.* See [43, Lemma 1.13]. □

Now, we are able to prove Theorem 2.3.3.

**Theorem 2.3.3.** Let $\mathcal{G}$ be a finitely generated abelian group. Then the following conditions on $\mathcal{G}$ are equivalent.

1. $\mathcal{G}$ is torsion free.

2. There exists a ring $\mathcal{R}$ with $|\mathcal{R}| \geq 2$ such that $\mathcal{R}$ is weak $\mathcal{G}$-Armendariz.

*Proof.* (2)$\Rightarrow$(1) If $g \in T(\mathcal{G})$ and $g \neq e$, then $\mathcal{T} = \langle g \rangle$ is a cyclic group of finite order. If a ring $\mathcal{R} \neq \{0\}$ is weak $\mathcal{G}$-Armendariz, then, by the Lemma 2.3.3, $\mathcal{R}$ is weak $\mathcal{T}$-Armendariz, a contradiction with Lemma 2.3.2. Thus every ring $\mathcal{R} \neq \{0\}$ is not weak $\mathcal{G}$-Armendariz.

(1)$\Rightarrow$(2) If $\mathcal{G}$ is finitely generated abelian group with $T(\mathcal{G}) = \{e\}$. Then $\mathcal{G} \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \ldots \times \mathbb{Z}$, a finite direct product group $\mathbb{Z}$, by Lemma 2.3.4, $\mathcal{G}$ is u.p. monoid.
Let $\mathcal{R}$ be a commutative reduced ring. Then by proposition 2.3.5, $\mathcal{R}$ is weak $\mathcal{G}$-Armendariz.

Anderson and Camillo [2] assert that it does not seem possible for $\mathcal{R}$ to reduced but classical right quotient ring $Q(\mathcal{R})$ of $\mathcal{R}$ is not reduced. Whereas, Kim and Lee [31] gave an affirmative answer and proved that there exists a classical right quotient ring $Q$ of a ring $\mathcal{R}$, then $\mathcal{R}$ is reduced if and only if $Q$ is reduced. Further, Huh et al. [26] extended the above result and showed that there exists the classical right quotient ring $Q$ of a ring $\mathcal{R}$, then $\mathcal{R}$ is Armendariz if and only if $Q$ is Armendariz. Afterwards, Jeon et al. [27] showed that a ring $\mathcal{R}$ is weak Armendariz if and only if $Q$ is weak Armendariz, when $\mathcal{R}$ is a right Ore ring with classical right quotient ring $Q$. In following Theorem, we generalize all the above mentioned results.

**Theorem 2.3.4.** Let $S$ be monoid and $\mathcal{R}$ a right Ore ring with classical right quotient ring $Q$. Then $\mathcal{R}$ is weak $S$-Armendariz if and only if $Q$ is weak $S$-Armendariz.

**Proof.** Suppose $\mathcal{R}$ is weak $S$-Armendariz ring. Let

$$0 \neq \alpha = \sum_{i=1}^{2} a_i g_i, 0 \neq \beta = \sum_{j=1}^{2} b_j h_j \in Q[S]$$

such that $\alpha \beta = 0$. Since $\mathcal{R}$ is right Ore ring with classical right quotient ring $Q$. By [52, Proposition 2.1.16], we can assume that $a_i = p_i u^{-1}, b_j = q_j v^{-1}$ with $p_i, q_j \in \mathcal{R}$ for all $i, j$ and regular elements $u, v \in \mathcal{R}$. Also by [52, Proposition 1.1.16], for each $j$ there exist $r_j \in \mathcal{R}$ and regular element $w \in \mathcal{R}$ such that $u^{-1} q_j = r_j w^{-1}$. Denote

$$\alpha_1 = \sum_{i=1}^{2} p_i g_i, \beta_1 = \sum_{j=1}^{2} q_j h_j, \beta_2 = \sum_{j=1}^{2} r_j h_j \in \mathcal{R}[S].$$

Then we have

$$0 = \alpha \beta = \left( \sum_{i=1}^{2} a_i g_i \right) \left( \sum_{j=1}^{2} b_j h_j \right) = \alpha_1 \beta_2 (vw)^{-1},$$

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hence $\alpha_1\beta_2 = 0$. Since $\mathcal{R}$ is weak $\mathcal{S}$-Armendariz, $p_ir_j = 0$ for all $i, j$ and so $a_i b_j = p_i u^{-1} q_j v^{-1} = p_i r_j w^{-1} v^{-1} = p_i r_j (uv)^{-1} = 0$ for all $i, j$. Therefore $\mathcal{Q}$ is weak $\mathcal{S}$-Armendariz. Converse is trivial.

\section{2.4 Weak McCoy Rings Relative To Monoid}

In this section, we discuss the concept of weak $\mathcal{S}$-McCoy ring (weak McCoy ring relative to monoid) which was introduced by Hashemi [20]. Weak $\mathcal{S}$-McCoy ring is a common generalization of weak McCoy ring, $\mathcal{S}$-McCoy ring, McCoy ring, Weak $\mathcal{S}$-Armendariz ring, weak Armendariz ring, $\mathcal{S}$-Armendariz ring and Armendariz ring.

Further, we generalize some well known results of above mentioned classes of rings to Weak $\mathcal{S}$-McCoy ring.

The following definition introduced by Hashemi [20].

**Definition 2.4.1.** Let $\mathcal{S}$ be a monoid. a ring $\mathcal{R}$ is said to be right $\mathcal{S}$-McCoy if whenever $0 \neq \alpha = a_1 g_1 + a_2 g_2 + \ldots + a_n g_n$, $0 \neq \beta = b_1 h_1 + b_2 h_2 + \ldots + b_m h^m \in \mathcal{R}[\mathcal{S}]$, with $g_i, h_j \in \mathcal{S}$, $a_i, b_j \in \mathcal{R}$ satisfy $\alpha \beta = 0$, then $\alpha r = 0$ for some nonzero $r \in \mathcal{R}$. Left $\mathcal{S}$-McCoy rings be defined similarly. If $\mathcal{R}$ is both right and left $\mathcal{S}$-McCoy, then we say $\mathcal{R}$ is $\mathcal{S}$-McCoy.

Let $M = (N \cup \{0\}, +)$. Then $\mathcal{R}$ is right McCoy if and only if $\mathcal{R}$ is right $\mathcal{S}$-McCoy. Clearly, $\mathcal{S}$-Armendariz rings are $\mathcal{S}$-McCoy.

**Definition 2.4.2.** Let $\mathcal{S}$ be a monoid. A ring $\mathcal{R}$ is said to be right weak $\mathcal{S}$-McCoy if for given $0 \neq \alpha = a_1 g_1 + a_2 g_2$, $0 \neq \beta = b_1 h_1 + b_2 h_2 \in \mathcal{R}[\mathcal{S}]$ with $\alpha \beta = 0$, then there exists a nonzero $r \in \mathcal{R}$ with $\alpha r = 0$. Left weak $\mathcal{S}$-McCoy rings be defined similarly. If $\mathcal{R}$ is both right and left weak $\mathcal{S}$-McCoy, then we say $\mathcal{R}$ is Weak $\mathcal{S}$-McCoy.

Let $\mathcal{S} = (N \cup \{0\}, +)$. Then $\mathcal{R}$ is right weak $\mathcal{S}$-McCoy if and only if $\mathcal{R}$ is right weak $\mathcal{S}$-McCoy. Clearly right $\mathcal{S}$-McCoy rings are right weak McCoy.
Here, we give some examples based on weak $S$-McCoy ring.

**Example 2.4.1.** Let $S$ be a monoid with $|S| \geq 2$ and $R$ a ring. Take $e, g \in S$ such that $e \neq g$. Suppose $A = C = e_{12}$, $B = e_{11}$, $D = -e_{22}$ where $e_{ij}$'s are usual matrix units,

$$
\alpha = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} g
$$

$$
\beta = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} D & 0 \\ 0 & I_{n-2} \end{pmatrix} g
$$

$$
\beta_1 = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} g
$$

$$
\alpha_1 = \begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix} e + \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} g
$$

are elements of $M_n(R)[S]$ [30, Example 2.6]. So $\alpha \beta = 0$ and $\alpha_1 \beta_1 = 0$. But if $S \beta = 0$ or $\alpha_1 T = 0$, for some $S, T \in M_n(R)/\{0\}$ then $S = T = 0$. Therefore $M_n(R)$ is neither right weak $S$-McCoy nor left weak $S$-McCoy ring.

**Example 2.4.2.** Let $R = \mathbb{Z}_3[x, y]/(x^3, x^2y^2, y^3)$, where $\mathbb{Z}_3$ is Galois field of order 3, $\mathbb{Z}_3[x, y]$ is the polynomial ring with two indeterminates $x, y$ over $\mathbb{Z}_3$ and $(x^3, x^2y^2, y^3)$ is an ideal of $\mathbb{Z}_3[x, y]$ generated by $x^3, x^2y^2, y^3$ [37, Example 3.2]. Suppose $S$ be a monoid and $R[S]$ be a monoid ring over $R$. Since $(\bar{x}e + \bar{y}g)^3 = (\bar{x}e + \bar{y}g)(\bar{x}^2e^2 + 2\bar{x}\bar{y}eg + \bar{y}^2g^2) = 0$ with $\bar{x}\bar{y} \neq 0$, where $e, g \in S$ such that $e \neq g$, then $R$ is not $S$-Armendariz but $R$ is weak $S$-Armendariz. Therefore $R$ is right weak $S$-McCoy.
Example 2.4.3. Let \( \mathcal{R} \) be an abelian ring and

\[
\mathcal{R}_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in \mathcal{R} \right\}.
\]

Therefore \( \mathcal{R}_4 \) is abelian by [27, Example 1.2(4)]. Let \( \mathcal{S} \) be a monoid with \( |\mathcal{S}| \geq 2 \).

Take \( e, g \in \mathcal{S} \) such that \( e \neq g \). Consider

\[
\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} g
\]

and

\[
\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} e + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} g
\]

are elements of \( \mathcal{R}_4[\mathcal{S}] \). Then \( \alpha \beta = 0 \), but

\[
\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0.
\]

So \( \mathcal{R}_4 \) is not weak \( \mathcal{S} \)-Armendariz ring. If take an element

\[
\mathcal{S} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{R}/\{0\}.
\]
Therefore
\[
\alpha S = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} g \right\} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 0
\]

Thus \( R_4 \) is right weak \( S \)-McCoy.

Liu [43] showed that a finitely generated abelian group \( G \) is torsion-free if and only if there exists a ring \( R \) with \(|R| \geq 2\) such that \( R \) is \( G \)-Armendariz. Further, Hashemi [20] extended the above result and proved that a finitely generated abelian group \( G \) is torsion-free if and only if there exists a ring \( R \) with \(|R| \geq 2\) such that \( R \) is right \( G \)-McCoy. In Theorem 2.4.1, we generalize these results to right weak \( G \)-McCoy which is also a generalization of Theorem 2.3.3. Before proving Theorem 2.4.1, we need to prove some important Lemmas and Propositions essential for Theorem 2.4.1.

**Proposition 2.4.1.** Let \( S \) be a cancellative monoid and \( T \) an ideal of \( S \). If \( R \) is right weak \( T \)-McCoy, then \( R \) is right weak \( S \)-McCoy.

**Proof.** Let \( \alpha = a_1g_1 + a_2g_2, 0 \neq \beta = b_1h_1 + b_2h_2 \in R[S], \) where \( a_1, a_2, b_1, b_2 \in R \) and \( g_1, g_2, h_1, h_2 \in S \) such that \( \alpha \beta = 0 \). Take \( g \in T \). Then \( gg_1, gg_2, h_1g, h_2g \in T \) and \( gg_i \neq gg_j \) and \( h_ig \neq h_jg \) when \( i \neq j \). Now, from
\[
\left( \sum_{i=1}^{2} a_igg_i \right) \left( \sum_{j=1}^{2} b_jh_jg \right) = 0
\]
and from the hypothesis, \( R \) is right weak \( T \)-McCoy, it follows that \( \alpha gr = 0 = \alpha r \), for some nonzero \( r \in R \). Therefore \( R \) is a right weak \( S \)-McCoy ring.

**Lemma 2.4.1.** Let \( S \) be a cyclic group of order \( n \geq 2 \) and \( R \) a ring with \( 0 \neq j \).

Then \( R \) is not a right weak \( S \)-McCoy ring.
Proof. Suppose that $\mathcal{S} = \{e, g\}$. Let $\alpha = 1e + 1g$ and $\beta = 1e + (-1)g \in \mathcal{R}[\mathcal{S}]$ then $\alpha \beta = 0$, but $\alpha r \neq 0$ for each nonzero $r \in \mathcal{R}$. Therefore $\mathcal{R}$ is not weak right $\mathcal{S}$-McCoy ring.

Lemma 2.4.2. Let $\mathcal{S}$ be a monoid and $\mathcal{T}$ a submonoid of $\mathcal{S}$. If $\mathcal{R}$ is a right weak $\mathcal{S}$-McCoy ring then $\mathcal{R}$ is a right weak $\mathcal{T}$-McCoy ring.

Proof. Proof is trivial.

Now, we are able to prove Theorem 2.4.1.

Theorem 2.4.1. Let $\mathcal{G}$ be a finitely generated abelian group. Then the following conditions on $\mathcal{G}$ are equivalent:

1. $\mathcal{G}$ is torsion-free

2. There exists a ring $\mathcal{R}$ with $|\mathcal{R}| \geq 2$ such that $\mathcal{R}$ is right weak $\mathcal{G}$-McCoy ring.

Proof. (1)⇒(2). If $\mathcal{G}$ is finitely generated Abelian group with $T(\mathcal{G}) = \{e\}$, then $G \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \ldots \times \mathbb{Z}$, a finite direct product of group $\mathbb{Z}$. By Lemma 2.3.4, $\mathcal{G}$ is u.p. monoid. Let $\mathcal{R}$ be a semicommutative ring. Then by Proposition 2.4.1, $\mathcal{R}$ is a right weak $\mathcal{G}$-McCoy ring.

(2)⇒(1) If $g \in T(\mathcal{G})$ and $g \neq e$, then $\mathcal{T} = \langle g \rangle$ is a cyclic group of finite order. If a ring $R \neq \{0\}$ is a right weak $\mathcal{G}$-McCoy ring, then by the Lemma 2.4.2, $\mathcal{R}$ is a right weak $\mathcal{T}$-McCoy ring, a contradiction with Lemma 2.4.1. Thus every ring $R \neq \{0\}$ is not a right weak $\mathcal{G}$-McCoy ring.

Anderson and Camillo [2] assert that it does not seem possible for $\mathcal{R}$ to reduced but classical quotient ring $Q(\mathcal{R})$ of $\mathcal{R}$ is not reduced. Whereas, Kim and Lee [31] gave an affirmative answer and proved that there exists a classical right quotient ring $Q$ of a ring $\mathcal{R}$. Then $\mathcal{R}$ is reduced if and only if $Q$ is reduced. Further, Huh et al. [26] extended the above result and showed that there exists the classical right quotient
ring $Q$ of a ring $R$. Then $R$ is Armendariz if and only if $Q$ is Armendariz. Afterwards, Jeon et al. [27] showed that a ring $R$ is a weak Armendariz ring if and only if $Q$ is a weak Armendariz ring, when $R$ is a right Ore ring with classical right quotient ring $Q$. Later, Ying et al. [62] studied the above results to right McCoy ring relative to polynomial. Furthermore, Khoramdel and Pishhesari [30] proved the theorem of Ying et al. [62] to right $S$-McCoy ring. In the preceding section, we have also done that a ring $R$ is weak $S$-Armendariz if and only if $Q$ is weak $S$-Armendariz when $R$ be a right Ore ring with classical right quotient ring $Q$ and $S$ a monoid. Here, we investigate the above results to class of right weak $S$-McCoy rings which is an extension of all these results.

**Theorem 2.4.2.** Let $S$ be a monoid and $R$ a right Ore ring with classical right quotient ring $Q$. Then $R$ is a right weak $S$-McCoy ring if and only if $Q$ is a right weak $S$-McCoy ring.

**Proof.** Suppose $R$ is right weak $S$-McCoy ring. Let

$$0 \neq \alpha = \sum_{i=1}^{2} a_i g_i, 0 \neq \beta = \sum_{j=1}^{2} b_j h_j \in Q[S]$$

such that $\alpha \beta = 0$. Since $R$ is a right ore ring with classical right quotient ring $Q$. By [52, Proposition 2.1.16], we can assume that $a_i = p_i u^{-1}, b_j = q_j u^{-1}$ with $p_i, q_j \in R$ for all $i, j$ and regular elements $u, v \in R$. Also by [52, Proposition 2.1.16], for each $j$ there exists $r_j \in R$ and regular element $w \in R$ such that $u^{-1} q_j = r_j w^{-1}$. Denote

$$\alpha_1 = \sum_{i=1}^{2} p_i g_i, \beta_1 = \sum_{j=1}^{2} q_j h_j, \beta_2 = \sum_{j=1}^{2} r_j h_j \in R[S].$$

Then we have

$$0 = \alpha \beta = \left(\sum_{i=1}^{2} a_i g_i\right) \cdot \left(\sum_{j=1}^{2} b_j h_j\right) = \alpha_1 \beta_2(vw)^{-1}.$$

So $\alpha_1 \beta_2 = 0$, hence $\alpha_1 \beta_1 = 0$. Thus there exists a nonzero element $s \in R/\{0\}$ such that $\alpha s = 0$, then $a_i us = 0$ for each $i$. Thus $\alpha(us) = 0$ and $us$ be nonzero elements.
of \( Q \). Therefore \( Q \) is a right weak \( S \)-McCoy ring. Conversely, let \( \alpha = a_1g_1 + a_2g_2 \), \( \beta = b_1h_1 + b_2h_2 \) are elements of \( R[S] \) such that \( \alpha \beta = 0 \). Since \( Q \) is a right weak \( S \)-McCoy ring so there exists a non-zero element \( q \in Q \) such that \( \alpha q = 0 \). Since \( Q \) is classical right quotient ring, we can assume \( q = ru^{-1} \) for some \( r \in R \setminus \{0\} \) and regular elementary \( u \in R \). Then \( \alpha ru^{-1} = \alpha q = 0 \) implies \( \alpha r = 0 \) for some \( r \in R \). Therefore, \( R \) is a right weak \( S \)-McCoy ring. \( \square \)