Chapter 5

Cluster Dimension Two-A

Characterization

5.1 Introduction

Communication Networks contain nodes and links, a node acts as a server, workstation, gateway, router etc. The role of a server is different from that of a workstation similarly the role of a gateway is different from that of the router. A server has the role of storing huge resource and servicing the routers based on a demand. The capacity of a server is larger in a network; performance of a server is comparatively efficient one. The role of a gateway is different from that of the server or a router which interfaces to the other network or other cluster to forward a data from one cluster to another cluster through these gateway nodes. The problem we face is different from the common problem of location on networks. The problem is with locating the resources in a small network and with small modifying costs of the network elements, in contrast to the case of locating central resources in communication networks. Position of servers, gateways and routers in network identification is a classical one. There are many studies dealing with the design of optimal communication network topologies. However, in current methodologies, every user or node presents the
same constraint and behavior; while in reality networks resources have different
c characteristics from any other user or node. This work deals with the issue of
how to identify the position of network resource (i.e., servers, routers, gateways,
etc) in an optimal way, i.e. minimizing the hop count in the entire network. This
approach is applied on some standard popular LAN communication topologies
in small networks and for small number of resources; it is feasible to find the
optimal locations of the resources by checking all the possibilities of identifying
the position of the network resources.

Networks can be modeled by graphs, each node in the graph represents the
processor or an agent who performs the process, and the edges are the links
between the processors. The problem of minimum number of machines (or
Robots) to be placed at certain nodes to trace each and every node uniquely is
a classical one.

This problem can be solved by a graph-structured framework in which the
navigating agent moves from one node to another in a graph space. The moti-
vation is to place landmarks in as few vertices of the graph as possible in such
a way that each vertex of the graph is uniquely identified by the vector of its
distances to the landmarks [30]. The places or nodes of a graph where we place
the machines are called landmarks.

Throughout this chapter we write $G(V, E)$ or simply $G$, to denote a graph
on a finite non empty set $V$ of nodes having its edge set $E$. All the graphs con-
sidered in this paper are simple, finite, undirected and connected. The terms
not defined here may be found in [58], [59], [60]. For any two nodes $u$ and $v$ the
distance between $u$ and $v$, denoted by $d(u, v)$, is the length of the shortest path
between them. For a given graph $G$, there are a number of properties related
to the distance between two nodes that have been widely studied by various
authors.

In this chapter the distance partition of vertex set of a graph $G$ is defined
with reference to a vertex and with the help of the same, we characterize graphs
with cluster dimension two (i.e. \( \beta_c(G) = 2 \)). In the process, a polynomial
time algorithm is developed which verifies that if the cluster dimension of \( G \)
is two. The same algorithm explores all the cluster basis of graph \( G \) whenever
\( \beta_c(G) = 2 \). A bound for cardinality of any distance partite set of \( G \) with
reference to given vertex whenever \( \beta_c(G) = 2 \) is found.

5.2 Definitions

A graph or directed graph together with a function which assigns a positive
real number to each edge (i.e., an oriented edge-labeled graph) is known as a
network.

5.2.1 Resolving Set

Definition 5.1. A vertex \( x \in V(G) \) resolves a pair of vertices \( v, w \in V(G) \) if
d\( (v, x) \neq d(w, x) \). A set of vertices \( S \subseteq V(G) \) resolves \( G \), and \( S \) is a resolving
set of \( G \), if every pair of distinct vertices of \( G \) are resolved by some vertex in
\( S \).

5.2.2 Metric Dimension

The now give the formal definition of the notion of metric dimension\(^{15}\).

Definition 5.2. The metric dimension of a graph \( G = (V, E) \) is the cardinality
of a smallest subset \( S \subseteq V \), such that, for each pair of vertices \( u, v \in V \), there
is a \( w \in S \) such that the length of a shortest path from \( w \) to \( u \) is different from
the length of a shortest path from \( w \) to \( v \) \(^{15}\).

A resolving set \( S \) of \( G \) with the minimum cardinality is a metric basis of
\( G \), and \( |S| \) is the metric dimension of \( G \), denoted by \( \beta(G) \).
5.2.3 Strongly Resolving set

Definition 5.3. A subset $S$ of $V(G)$ is defined as a strongly resolving set if it satisfies the following properties:

1. For any two distinct nodes $x$ and $y$ in $G$ there exist nodes $s_1$ and $s_2$ (need not be distinct) in $S$ such that $|d(x, s_2) - d(y, s_1)| \geq 1$ and

2. $d(x, s_2) < d(x, s)$ for all $s \in S - \{s_2\}$.

In fact the first part of the definition gives the concept of resolving set defined by [15] and the second part is added so that the concept can be used to make clusters in $V(G)$.

5.2.4 Cluster Dimension

Definition 5.4. The Cluster Dimension of a network is defined as the minimum cardinality of a subset $S$ of the set of nodes having the property that for any two distinct nodes $x$ and $y$, there exist the nodes $s_1, s_2$ (need not be distinct) in $S$ such that $|d(x, s_1) - d(y, s_1)| \geq 1$ and $d(x, s_2) < d(x, s)$ for all $s \in S - \{s_2\}$. The elements of the set $S$ are called the routers or route node or (resource locators) and the set $S$ is called a Cluster basis. The Cluster dimension of a graph $G$ is denoted by $\beta_c(G)$.

Every element in a cluster basis is treated as a cluster head and it is denoted by $CH_i$.

5.2.5 N-Tuple

Unique ID is an $N$ Tuple $C(d_1, d_2, \ldots, d_n)$ associated with every node of the network where $d_i$ represents the distance of the node from the $i^{th}$ cluster head $CH_i$ for $i = 1, 2, \ldots, n$.

Note 5.1. The $i^{th}$ position value of length of the Tuple, say $n$, the $n$ Tuple associated with the cluster head $CH_i$ is zero. For $i = 1, 2, \ldots, n$ defines the
number of cluster heads in the cluster basis. The length of the Tuple, say \( n \), is the cluster dimension of \( G \).

5.2.6 Cluster

**Definition 5.5.** Clustering a set \( S \) is dividing the set into smaller subset \( s_1, s_2, s_3, \ldots, s_k \) such that \( s_1 \cup s_2 \cup s_3, \ldots, \cup s_k = S \). Each set \( s_i \) is called clusters.

In our case, the \( i^{th} \) cluster is the collection of all the vertices which are at the minimum distance from the \( i^{th} \) cluster head \( 1 \leq i \leq n \). Equivalently it is the collection of these \( n-Tuples \) which have the least entry in the \( i^{th} \) position for \( 1 \leq i \leq n \).

*Note 5.2.* The clusters obtained by the above procedures are non-overlapping and non empty.

5.2.7 Classification of nodes based on Unique n-Tuple

The classification of the nodes inside a cluster using the \( n-Tuple \) associated with the nodes as follows:

1. **Cluster Head:** It is that node in the cluster which contains a ‘0’ in its \( n-Tuple \)

2. **Member Node:** Every node which has least entry in the \( i^{th} \) position of its \( n-Tuple \) is a member node of \( i^{th} \) cluster for \( i = 1, 2, \ldots, n \)

3. **Gateway Node:** A gateway node in the \( i^{th} \) cluster, \( i = 1, 2, \ldots, n \) is a member node in that cluster which has an adjacent node in the \( j^{th} \) cluster \( i \neq j, j = 1, 2, \ldots, n \)

*Remark 5.3.* Note that \( V \) itself is a strongly resolving set of \( V \).
**Definition 5.6.** A strongly resolving set $T$ with minimum cardinality amongst all strongly resolving sets is known as a cluster basis of a graph $G$. Further, the cardinality of any cluster basis is the cluster dimension of $G$ and is denoted by $\beta_c(G)$.

**Remark 5.4.** Clearly, cluster basis exists for every given graph (by definition) and it need not be unique. For example, given a path graph $G$ with pendant vertices $u$ and $v$, $\{u\}$ and $\{v\}$ are cluster bases for $G$.

**Remark 5.5.** A cluster basis of a graph $G$ is minimal (set theoretic sense) among all strongly resolving sets, but the converse need not be true. For example, in case of $u$ and $v$ adjacent vertices in a path graph $G$ and neither among $u$ and $v$ are pendant vertices, then $\{u,v\}$ is a minimal strongly resolving set but not a cluster basis.

**Definition 5.7.** Let $G$ be a graph with vertex set $V(G)$ and $v$ be a vertex in it. Then $\{V_0, V_1, V_2, \ldots, V_k\}$ is called a distance partition of $V(G)$ with reference to the vertex $v$ if $V_0 = \{v\}$ and $V_i$ contains those vertices which are at distance $i$ from $v$ for $0 < i \leq k$, where $k$ is the eccentricity of $v$ in $G$. The sets $V_0, V_1, V_2, \ldots, V_k$ are called distance partite sets.

The result given in the following proposition was observed by Samir Khuller et al [30], and is an important tool in deriving several interesting results of the present chapter.

**Proposition 5.6.** (Samir khuller et al) [30]. In a graph $G(V, E)$, consider any three vertices $u, v$ and $w$ such that $(u, v) \in E$. If $d = d(u, w)$ then $d(v, w)$ is one of $d - 1, d$ and $d + 1$.

In the following corollaries the graph $G$ with $\beta_c(G) = 2$ is considered, cluster basis $\{v_1, v_2\}$ and distance partite sets $V_0, V_1, V_2, \ldots, V_k$ with reference to $v_1$. Proof is immediate from the above proposition and the definition of $\beta_c(G) = 2$.

**Corollary 5.7.** Given any vertex $v \in V_i$ there exist at most three vertices in $V_{i+1}$.
adjacent to $v$, where $0 \leq i \leq k - 1$. Similarly there exist at most three vertices in $V_{i-1}$ adjacent to $v$ when $1 \leq i \leq k$.

**Corollary 5.8.** Every pair of vertices $w_1$ and $w_2$ from different distance partite sets are strongly resolved by at least $v_1$ and when $u_1$ and $u_2$ are from same distance partite set then $v_2$ strongly resolves them.

### 5.3 Properties of Distance Partitions

This section discusses some characteristics of the graph due to the properties of distance partition, proofs are straight forward. Let $v$ be a vertex in $V(G)$ and $\{V_0, V_1, V_2, \ldots, V_k\}$ be the distance partition of $V(G)$ with reference to the vertex $v$.

**Theorem 5.9.** Every vertex in $V_j$ is adjacent to at least one vertex in $V_{j-1}$ for every $j$ with $2 \leq j \leq k$ and every vertex in $V_1$ is adjacent to $v$.

**Theorem 5.10.** Let $G$ be a graph and $|G| = n$. Then the following are equivalent:

1. There exists a $v \in V(G)$, such that $|V_i = 1|$ for each of distance partite set $V_i$ of $G$ with reference to the vertex $v$.

2. $G$ is a path graph and $v$ is a pendant vertex in it.

3. There exists $v \in V(G)$ such that $e(v) = n - 1$.

In fact, in the above there exist exactly two vertices in $V(G)$ such that with reference to each of them the number of distinct distance partite sets of $G$ is equal to $n$.

**Theorem 5.11.** Let $G$ be a graph and $k_v$ be the number of distance partite sets with reference to a vertex $v \in V(G)$. Then $K_v = 2$ for every $v \in V(G)$ if and only if $G$ is a complete graph.
The following corollary is a result given by Samir Khuller et al. [30] and is immediate from Theorem 5.10.

**Corollary 5.12.** Cluster dimension of a graph $G$ is one if and only if $G$ is a path graph.

**Theorem 5.13.** The number $k$ is the eccentricity of vertex $v$ (i.e. $e(v)$). Every vertex in $V_j$ is adjacent to at least one vertex in $V_{j-1}$ for every $j$ with $2 \leq j \leq n$ and every vertex in $v_1$ is adjacent to $v$. If $|V(G)| = n$, $n - 1$ then maximum value of $k$ is $n - 1$ and the minimum value is two.

**Theorem 5.14.** Let $G$ be a graph and $v \in V(G)$ with distance partition 
\[
\{V_0, V_1, V_2, \ldots, V_k\} = 1 \text{ for every } i \text{ if and only if, } G \text{ is a path graph.}
\]

**Theorem 5.15.** Let $G$ be a graph and $k_v$ be the number of distance partite sets with reference to a vertex $v \in V(G)$. Then $k_v = 2$ for every $v \in V(G)$, if and only if $G$ is a complete graph.

**Theorem 5.16.** Cluster Dimension of a graph $G$ is one if and only if $G$ is a path. For any graph which is not a path $\beta_c(G) \geq 2$.

**Theorem 5.17.** Let $G$ be a graph and $v \in V(G)$. If $v$ is a full degree vertex then $v$ must belong to every cluster basis.

**Proof.** Suppose that $v$ is outside a cluster basis $C$ of $G$. As $v$ is at distance one from every vertex in $C$, $v$ is not strongly resolved by $C$. Hence $v$ must be inside every cluster basis of $G$. \hfill \Box

**Theorem 5.18.** Let $G$ be a graph with cluster dimension two and \{v_1, V_2\} be a cluster basis of $G$. Then $e(v_1) \leq 2e(v_2)$ and $e(v_2) \leq 2e(v_1)$ where $e(v)$ denotes the eccentricity of the vertex $v$ in $V(G)$.

**Proof.** Let \{V_0, V_1, V_2, \ldots, V_k\} be the distance partition of $V(G)$ with reference to the vertex $v_1$ where $e(v_1) = k$. Let $v_2 \in V_j$, $2 \leq j \leq k$. As all the vertices in the same distance partite set are resolved by the vertex $v_2$. We have $e(v_2) \geq \frac{1}{2}e(v_1)$.
Either before the partite set $V_j$ or after the partite set $V_j$ till $V_k$ there are at least half of the total number of partite sets. Similarly the other part follows.

From the Corollary 5.8 the following results are observed.

**Note 5.19.** If $\{v_1, v_2\}$ is a cluster basis for $G$ with eccentricity $e(v_1)$ and $e(v_2)$ then the maximum number of vertices in $V_j$ where $v_2 \in V_j$ is $e(v_2) + 1$.

**Note 5.20.** If $e(v_1) = k$ and $e(v_2) = m$, then $v_2$ must be placed in one of the partite sets $V_i$ for $i = k - m, k - m + 1, \ldots m$ with $i$ an odd integer.

**Note 5.21.** In any partite set $V_i$ there cannot exist a vertex $u$ which is at distance $i$ from $v_2$.

**Proof.** Let $u$ be a vertex in $V_i$ which is at distance $i$ from $v_2$, then $u$ is not strongly resolved since $d(u, v_1) = d(u, v_2) = i$.

**Theorem 5.22.** If $G$ is a graph with diameter two which is not a path and has a full degree vertex then $\beta_c(G) \neq 2$.

**Proof.** By Theorem 5.17 full degree vertex $v$ must be in the cluster basis. If we consider the distance partition of $V(G)$ with reference to $v$ then all the other vertices in $V(G)$ must be in $V_1$. If $v_2$ is an element in $V_1$, it can be adjacent to $v_2$. Any vertex in $V_1$ non adjacent to $v_2$ is not resolved by either $v_1$ or $v_2$. Hence $\beta_c(G) \neq 2$.

Graph with cluster dimension two- A characterization

**5.4 Results**

In this section, some results are established pertaining to the structure of a graph $G$ with $\beta_c(G) = 2$. Throughout this section the following are assumed to be true.

Let $G$ be a graph with $\beta_c(G) = 2$ and $\{v_1, v_2\}$ be a cluster basis of $G$. Further, let $\{V_0, V_1, V_2, \ldots, V_k\}$ be the distance partition of $V(G)$ with reference to the vertex $v_1$ and $v_2 \in V_j, 2 \leq j \leq k$. 

75
Theorem 5.23. For any vertex \( v \in V_j \) there exists a shortest path of length \( j \) between \( v_1 \) and \( v \). In fact, the shortest path from \( v_1 \) to \( v \) contains exactly one vertex \( w_i \in V_i \) for \( 1 \leq i \leq j \), and the distance \( d(w_i, v) = j - i \).

Proof. Consider any vertex \( v \in V_j \) by Theorem 5.9, the vertex \( v \) is adjacent to a vertex, say \( w \), in \( V_{j-1} \) and the vertex \( w \) is adjacent to a vertex, say \( x \), in \( V_{j-2} \) and so on. Finally every vertex in \( V_1 \) is adjacent to the vertex \( v_1 \). Thus there exists a path of length \( j \) between \( v_1 \) and \( v \) and it follows that \( d(w_i, v) = j - i \).

Theorem 5.24. Let \( P \) be a path between \( v_1 \) and \( v_2 \) which is vertex disjoint with any other path between them. Then \( P \) must be of odd length.

Proof. Suppose that the path \( P \) is of even length \( 2t \). Then on \( P \) there exists a vertex \( u \) which is at equal distance, \( t \) from both \( v_1 \) and \( v_2 \) along \( P \). Because \( P \) is vertex distinct with every other path between \( v_1 \) and \( v_2 \), \( d(v_2, u) = d(v_1, u) = t \). This implies that \( u \) is not strongly resolved by any of \( v_1 \) and \( v_2 \), a contradiction.

Theorem 5.25. Between strongly resolving vertices there exist a unique shortest path.

Proof. By theorem 5.23, shortest path between \( v_1 \) and \( v_2 \) contains only one vertex from each distance partite set \( V_0, V_1, V_2, \ldots, V_j \), where \( v_2 \in V_j \). Suppose that \( P_1 \) and \( P_2 \) are two distinct paths between \( v_1 \) and \( v_2 \). Let \( V_j \) be the first partite set, while moving from \( v_2 \) to \( v_1 \), in which \( P_1 \) and \( P_2 \) pass through two distinct vertices \( u_1 \) and \( u_2 \) respectively. Then \( d(v_2, u_1) = d(v_2, u_2) \) and hence by Corollary 5.8, \( u_1 \) and \( u_2 \) are not strongly resolved by any of \( v_1 \) and \( v_2 \), a contradiction.

Theorem 5.26. Let \( \{v_1, v_2\} \) be a cluster basis of \( G \) with \( \beta_c(G) \) then degree of both \( v_1 \) and \( v_2 \) are less than or equal to three.

Proof. Let \( d(v_1, v_2) = d \). Then any vertex adjacent to \( v_1 \) is at distance \( d - 1, d \) or \( d + 1 \) from \( v_2 \). Since any pair of vertices that are adjacent to \( v_1 \) are not...
resolved by \( v_1 \) and are to be resolved by \( v_2 \), the distances from these vertices to \( v_2 \) are different. Hence the number of vertices adjacent to \( v_1 \) does not exceed three. In other words, \( \deg(v_1) \leq 3 \). Similarly \( \deg(v_2) \leq 3 \).

**Theorem 5.27.** Let \( \{v_1, v_2\} \) be a cluster basis of \( G \) where \( \beta_c(G) = 2 \). For any vertex \( v \) on the unique shortest path between \( v_1 \) and \( v_2 \), there exists at most one vertex adjacent to it in the distance partite set with respect to \( v_1 \) to which it belongs to. Further, \( v \) has exactly one vertex adjacent to it in the preceding distance partite set.

**Proof.** The proof is immediate from the fact that the shortest path between \( v_1 \) and \( v_2 \) is unique and \( \beta_c(G) = 2 \).

**Theorem 5.28.** Let \( u \) be a vertex on the unique shortest path, say \( p \), between \( v_1 \) and \( v_2 \), which lies in \( V_i \) with \( 2 \leq i \leq k \). Let \( d(V_j, u) = d_j \), \( j = 1, 2 \). Then

1. **Case (i)** If \( d_1 = d_2 + 1 \) then degree of \( u \) in the induced graph \( \langle V_j \rangle \) is 0.
2. **Case (ii)** If \( d_1 \neq d_2 + 1 \) then \( u \) is adjacent to at most one vertex say \( w \), in the induced graph \( \langle V_i \rangle \) with \( d(v_2, w) = d_2 + 1 \). Degree of \( u \) in \( \langle V_j \rangle \) is 1.

**Proof.**
1. **Case (i):** Let \( d_1 = d_2 + 1 \) and \( u \) is adjacent to a vertex \( x \) in \( V_i \). Then \( d(v_1, x) = d(v_2, x) = d_1 \) and is not strongly resolved by any of \( v_1 \) and \( v_2 \), a contradiction.

2. **Case (ii):** Let \( d_1 \neq d_2 + 1 \) then, because the path between \( v_1 \) and \( v_2 \) is unique, \( u \) is adjacent to exactly one vertex in \( V_i \) and hence degree \( u \) in \( \langle V_i \rangle \) is equal to one.

**Theorem 5.29.** Let \( u \) be any vertex in \( V_i \) with \( 2 \leq i \leq k \) and \( d(v_j, u) = d_j \) for \( j = 1, 2 \).

1. **Case (i)** If \( d_1 = d_2 + 1 \) or \( d_2 = d_1 + 1 \) then degree of \( u \) in \( \langle V_j \rangle \) is 1.
2. **Case (ii)** If \( d_1 = d_2 + k \) or \( d_2 = d_1 + k \) for \( k \geq 2 \) then degree of \( u \) in \( \langle V_j \rangle \) is 2.

**Proof.**
1. **Case (i):** let \( u \) be a vertex in \( V_i \) with \( 2 \leq i \leq k \) and \( v \) be a vertex adjacent to \( u \) in \( V_i \) then the distance of \( v \) from \( v_2 \) is either \( (d_2, d_2 - 1, d_2 + 1) \). When
$d_2 = d_1 + 1$ except for $d(V_2, v) = d_2 + 1$ the vertex $V$ is either not resolved or not strongly resolved by $v_2$ then degree of $u$ in $\langle V_j \rangle$ is $\leq 1$.

Case (ii): let $u$ be a vertex in $V_i$ and $v$ be a vertex adjacent to $u$ in $V_i$ for $2 \leq i \leq k$ then $d(V_2, v) = d$ is not possible. Then, both the cases in which $d(V_2, v) = d_2 + 1$ or $d_2 - 1$ are possible. Hence the degree of $u$ in $\langle V_j \rangle$ is $\leq 2$.

**Theorem 5.30.** The Maximum degree of any vertex on the unique shortest path is five and

Case (i): if $(d(u, v_1) - d(u, v_2)) = 1$ then maximum degree of $u$ is four

Case (ii): if $(d(u, v_1) - d(u, v_2)) > 2$, and odd then the maximum degree of $u$ is five which can be attained through the following structure.

**Proof.** Proof is immediate from the above Theorem 5.29.

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**Theorem 5.31.** Let $\{v_1, v_2\}$ be a cluster basis of $G$, where $\beta_c(G) = 2$. Consider distance partite sets $V_0, V_1, V_2, \ldots, V_k$ with reference to $v_1$. Any connected component of the graph induced by a distance partite set is a path and in fact, the degree of any vertex in the graph induced by the distance partite set is at most two.
Proof. Let $V_j$ be a distance partite set and $C$ be a connected component in the induced graph $\langle V_j \rangle$. Further, let $u$ be among vertices in $C$ such that $d = d(u,v_2) = \min_{v \in V} d(v,v_2)$. Since $v_2$ resolves every pair of vertices in $V_j$, the choice of $u$ is unique. Then any vertex adjacent to $u$ say $w$, is at distance $d + 1$ from $v_2$, any vertex adjacent to $w$, say $x$, is at distance $d + 2$ from $v_2$ and so on. In fact, for any $v \in C$, $d(v_2,v) = d(v_2,u) + d(u,v)$. Thus the component $C$ is a path and the second part is trivial.

Theorem 5.32. In a graph $G$ with $\beta_c(G) = 2$ and for a triangle $T$ in $G$, if any, all the vertices of $T$ cannot be at the same distance from $v_1$ or $v_2$.

Proof. Otherwise, all the three vertices of $T$ are found in the same distance partite set and the graph induced by it consists of a triangle $T$, a cycle, a contradiction to Theorem 5.31.

Theorem 5.33. For any graph $G$ with $\beta_c(G) = 2$, the cluster basis of $G$ cannot have a vertex $v$ of a sub graph $K_4$ of $G$.

Proof. Take $v_1$ as one of the vertices of $K_4$ and consider the distance partition of $V(G)$ with reference to $v_1$. Then the distance partite set $V_1$ has the other three vertices of $K_4$ which induce a cycle, a contradiction to Theorem 5.31.

Theorem 5.34. A graph $G$ with $\beta_c(G) = 2$ cannot have $K_5$.

Proof. As diameter of $K_5$ is one, vertices of $K_5$ are to be there in at most two consecutive distance partite sets. Then at least one among the possible two sets contain three or more vertices of $K_5$ which induces a cycle, which is not a path. Hence $G$ cannot have $K_5$.

Definition 5.8. The shortest path from vertex $v_1$ to a vertex $u_j$ of $V_j$ is said to be downward extendable if there exists vertex $u_{j+1} \in V_{j+1}$ such that $u_{j+1}$ is adjacent to $u_j$. Note that the path $v_1 \rightarrow, \cdots, \rightarrow u_j, \rightarrow u_{j+1}$ is the shortest path from $v_1$ to $u_{j+1}$. Any path $v_1 \rightarrow, \cdots, \rightarrow u_j \rightarrow u_{j+1} \rightarrow, \cdots, \rightarrow u_{j+1}$ is said to
be a downward extension of a path \( v_1 \to \cdots \to u_j \). In the following theorem, we shall observe that a maximal downward extension of the unique shortest path \( v_1 \to \cdots \to v_2 \) is unique.

**Theorem 5.35.** Let \( \{v_1, v_2\} \) be a cluster basis of \( G \) where \( \beta_c(G) = 2 \). The maximal downward extension of the unique shortest path \( v_1 \to v_2(v_2 \in V_j) \) is unique and has at most one vertex \( u_{j+1} \) from \( V_{j+1} \) where \( 0 \leq t \leq k - j \).

**Proof.** Suppose that \( v_1 \to v_2 \) is downward extendable. Note that if any downward extended path \( v_1 \to \cdots \to v_2 \to \cdots \to u_{j+1} \) is further extendable, then such an extension is unique. Otherwise, for two different extensions \( v_1 \to \cdots \to v_2 \to \cdots \to u_{j+1} \) and \( v_1 \to \cdots \to v_2 \to \cdots \to u_{j+1} \to \hat{u}_{j+t+1} \), where \( u_{j+t+1} \) and \( \hat{u}_{j+t+1} \in V_{j+t+1} \), we obtain \( d(v_2, u_{j+t+1}) = d(v_2, \hat{u}_{j+t+1}) = t + 1 \), and hence are not resolved by any of \( v_1 \) and \( v_2 \), a contradiction. Hence the maximal downward extension of the unique shortest path \( v_1 \to \cdots \to v_2 \) is unique.

**Theorem 5.36.** The maximum degree of any vertex in a graph \( G \) with \( \beta_c(G) = 2 \) is

- **Case (i):** If \( (d(u, v_1) - d(u, v_2)) = 1 \), then maximum degree of \( u \) is six
- **Case (ii):** If \( (d(v, v_1) - d(v, v_2)) = 2 \), then maximum degree of \( v \) is seven
- **Case (iii):** If \( (d(w, v_1) - d(w, v_2)) > 2 \), then maximum degree of \( w \) is Eight

**Proof.** Case(i): Let \( u \) be a vertex in \( V_i \) then by Theorem 5.31 the degree of vertex \( u \) in \( \langle V_i \rangle \leq 1 \). Hence the maximum degree possible in same partite set is 1. i.e; it can be adjacent to one of the vertex in \( V_i \). Only two vertices in \( V_{i-1} \) can be adjacent to \( u \) and three vertices in \( V_{i+1} \) can be adjacent to \( u \). Hence maximum degree of \( u \) in \( V_i \) is six. Case(ii): Let \( v \) be a vertex in \( V_i \) then by Theorem 5.31 the degree of vertex \( v \) in \( \langle V_j \rangle \leq 2 \). Hence the maximum degree possible in same partite set is 2 i.e; it can be adjacent to two vertices in \( V_j \). Only two vertices in \( V_{i-1} \) can be adjacent to \( v \) and three vertices in \( V_{i+1} \) can be adjacent to \( v \). Hence maximum degree of \( v \) in \( V_i \) is six. Case(iii): By corollary
Figure 5.2: The Maximum Degree of any vertex in graph $G$ when $\beta_c(G) = 2$

and by Theorem 5.33, given any vertex $w \in V_i$, it can be adjacent to at most three vertices each from $V_{i-1}$ and $V_{i+1}$ and at most two vertices from $V_i$. Hence degree of $w$ is at most eight. In the following, we observe a graph $G$ with $\beta_c(G) = 2$ and a vertex of $G$ having degree eight. As shown in Figure 5.2.

Note 5.37. The above theorem gives an upper bound for degree of any vertex in a graph $G$ with $\beta_c(G) = 2$.

**Theorem 5.38.** Let $G$ be a graph with $\beta_c(G) = 2$ and $\{v_1, v_2\}$ be a cluster basis for $G$. If a triangle in $G$ exists it should have all its vertices strongly resolved by exactly one of the vertices $v_1$ or $v_2$.

**Proof.** Let $u, v$ and $w$ be the vertices of the triangle. Let $d(v_1, u) = d_1$ and $d(v_2, u) = d_2$. Let $d_1 < d_2$, if $d_1 = d_2 - 1$ then $(d_1, d_2 - 1)$ cannot be the distance pair for vertex $v$ because, when $d_1 = d_2 - 1$ and $(d_1, d_2 - 1)$ is the distance pair for $v, v$ will not be strongly resolved either by $v_1$ or $v_2$. Hence the distance pair for $v$ is $(d_1, d_2 + 1)$. As $w$ cannot be in the same partite set of $u$ & $v$, suppose that $w$ belongs to the next partite set as shown in the Figure 5.3. The possible distance pair for $w$ are $(d_1 + 1, d_2)$ and $(d_1 + 1, d_2 + 1)$.
• Case(i): Suppose the distance pair for \( w \) is \( d_1 + 1, d_2 \). We have \( d_1 < d_2 \), therefore \( d_1 + 1 \leq d_2 \). But \( d_1 + 1 \neq d_2 \) for the coordinates when \( d_1 + 1 = d - 2 \), \( w \) is not strongly resolved by either \( v_1 \) or \( v_2 \), therefore \( d_1 + 1 < d_2 \).

• Case(ii): Let \((d_1 + 1, d_2 + 1)\) be the distance pair for \( w \). We have \( d_1 < d_2 \). When \( d_1 + 1 < d_2 + 1 \). Hence in all possible cases with \( d_1 < d_2 \), the vertex \( u, v \) and \( w \) are all in the same cluster which is strongly resolved by \( v_1 \).

From the following theorem, this can easily draw conclusion. As any three vertices in a complete graph induce a triangle, the entire complete graph, whenever it exists should belong to exactly one of the clusters. Similarly any wheel graph \( W_{1,n} \) and the graph \( K_5 - 2e \), whenever it exists, should belong to exactly one of the two clusters.

**Note 5.39.** If \( G \) is any graph with \( \beta_c(G) = 2 \) then there exist a pair of adjacent vertices \( u \) and \( v \) such that there is no vertex \( w \) which is adjacent to both \( u \) and \( v \).

**Theorem 5.40.** Let \( \{v_1, v_2\} \) be a cluster basis of \( G \), where \( \beta_c(G) = 2 \); then \( G \) cannot have \( K_5 - e \) as a sub graph.
Proof. Since the graph induced by any distance partite set can have only component of paths (not triangles) and isolated vertices, vertices of \( K_5 - e \) are distributed as three \((u_1, u_2, u_3)\) in one distance partite set, say \( V_i \), and other two \((u_4, u_5)\) in an adjacent distance partite set, \( V_{i-1} \) or \( V_{i+1} \) as shown in the Figure 3.11 in which case two of the three vertices \( u_1, u_2, u_3 \) are of degree three in \( K_5 - e \) and the remaining vertices are of degree four in \( K_5 - e \). Without loss of generality, assume that \( u_1 \) and \( u_3 \) are of degree three and \( u_2, u_4, u_5 \) are of degree four in \( K_5 - e \), as shown in the Figure 3.11. Note that \( u_1, u_2, u_3 \) are pair wise resolvable by \( v_2 \) and so are \( u_4 \) and \( u_5 \). Now consider \( u_4 \) which is adjacent to all the remaining four vertices and let \( d(v_2, u_4) = d \). Further, as \( u_1, u_2 \) and \( u_3 \) are resolved pairwise by \( v_2 \) and are adjacent to \( u_4 \) \( d(v_2, u_j), (j = 1, 2, 3) \) takes distinct values among \( d + 1, d, d - 1 \) since \( u_5 \) is also adjacent with all three vertices \( u_1, u_2 \) and \( u_3 \) for \( i = 1, 2, 3 \). We get \( d(v_2, u_5) = d \), a contradiction to the fact that \( u_4 \) and \( u_5 \) are resolved by \( v_2 \).

Remark 5.41. From Theorem 5.40 it is clear that neither \( K_5 \) nor \( K_5 - e \) can be a sub graph of a graph with cluster dimension two. So it is of natural curiosity as to how further smaller sub graph of \( K_5 \) can be excluded from being a sub graphs of a graph from the class of graphs with cluster dimension two. In the following we realize that \( K_5 - 2e \) could be a sub graph of the same graph \( G \) with \( \beta_c(G) = 2 \).

Theorem 5.42. Let \( \{v_1, v_2\} \) be a cluster basis of \( G \), where \( \beta_c(G) = 2 \). Suppose \( e(v_1) = k \) and \( |V(G)| = n \). Then the eccentricity of the second strongly resolving vertex \( v_2 \) is greater than or equal to \( \left\lfloor \frac{n-4}{k-1} \right\rfloor \), \( k \neq 1 \), where \( [x] \) is the integer part of the number \( x \).

Proof. Let \( e(v_1) = k \) and \( \{V_0, V_1, \ldots, V_k\} \) be the distance partition of \( V(G) \) with reference to \( v_1 \). Then there is at least one distance partite set with a number of vertices greater than or equal to \( \left\lfloor \frac{n-4}{k-1} \right\rfloor \), \( k \neq 1 \), which are all to be resolved by \( v_2 \). Hence \( e(v_2) \geq \left\lfloor \frac{n-4}{k-1} \right\rfloor \), \( k \neq 1 \).
Remark 5.43. In a graph $G$ with $\beta_c(G) = 2$ and cluster basis $\{v_1, v_2\}$, if $e(v_2) = s$ then any distance partite set $V$ with respect to the vertex $v_1$ other than the one with $v_2$ in it contains at most $s$ vertices (to be precise, not more than $s + 1 - d(V, v_2)$ where $d(V, v_2)$ is the shortest of distances between any vertex $v$ of $V$ and $v_2$, distance partite set that contains $v_2$ may have $s + 1$ vertices.

Proof. Proof is presented in Corollary 5.8 and Theorem 5.31. \qed

Note 5.44. In other words, if $e(v_2) = s$ then there is no partite set with a number of vertices greater than $s$, except the one which contains $v_2$. It can have at most $s + 1$ vertices.

Though the theorem and the corollary provide much stronger result, for a general understanding of the structure of $G$ with $\beta_c(G) = 2$, we give the following results:

**Theorem 5.45.** Let $G$ be a graph with $\beta_c(G) = 2$ and $\{v_1, v_2\}$ be a cluster basis of $G$. Let $P$ be the Petersen graph. Then neither of $v_1$ and $v_2$ are in $V(P)$. Further, if eccentricity of any of $v_1$ and $v_2$ is not more than three, then $P$ cannot be a sub graph of $G$.

Proof. Consider distance partite sets $\{V_0, V_1, V_2, \ldots, V_k, \}$ with reference to $v_1$. If $v_1 \in V(P)$, then $V_2$ consists of at least six vertices of $V(P)$ which induces a cycle in $V_2$ which is a contradiction. Hence $v_1 \notin V(P)$. Similarly $v_2 \notin V(P)$.

Suppose that $P$ is a sub graph of $G$ and $e(v_2) = 3$. Now consider distance partite sets with reference to $v_1$. From the remark 5.43 at most one $V_j$ which contains $v_2$ may have four vertices and the remaining $V_j$’s have no more than three vertices. As $v_1 \notin V(P)$ and diameter of $P = 2$, $V(P)$ is distributed among three $V_j$’s so that one has four vertices of $V(P)$ and other two have three each. This implies $v_2 \notin V(P)$, a contradiction. \qed

**Theorem 5.46.** Let $G$ be a graph with $\beta_c(G) = 2$ then there is no connected sub graph $H$ of $G$ such that diameter of $H < \sqrt{m - 1}$, where $m$ is the cardinality.
of $V(H)$.

**Proof.** Consider a cluster basis $\{v_1, v_2\}$ of $G$, where $\beta_c(G) = 2$, and the distance partition $\{V_0, V_1, V_2, \ldots, V_k\}$ of $V(G)$ with reference to one among the basis elements, say $v_1$. Let $H$ be any connected sub graph of $G$ with diameter of $H$ equal to $D$. Any pairs of vertices, among vertices of $H$ and in the same partite set, (say $V_j$), are resolved by $v_2$. Since the distance between any pair of vertices from $\{v_h|v_h \in V(H) \cap V_j\}$ is not more than diameter $D$ of $H$, $d(v_2,v_h)$ takes distinct values among $d, d+1, \ldots, d+D$ where $d = \min_{v \in H \cap V_j} \{d(v,v_2)\}$. So, the cardinality of $H \cap V_j$ is at most $D+1$. Further, as diameter $H = D$, the vertices of $H$ could be distributed among at most $D+1$ consecutive $V_i$’s. Hence the cardinality of $H$ is at most $(D+1)(D+1)$. That is $m \leq (D+1)^2$, where $m$ is cardinality of $V(H)$. Therefore $\sqrt{m-1} \leq D$. This proves the result. 

**Corollary 5.47.** The complete graph $K_5$ or the Petersen graph $P$ cannot be a sub graph of a graph $G$ with $\beta_c(G) = 2$.

**Proof.** Proof is directly available in the Theorem 5.46 and $K_5$ is of diam 1 with order 5, and Peterson graph is of diam 2 with order 10.

**Lemma 5.48.** Let $G$ be a graph with $\beta_c(G) = 2$ and $\{v_1, v_2\}$ be a cluster basis of $G$. Further, let $\{V_0, V_1, V_2, \ldots, V_k\}$ be the distance partition of $V(G)$ with reference to the vertex $v_1$. Then every distance partite set can have at most two vertices more than the maximum possible cardinality of the preceding distance partite set.

**Proof.** Consider a distance partite set $V_i$ and let $V_{i-1}$ have $m$ vertices. Let $d(v_2,u_j) = d-i, d-i+1, \ldots, d-i+m, (\hat{m} \geq m)$, where $u_j \in V_{i-1}$. As every vertex in $V_i$ is adjacent to one or the other vertices in $V_{i-1}$, $d(v_2, w_i)$ where $w_i \in V_i$ can take one of the distinct values $d-i-1, d-i, \ldots, d-i+\hat{m}+1$. Thus, if $V_{i-1}$ has a maximum of $\hat{m}+1$ vertices then $V_i$ has a maximum of $\hat{m}+1+2$ vertices.
Theorem 5.49. Let $G$ be a graph with $\beta_c(G) = 2$ and $\{v_1, v_2\}$ be a cluster basis of $G$. Further, let $\{V_0, V_1, V_2, \ldots, V_k\}$ be the distance partition of $V(G)$ with reference to one of the vertices in the cluster basis. Then the maximum number of vertices in any distance partite set, say $V_i$, for $0 \leq i \leq k$ is $(2i + 1)$.

Proof. Proof is by mathematical induction and induction is applied on $i$, the suffix of $V_i$ for $0 \leq i \leq k$. The result is true for $i = 0$ and 1. Assume the result for $i$. i.e., $V_i$ has at most $(2i + 1)$ vertices. By the previous lemma 5.48 $V_{i+1}$ can have at most the vertices more than $(2i + 1)$. Hence $V_{i+1}$ can have at most $2i + 3 = 2(i + 1) + 1$ vertices. By mathematical induction the result follows for any positive integer $i$.

5.5 Bound For Number of Vertices in a Graph $G$ With $\beta_c(G) = 2$

In the following a sharper bound for a number of vertices in a graph $G$ with $\beta_c(G) = 2$ is given.

Let $G$ be a graph with $\beta_c(G) = 2$ and $\{v_1, v_2\}$ be a cluster basis of $G$. Further, let $\{V_0, V_1, V_2, \ldots, V_k\}$ be the distance partition of $V(G)$ with reference to $v_1$ and $v_2 \in V_i$ where $i$ is odd. Let $e(v_2) = s$. $V_i$ can have at most $(s + 1)$ vertices only if $(s + 1) \leq 2i + 1$. Thus to find the maximum number of vertices that $G$ with $\beta_c(G) = 2$ can have, the following two cases are considered.

Case (i): $(s + 1) \geq 2i + 1$, i.e. $s \geq 2i$.

Let $v_2$ be placed in $V_i$. the partite sets $V_i, V_{i-1}, V_{i-2}, \ldots, V_0$ can have respectively at most $2i + 1, 2i - 1, \ldots, 2(i - 1) + 1, \ldots, 3, 1$ vertices. Find $l$ such that $l \leq \frac{s - 2i}{2}$. Then the distance partite sets $V_{i+l}, V_{i+2}, V_{i+3}, \ldots, V_{i+l}$ can have respectively at most $2(i + 1) + 1, 2(i + 2) + 1, \ldots, 2(i + l) + 1$ vertices. Beyond $V_{i+1}$ the distance partite sets $V_{i+l+1}, V_{i+l+2}, \ldots, V_k$ can have respectively at most $s - l, s - l - 1, \ldots, 3, 2, 1$ vertices. But no distance partite set can have a vertex.
which is equidistance from both the cluster basis elements $v_1$ and $v_2$. Immediately above $V_i$, each of the $\frac{i-1}{2}$ distance partite sets contains exactly one vertex which is equidistant from both $v_1$ and $v_2$. Each of the distance partite sets $V_i, V_{i+1}, \ldots, V_s$ contains exactly one vertex which is equidistant from both $v_1$ and $v_2$. These many vertices are to be subtracted from the total number of vertices. Thus formula $e$ for maximum number of vertices possible in a graph $G$ with $\beta_c(G) = 2$ is given by,

$$|V| = \left( \sum_{r=1}^{i} (2r + 1) + \sum_{r=1}^{k} (2i + 2r + 1) + \sum_{n=1}^{s-k} n \right) - \left( \frac{i-1}{2} + (s-i) + 1 \right)$$

$$= \left( (i-1)^2 + (2ik + (k+1)^2 - 1) + \frac{(s-k)(s-k+1)}{2} \right) - \left( \frac{i-1}{2} + (s-i) + 1 \right)$$

Case (ii): $(s+1) < 2i + 1$, i.e. $s < 2i$.

Let $l = s - i$. Then the distance partite sets $V_{i+1}, V_{i+2}, \ldots, V_i$ respectively can contain atmost $2(s-i) + 2, \ldots, (s-2), (s-1), s$ and $s+1$ vertices. And the partite sets $V_i, V_{i-1}, \ldots, V_0$ can respectively contain atmost $2l+1, 2l-1, \ldots, 3, 1$ vertices. The distance partite sets $V_{j+1}, V_{j+2}, \ldots, V_k$ respectively contain atmost $s, s-1, s-2, \ldots, 3, 2, 1$ vertices as the vertices are to be strongly resolved by either $v_1$ or $v_2$. We have to subtract $\left( \frac{i-1}{2} + (s-i) + 1 \right)$ from the total number of vertices possible. Thus the maximum number of vertices in a graph $G$ with $\beta_c(G) = 2$ is given by,

$$|V| = \left( \sum_{k=0}^{s-i+1} (2K + 1) + (2s + 1) + \sum_{k=1}^{2j-s-1} (s-k) + \sum_{k=1}^{s} k \right) - \left( \frac{i-1}{2} + s - i + 1 \right)$$

$$= \left( (s-i)^2 + 2s(2i-s+1) + 2 - (2i-s-1)(2i-s) + s(s+1) \right)$$

$$- \left( \frac{i-1}{2} + s - i + 1 \right).$$
5.6 Characterization of Graphs With Cluster Dimension Two

The following is the characterization of graphs with metric dimension two.

**Theorem 5.50.** Let $G$ be a graph which is not a path with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\{V_{v_0}, V_{v_1}, \ldots, V_{v_k}\}$ be the distance partition of $V(G)$ with reference to the vertex $v_i$ where $k_i$ is the eccentricity of $v_i$, $1 \leq i \leq n$. The metric dimension of $G$ is 2 if and only if there exist vertices $v_i$ and $v_j$ such that $|V_i \cap V_j| \leq 1$ for every $k \neq L$ and when $k = l$ then $|V_i \cap V_j| = 0$ with $1 \leq k \leq e(v_i)$ and $1 \leq l \leq e(v_j)$.

*Proof.* Given $v_p$ and $v_r$, $|V_{pq} \cap V_{rs}| > 1$ for some $p_q$ and $r_s$ it implies that there exists at least two vertices, say $u_1$ and $u_2$ in $V_{pq} \cap V_{rs}$ such that $d(v_p, u_1) = d(v_p, u_2) = q$ and $d(v_q, u_1) = d(v_q, u_2) = s$ and hence $u_1$ and $u_2$ are not resolved by both $v_p$ and $v_r$ so, $|V_{pq} \cap V_{rs}| > 1$ for all $p_q$ and $r_s$ implies no pair of vertices $v_p$ and $v_r$ resolves $V(G)$, in other words $\beta(G) \neq 2$.

Conversely if there exist $v_p$ and $v_r$ such that $|V_{pq} \cap V_{rs}| \leq 1$ for all $p_q$ and $r_s$, then given any pair of vertices $w_1$ and $w_2$ from $V(G)$ we have $w_1 \in V_{p_{q}} \cap V_{r_{s1}}$ and $w_2 \in V_{p_{q2}} \cap V_{r_{s2}}$ where at least $p_{q1}$ is different from $p_{q2}$ or $r_{s1}$ is different from $r_{s2}$. This implies that $w_1$ and $w_2$ are resolved by at least one of $v_p$ and $v_r$. Further, since $|V_{pq} \cap V_{rs}| = 0 \forall q$ all vertices in $G$ are strongly resolved by the set $\{v_p, v_r\}$.

So $\beta_c(G) \leq 2$ and in fact, $\beta_c(G) = 2$ as $G$ is not a path.

5.6.1 Algorithm to Check Whether the Cluster Dimension of a Given Graph $G$ is Two

The following algorithm follows from Theorem 5.50

**Step 1: Input: distance matrix**

Input is the distance matrix ordered according to vertices of a graph which is not a path.
Step 2: Selection of suitable cluster basis

1. If the number of vertices in $G$ i.e $|V|_{\beta(c)=2} > \begin{cases} D^2 - D + 2 & \text{for } D \leq 3 \\ D^2 - 3D + 10 & \text{Otherwise} \end{cases}$ where $D$ is the diameter of the graph $G$ then $\beta_c(G) \neq 2$.

2. If the degree of any vertex is greater than three reject that vertex (that vertex cannot be a part of cluster basis) (Samir Khuller et al).

3. If $d(v_1, v_2)$ is even reject it.

4. If eccentricity of a vertex $v_2$ is less than the number of vertices at distance $j$ from the vertex $v - 1$, then $v_2$ cannot form a cluster basis for $G$ with $v_1$ (Theorem).

5. If $e(v_1) \leq \frac{1}{2}e(v_1)$, then reject it.

Step 3: Selection of vertices for finding metric basis

i) Select only those vertices in $G$ with a degree less than or equal to three.

Step 4: Formation of distance partitions

Form distance partition $\{V_{i_0}, V_{i_1}, V_{i_2}, \ldots, V_{i_k}\}$ of $G$ with reference to every vertex $v_1$ having degree less than or equal to three $1 \leq i \leq n$. With reference to the cluster basis if the vertices in the $j^{th}$ partite set is greater than $(2j + 1)$. Then reject it.

Step 5: Identify the pair of vertices for finding cluster basis

1. Given a pair $(u_1, u_2)$ if the eccentricity of a vertex $u - 2$ is less than number of vertices at distance $d(u_1, u_2)$ and vice versa then $\{u_1, u_2\}$ cannot be a cluster basis for $G$. Consider only a remaining pairs.
2. Among the pairs \((u_i, u_j)\) remaining, consider only the pairs with the unique shortest path between them.

**Step 6: Find intersection**

If there exists vertices \(v_i\) and \(v_j\) \((i \neq j)\) with \(|V_{v_i} \cap v_{v_j}| = 0\) for every \(k\) and \(l\) with \(1 \leq k \leq e(v_i)\) and \(1 \leq l \leq e(v_j)\), then \(\{v_1, v_2\}\) is a cluster basis for the graph \(G\). Otherwise the cluster dimension of \(G\) is not equal to two.
5.7 Complexity

Let $G$ be a graph with diameter $D$ on $n$ vertices. Every set in distance partition of $V(G)$ with reference to a vertex $v$ is to be compared with at most $(n-1)D$ sets. Therefore totally there are $((n-1)D)D$ comparisons for $v$. For the next vertex the number of comparisons needed is $((n-2)d^2)$. Similarly for the last vertex the number of comparisons needed is $(n-(n-1))D^2$. Therefore the total number of set comparisons required is $((n-1)D^2+(n-2)D^2+\ldots+1.D^2) = D^2 \left[ \frac{n(n-1)}{2} \right]$. In every comparison of two sets there can be at most $D^2$ comparisons of elements. Hence the total number of element comparisons is $D^2 \left[ \frac{n(n-1)}{2} \right] D^2$. Thus the complexity of the algorithm is $\left[ \frac{n(n-1)}{2} \right] D^4$.

5.8 Conclusion

In this chapter, the distance partition of vertex set of a graph $G$ is defined with reference to a vertex in it and with the help of the same, a graph with cluster dimension two (i.e. $\beta_c(G) = 2$) is characterized. In the process, a polynomial time algorithm is developed, which verifies that if the cluster dimension of a given graph $G$ is two. The same algorithm explores all cluster bases of graph $G$ whenever $\beta_c(G) = 2$. A bound for cardinality of any distance partite set with reference to a given vertex is found, whenever $\beta_c(G) = 2$. Also, in a graph $G$ with $\beta_c(G) = 2$, a bound for cardinality of any distance partite set as well as a bound for number of vertices in any sub graph $H$ of $G$ is obtained in terms of $\text{diam } H$. 