CHAPTER 5

REDUCTION OF QUANTUM PHASE FLUCTUATIONS IN INTERMEDIATE STATES

In the introductory quantum mechanics, it is often told that for every physical observables of classical world there exists a Hermitian quantum mechanical operator. This is not true for phase. Dirac was the first person who tried to introduce a Hermitian phase operator in 1927 [96]. Immediately after Dirac’s introduction of Hermitian phase operator, it was realized that Hermitian phase operators have some ambiguities (interested readers can see the review [97]). The primary observation in this regard was due to Louisell [98] who had shown that most of the problems are associated with the bare phase operators but sine and cosine operators can circumvent some of the problems. This observation lead to several formalisms ([99-101] and references there in) of quantum phase. Among the different formalisms, Susskind Glogower (SG) [99], Pegg Barnett [100] and Barnett Pegg (BP) [101] formalisms played most important role in the studies of phase properties and the phase fluctuations of various physical systems (see [71] and references there in). Quantum phase problem has attracted attention of physicist from several decades and that lead to many interesting studies [96-110]. These studies may be classified as introduction of phase operators [99-101] and study of quantum phase fluctuations [102-106].
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All the studies on quantum phase fluctuation properties essentially use the parameters introduced by Carruthers and Nieto [102]. For example, Pathak and Mandel [106], Gerry [103, 104] and Lynch [105] have studied quantum phase fluctuations of different physical systems by using Carruthers Nieto parameters and recently Pathak and Gupta [71] have shown that the reduction of the Carruthers-Nieto symmetric quantum phase fluctuation parameter \( U \) with respect to its coherent state value corresponds to an antibunched state, but the converse is not true. Consequently reduction of \( U \) is a stronger criterion of nonclassicality than the lowest order antibunching. In this chapter we have studied the possibilities of observing reduction of \( U \) in intermediate states by using the Barnett Pegg formalism. We have shown that the reduction of phase fluctuation parameter \( U \) can be observed in different intermediate states, such as binomial state, generalized binomial state, hypergeometric state, negative binomial state, and photon added coherent state. It is also shown that the depth of nonclassicality can be controlled by various parameters related to intermediate states. Further, in this chapter we have provided specific examples of antibunched states, for which \( U \) is greater than its Poissonian state value.

5.1 Introduction

In the earlier chapters we have already mentioned that the standard deviation of an observable is normally considered to be the most natural measure of quantum fluctuation [92] associated with that observable and the reduction of quantum fluctuation below the coherent state level corresponds to a nonclassical state. But standard deviations can also be combined to form some complex measures of nonclassicality, which may increase with the increasing nonclassicality. For example, the total noise of a quantum state, which is a measure of the total fluctuations of the amplitude of two non-commuting quadrature components of the field, increases with the increasing nonclassicality in the system [92]. Analogous complex measure of quantum phase fluctuations was introduced by
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Carruthers and Nieto [102]. Carruthers-Nieto parameters have been used to study quantum phase fluctuations of coherent light interacting with a nonlinear non-absorbing medium of inversion symmetry [103-106].

The intermediate states are nonclassical in general. In the previous chapters it is already shown that most of the intermediate states satisfy the conditions of higher order antibunching, higher order squeezing and higher order subpoissonian photon statistics. Normally all the criteria of higher order nonclassicality are stronger than their lower order counter part. As the reduction of $U$ is also stronger than the usual antibunching criterion. It is of an outstanding curiosity to check whether this particular stronger criterion of nonclassicality is satisfied by intermediate states or not. In this chapter we shall show that the intermediate states may satisfy this stronger nonclassical criterion (i.e. reduction of $U$ criterion).

The importance of a systematic study of quantum phase fluctuation of intermediate state has also increased with the recent observations of quantum phase fluctuations in quantum computation [108, 109] and superconductivity [107, 110] and with the success in experimental production of photon added coherent state [67]. These observations along with the fact that intermediate states satisfy stronger criteria of nonclassicality have motivated us to study quantum phase fluctuation of intermediate states. In next section we briefly introduce quantum phase fluctuation parameter ($U$) and the meaning of reduction of $U$. In section 5.3, it is shown that the reduction of phase fluctuation parameter $U$ can be seen in different intermediate states, such as binomial state, hypergeometric state, generalized binomial state, negative binomial state and photon added coherent state. Role of various parameters in controlling the depth of nonclassicality is also discussed. Finally in section 5.4 we conclude.
5.2 Measures of quantum phase fluctuations

Dirac [96] introduced the quantum phase operator in 1927. Immediately after Dirac’s introductory work, it was realized that the uncertainty relation $\Delta N \Delta \phi \geq \frac{1}{2}$ associated with Dirac’s quantum phase has many problems [97]. Later on Louisell [98] had shown that most of the problems can be solved if instead of bare phase operator we consider sine ($S$) and cosine ($C$) operators which satisfy

$$[N, C] = -iS \quad (5.1)$$

and

$$[N, S] = iC. \quad (5.2)$$

Therefore, the uncertainty relations associated with them are

$$\Delta N \Delta C \geq \frac{1}{2} |\langle S \rangle| \quad (5.3)$$

and

$$\Delta N \Delta S \geq \frac{1}{2} |\langle C \rangle|. \quad (5.4)$$

There are several formalism of quantum phase, and each formalism defines sine and cosine in an unique way. Most convenient way is to rescale an appropriate quadrature operator with the averaged photon number. Barnett and Pegg followed this convention and defined the exponential of phase operator $E$ and its Hermitian conjugate $E^\dagger$ as [101]

$$E = (\bar{N} + \frac{1}{2})^{-1/2} a$$

$$E^\dagger = (\bar{N} + \frac{1}{2})^{-1/2} a^\dagger \quad (5.5)$$
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where $\overline{N}$ is the average number of photons present in the radiation field after interaction. The usual cosine and sine of the phase operator are defined as

\[
C = \frac{1}{2} (E + E^\dagger) \\
S = -\frac{i}{2} (E - E^\dagger)
\]  

(5.6)

which satisfy

\[
\langle C^2 \rangle + \langle S^2 \rangle = 1.
\]  

(5.7)

Squaring and adding (5.3) and (5.4) we obtain

\[
(\Delta N)^2 \left[ (\Delta S)^2 + (\Delta C)^2 \right] / \left[ \langle S \rangle^2 + \langle C \rangle^2 \right] \geq \frac{1}{4}.
\]  

(5.8)

Carruthers and Nieto [102] introduced (5.8) as measure of quantum phase fluctuation and named it as $U$ parameter. To be precise, Carruthers and Nieto defined following parameter as a measure of phase fluctuation:\footnote{They had also introduced two more parameters $S$ and $Q$ for the purpose of calculation of the phase fluctuations. But these parameters are not relevant for the present work.}

\[
U (\theta, |\alpha|^2) = (\Delta N)^2 \left[ (\Delta S)^2 + (\Delta C)^2 \right] / \left[ \langle S \rangle^2 + \langle C \rangle^2 \right]
\]  

(5.9)

where, $\theta$ is the phase of the input coherent state $|\alpha\rangle$, and $|\alpha|^2$ is the mean number of photon prior to the interaction. Later on this parameter draw more attention and many groups [103-106] have used these parameters as a measure of quantum phase fluctuation.

The total noise of a single mode quantum state ($T = (\Delta X)^2 + (\Delta \dot{X})^2$) is a measure of the total fluctuations of the amplitudes [92] of two noncommuting quadrature components of the field. In analogy to it we can define the total phase fluctuation as

\[
T = (\Delta S)^2 + (\Delta C)^2.
\]  

(5.10)

Now using the relations (5.3), (5.4), (5.7), (5.8) and (5.10) we obtain
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\[ U = \frac{\left( (\Delta S)^2 + (\Delta C)^2 \right) (\Delta N)^2}{\left( 1 - \left( (\Delta S)^2 + (\Delta C)^2 \right) \right)} = \frac{T (\Delta N)^2}{(1 - T)} \geq \frac{1}{4} \quad (5.11) \]

and

\[ [C, S] = i \left( \frac{1}{2} \left( N + \frac{1}{2} \right) \right)^{-\frac{1}{2}}. \quad (5.12) \]

Therefore,

\[ (\Delta C)^2 (\Delta S)^2 \geq \frac{1}{16} \frac{1}{(N + \frac{1}{2})}. \quad (5.13) \]

Now we can write,

\[ T = (\Delta C)^2 + (\Delta S)^2 \geq (\Delta C)^2 + \frac{1}{16 \left( N + \frac{1}{2} \right)^2 (\Delta C)^2}. \]

The function \( T = (\Delta C)^2 + \frac{1}{16 \left( N + \frac{1}{2} \right)^2 (\Delta C)^2} \) has a clear minima at \( (\Delta C)^2 = \frac{1}{4 \left( N + \frac{1}{2} \right)^2} \), which corresponds to a coherent state and thus the total fluctuation in quantum phase variables \( (\Delta C)^2 + (\Delta S)^2 \) cannot be reduced below its coherent state value \( \frac{1}{2 \left( N + \frac{1}{2} \right)^2} \). Now since \( (\Delta N)^2 \) is positive and the

\[ U = \frac{T (\Delta N)^2}{(1 - T)} = b(\Delta N)^2 \geq \frac{1}{4}, \] therefore \( b = \frac{T}{(1 - T)} \) is positive and consequently \( 0 < T < 1 \) as \( b \) will be negative for other values of \( T \). Further, we would like to note that \( T \leq 0 \) is unphysical and \( b \) blows up at \( T = 1 \). Under the above conditions (i.e. when \( 0 < T < 1 \)) \( b \) increases monotonically with the increase in \( T \). Thus the minima of \( T \) corresponds to the minima of \( b \) too and consequently, \( b \) is minimum for coherent state. In other words \( b \) cannot be reduced below its coherent state value. Therefore any reduction in \( U = b(\Delta N)^2 \) with respect to its Poissonian state value will mean a decrease in \( (\Delta N)^2 \) with respect to its Poissonian state counterpart. Thus the reduction of \( U \) with respect to its Poissonian state value implies antibunching but the converse is not true. Earlier Gupta and Pathak [71] have reported reduction of \( U \) with respect to coherent state in some simple optical processes and verified that the states are antibunched for
the corresponding parameters. But the fact that every antibunched state is not associated with the reduction of $U$ was not verified in the earlier work. Here we have shown that reduction of $U$ is possible for various intermediate states and have also provided examples of intermediate states, which are antibunched for specific values of parameters but do not show reduction of $U$ for the same values of the parameters. The intermediate states are studied under BP formalism because of the inherent computational simplicity of these formalism over the others. As the value of $U$ in coherent (Poissonian) state is $\frac{1}{2}$, our requirement of strong nonclassicality reduces to

$$d_u = U - \frac{1}{2} < 0.$$  \hspace{1cm} (5.14)

Further simplification of the criterion (5.14) is possible in BP formalism since the symmetric phase fluctuation parameter in BP formalism reduces to

$$U = [\langle a^\dagger a^2 \rangle + \langle a^\dagger a \rangle - \langle a^\dagger a \rangle^2][\frac{\langle a^\dagger a \rangle - \langle a^\dagger \rangle \langle a \rangle + \frac{1}{2}}{\langle a^\dagger \rangle \langle a \rangle}]$$  \hspace{1cm} (5.15)

and consequently our requirement for strong nonclassicality is

$$d_u = [\langle a^\dagger a^2 \rangle + \langle a^\dagger a \rangle - \langle a^\dagger a \rangle^2][\frac{\langle a^\dagger a \rangle - \langle a^\dagger \rangle \langle a \rangle + \frac{1}{2}}{\langle a^\dagger \rangle \langle a \rangle}] - \frac{1}{2} < 0.$$  \hspace{1cm} (5.16)

Now in light of this criterion we would like to study the nonclassical behavior of intermediate states.

### 5.3 Quantum phase fluctuations in intermediate states

In Chapter 1, we have already defined different kind of intermediate states. It is also observed in previous chapters that most of these intermediate states show antibunching, squeezing, subpoissonian photon statistics and higher order nonclassicalities (HOA, HOS and HOSPS etc.). Keeping these facts in mind, in the following subsections we will investigate the possibility of satisfaction of the criterion of nonclassicality introduced in the present chapter (i.e. (5.16))
for different intermediate states.

![Graph showing variation of quantum phase fluctuation of binomial state.](image)

**Figure 5.1:** Variation of quantum phase fluctuation of binomial state.

### 5.3.1 Binomial state

Binomial state is already defined in (1.46), from which it is straightforward to show that

\[
\langle M, p|a^\dagger a|p, M \rangle = Mp, \tag{5.17}
\]

similarly, we can obtain

\[
\langle M, p|a^\dagger a^2|p, M \rangle = M(M - 1)p^2, \tag{5.18}
\]

and

\[
\langle a^\dagger \rangle \langle a \rangle = Mp \left( \sum_{n=0}^{M-1} B_{n}^{M-1} B_{n}^{M} \right)^2. \tag{5.19}
\]

Now using equations (5.16) and (5.17-5.19) one can obtain,

\[
d_{U(BS)} = \left[ \frac{Mp(1-p)}{(\sum_{n=0}^{M-1} B_{n}^{M-1} B_{n}^{M})^2} \left[ \frac{1}{2Mp} + 1 - \left( \sum_{n=0}^{M-1} B_{n}^{M-1} B_{n}^{M} \right)^2 \right] \right]^{1/2}. \tag{5.20}
\]

Variation of quantum phase fluctuation in binomial state is shown in Fig. 5.1. From this figure it is clear that the binomial state shows reduction of fluctuation of quantum phase with respect to its coherent state counterpart and thus it
satisfies this stronger criterion of nonclassicality. But it does not satisfy the criterion for higher values of $p$ (i.e. when $p$ is close to 1). In Chapter 2 we have shown that Binomial state is always antibunched up to any order. When $p$ is close to 1, then binomial state is antibunched in every order and thus satisfies the other strong criterion of nonclassicality but do not satisfy the criterion laid down on the basis of quantum phase fluctuations. Earlier Gupta and Pathak had reported [71] that reduction of quantum phase fluctuation means antibunching but the converse is not true. This is the first time when an example of such a state which is antibunched but reduction of quantum phase fluctuation with respect to coherent state does not happen, is found.

![Figure 5.2: Variation of quantum phase fluctuation with respect to $\alpha$ and $\beta$ for Roy and Roy generalized binomial state.](image)

**5.3.2 Generalized binomial state**

As we have mentioned earlier there are different form of generalized binomial states [47-51], in the present section we have chosen generalized binomial state introduced by Roy and Roy [48]. The definition of Roy and Roy generalized binomial state is already mentioned in Chapter 1 (see (1.47)). Now with the help of properties of Pochhammer symbol and operator algebra we can obtain
following relations:

\[
\begin{align*}
\langle N, \alpha, \beta | a^\dagger a | N, \alpha, \beta \rangle &= N \frac{(\alpha+1)(\alpha+2)}{(\alpha+\beta+2)} \frac{N(N-1)(\alpha+1)}{(\alpha+\beta+2)^2} \sqrt{\omega(n, N-1, \alpha+1, \beta)} |n\rangle, \\
\langle N, \alpha, \beta | a^\dagger a^2 | N, \alpha, \beta \rangle &= N \frac{(\alpha+1)(\alpha+2)}{(\alpha+\beta+2)^2}, \\
\langle a^\dagger \rangle \langle a \rangle &= N^2 \frac{(\alpha+1)^2}{(\alpha+\beta+2)^2} \sum_{n=0}^{N-1} \frac{\omega(n, N-1, \alpha+1, \beta)}{(\alpha+\beta+2)^2},
\end{align*}
\]

Therefore,

\[
d_U(GBS) = \left[ \frac{(\beta+1)(\alpha+\beta+N+2)}{N^3(\alpha+1)(\alpha+\beta+3)(\alpha+\beta+3)^2} \sum_{n=0}^{N-1} \frac{(\alpha+1)(\alpha+2)}{(\alpha+\beta+2)^2} \right] - \frac{1}{2}.
\]

Variation of quantum phase fluctuation in generalized binomial state is shown in Fig 5.2. From Fig. 5.2 it is clear that the reduction of quantum phase fluctuation happens for Roy and Roy generalized binomial state. It is also observed that the depth of nonclassicality reduces with the increase in \(\alpha\) and \(\beta\). But as far as the higher (second) order antibunching is concerned, the depth of nonclassicality associated with it decreases with increase in \(\alpha\) and increases with increase in \(\beta\) (see Fig. 2.2 and Fig. 2.3). Farther it had been observed in Chapter 2 that for particular values of \(\alpha, \beta\) and \(N\) the state does not show second order antibunching. But here it is found that for the same values of \(\alpha, \beta\) and \(N\) we can obtain reduction of quantum phase fluctuation parameter with respect to its coherent state counterpart. Consequently we can say that, reduction of quantum phase fluctuation means antibunching but does not essentially mean higher order antibunching and therefore, it is not essential.
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that these two conditions of nonclassicality would appear simultaneously.

![Figure 5.3: Variation of quantum phase fluctuation of photon added coherent state.](image)

5.3.3 Photon added coherent state

Photon added coherent state or PACS which was introduced by Agarwal and Tara [66], is already defined in (1.56). Application of operator algebra yields following expectation values for PACS

\[
\langle a^\dagger a \rangle = \frac{\exp(-|\alpha|^2)}{L_m(-|\alpha|^2)m!} \sum_{n=0}^{\infty} \frac{(n + m)!\alpha^{2(n+1)}(m + n + 1)^2}{(n + 1)!^2}, \tag{5.26}
\]

\[
\langle a^\dagger a \rangle = \frac{\exp(-|\alpha|^2)}{L_m(-|\alpha|^2)m!} \sum_{n=0}^{\infty} \frac{(n + m)!\alpha^{n+2}(m + n + 1)^2(m + n + 2)^2}{(n + 2)!^2}, \tag{5.27}
\]

and

\[
\langle a^\dagger \rangle = \langle a \rangle = \frac{\exp(-|\alpha|^2)}{L_m(-|\alpha|^2)m!} \sum_{n=0}^{\infty} \frac{(n + m)!\alpha^{2n+1}(m + n + 1)}{(n + 1)(n!)^2}. \tag{5.28}
\]

By substituting equations (5.26-5.28) in (5.16) we can easily obtain a long expression of \( d_U \) as
\[ d_{U(PACS)} = \frac{\mathcal{L}_m(-|\alpha|^2|m!)}{\mathcal{L}_m(-|\alpha|^2|m!)} \left \{ \sum_{n=0}^{\infty} \frac{(n+m)!\alpha^{n+2}(m+n+1)^2(m+n+2)^2}{(n+2)!2} + \sum_{n=0}^{\infty} \frac{(n+m)!\alpha^{n+2}(m+n+1)^2}{(n+1)!2} \left( \sum_{n=0}^{\infty} \frac{(n+m)!\alpha^{n+1}(m+n+1)^2}{(n+1)!2} \right)^2 \right \} \right \}

\[ \sum_{n=0}^{\infty} \frac{(n+m)!\alpha^{n+2}(m+n+1)^2}{(n+1)!2} \left( \sum_{n=0}^{\infty} \frac{(n+m)!\alpha^{n+1}(m+n+1)^2}{(n+1)!2} \right)^2 + \frac{1}{2} - \frac{1}{2}. \]  

(5.29)

Essential characteristic of \( d_U \) of PACS can be seen in Fig. 5.3. It is easy to observe that the reduction of quantum phase fluctuation is possible in photon added coherent state. The depth of nonclassicality increases with the increase in \( m \). So we can conclude, the more photon are added to coherent state the more nonclassical it is as far as the depth of nonclassicality associated with quantum phase fluctuation is concerned. This particular characteristic is also been reflected in higher order antibunching [39].

5.3.4 Other intermediate states

As it is mentioned in the earlier sections, there exist several different intermediate states. For the systematic study of possibility of reduction of quantum phase fluctuation in intermediate states, we have studied all the well known intermediate states. Since the procedure followed for the study of different states is similar, mathematical detail has not been shown in the subsections below. But from the expression of \( d_U \) and the corresponding plots it would be easy to see that the reduction of quantum phase fluctuation can be observed in all the intermediate states studied below.

5.3.4.1 Negative binomial state

NBS is already defined in (1.50). Now application of operator algebra similar to the one used in previous subsections yields
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Figure 5.4: Variation of quantum phase fluctuation of negative binomial state.

\[
d_{U(NBS)} = \left\{ \begin{align*}
\frac{(M+1)(1-p)^{2M+1}}{p(M+2)} & \sum_{n=M}^{\infty} \binom{n+1}{M} \binom{n}{M} (1-p)^{(n+1)(n+1)} \\
(1-p)^{2(M+1)} & \sum_{n=M}^{\infty} \binom{n+1}{M} \binom{n}{M} (1-p)^{(n+1)(n+1)} \\
& + \frac{1}{2}
\end{align*} \right\} - \frac{1}{2}
\]

(5.30)

Variation of quantum phase fluctuation parameter with respect to the probability \( p \) for NBS is shown in the Fig. 5.4 and it has been observed that the state is more nonclassical for lower values of \( p \) and higher values of \( M \) [39]. This is consistent with the earlier observations on higher order antibunching of NBS.

Figure 5.5: Variation of quantum phase fluctuation of hyper geometric state.
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5.3.5 Hyper geometric state

Hyper geometric state (HS) [60] is defined in (1.51). Adopting similar method we obtain

\[ d_{U(HS)} = \left\{ M_p - \frac{\sqrt{L_p}}{L} \left( \sum_{n=0}^{M-1} \sqrt{\binom{L_p}{n}} \binom{L(1-p)}{M-n} \frac{L_p}{n} \binom{L(1-p)-n}{M-n-1} \right)^2 + \frac{1}{2} \right\} - \frac{1}{2} \right\} . \]

Fig. 5.5 depicts the characteristics of quantum phase fluctuation of HS. From this figure we can clearly observe that HS does not satisfy the reduction of quantum phase fluctuation criterion for higher values of \( p \) (i.e. when \( p \) is close to 1) but it satisfies the condition of higher order antibunching and consequently the condition of antibunching for those values of \( p \). This is consistent with the theoretical prediction that reduction of quantum phase fluctuation means antibunching but the converse is not true.

5.4 Conclusions

In essence, all the intermediate states described above show reduction of \( U \) with respect to its coherent state value\(^2\). This establishes that the intermediate states can satisfy the stronger criterion of nonclassicality compared to the criterion of usual antibunched state. This is consistent with our earlier observation (see Chapter 2 to Chapter 4) of higher order nonclassicalities in intermediate states. Further, we would like to note that the binomial state can show reduction of fluctuation of quantum phase with respect to its coherent state counterpart and thus it can satisfy this stronger criterion of nonclassi-

\(^2\)In reciprocal binomial state, we have not observed this phenomenon.
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cality. But it does not satisfy the criterion for values of $p$ close to 1 (see Fig. 5.1). In Chapter 2, we have shown that Binomial state is always antibunched up to any order. Thus for higher values of $p$ (i.e. when $p$ is close to 1) it is higher order antibunched but do not satisfy the criterion laid down on the basis of quantum phase fluctuations. Similar phenomenon is also observed in HS. Earlier Gupta and Pathak [71] had reported that reduction of quantum phase fluctuation means antibunching but the converse is not true. Example of such a state which is antibunched but does not show reduction of quantum phase fluctuation with respect to coherent state, is reported in this chapter. Further from the study of phase properties of Roy and Roy GBS and HS we have learnt that the reduction of quantum phase fluctuation mean antibunching but does not essentially mean higher order antibunching and therefore, it is not essential that these two conditions of nonclassicality appear simultaneously. In connection to PACS we have observed that the more photon are added to coherent state the more nonclassical the PACS, is as far as the depth of nonclassicality associated with quantum phase fluctuation is concerned. This particular characteristic has also been reflected in higher order antibunching, Higher order squeezing and Higher order subpoissonian photon statistics too (see Chapter 2 and 4). Further we have seen that the NBS show reduction of quantum phase fluctuation parameter for all values of $p$ and is more nonclassical for lower values of $p$. 