CHAPTER 3

GENERALIZED STRUCTURE OF HIGHER ORDER NONCLASSICALITY

A generalized notion of higher order nonclassicality (in terms of higher order moments) is introduced in this chapter. Under this generalized framework of higher order nonclassicality, conditions of higher order squeezing and higher order subpoissonian photon statistics are derived. A simpler form of the Hong-Mandel higher order squeezing criterion is derived under this framework by using an operator ordering theorem introduced by Pathak [91]. It is also generalized for multi-photon Bose operators of Brandt and Greenberg [65]. Similarly, condition for higher order subpoissonian photon statistics is derived by normal ordering of higher powers of number operator. Further, with the help of simple density matrices, it is shown that the higher order antibunching (HOA) and higher order subpoissonian photon statistics (HOSPS) are not the manifestation of the same phenomenon and consequently it is incorrect to use the condition of HOA as a test of HOSPS. It is also shown in this chapter that the HOA and HOSPS may exist even in absence of the corresponding lowest order phenomenon.
3.1 Introduction: The generalized notion of higher order nonclassicality

In Chapter 1, we have already mentioned that when the Glauber Sudarshan P function of a radiation field become negative or more singular than a delta function then the radiation field is said to be nonclassical. In these situations, quasi probability distribution P is not accepted as classical probability and thus we cannot obtain an analogous classical state. Lowest order nonclassical states (e.g. Squeezed state and antibunched state) have been studied since long but the interest in higher order nonclassical states is relatively new. Possibilities of observing higher order nonclassicalities in different physical systems have been investigated in recent past [11-37] (e.g. HOS, HOSPS and HOA). But the general nature of higher order nonclassicality and the mutual relation between these higher order nonclassical states have not been studied till now. In this chapter, we aim to provide a general and simplified framework for the study of higher order nonclassical state.

Commonly, second order moment (standard deviation) of an observable is considered to be the most natural measure of quantum fluctuation [92] associated with that observable and the reduction of quantum fluctuation below the coherent state (Poissonian state) level corresponds to lowest order nonclassical state. For example, an electromagnetic field is said to be electrically squeezed field if uncertainties in the quadrature phase observable $X$ reduces below the coherent state level (i.e. $(\Delta X)^2 < \frac{1}{2}$) and antibunching is defined as a phenomenon in which the fluctuations in photon number reduces below the Poisson level (i.e. $(\Delta N)^2 < \langle N \rangle$) [4, 93]. In essence, if we consider an arbitrary quantum mechanical operator $A$ and a quantum mechanical state $|\psi\rangle$ then the state $|\psi\rangle$ is lowest order nonclassical with respect to the operator $A$ if

$$(\Delta A)_{|\psi\rangle}^2 < (\Delta A)_{\text{poissonian}}^2.$$  (3.1)
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If \(|\psi\rangle\) corresponds to an electromagnetic field, this condition will mean that the
radiation field is nonclassical. This condition can now be further generalized
and we can say that a state \(|\psi\rangle\) has \(n\)th order nonclassicality with respect to
the operator \(A\) if the \(n\)th order moment of \(A\) in that state reduces below the
value of the \(n\)th order moment of \(A\) in a Poissonian state, i.e. the condition of
\(n\)th order nonclassicality is

\[(\Delta A)^n_{|\psi\rangle} < (\Delta A)^n_{|\text{poissonian}\rangle} , \tag{3.2}\]

where \((\Delta A)^n\) is the \(n\)th order moment defined as

\[\langle (\Delta A)^n \rangle = \sum_{r=0}^{n} n C_r (-1)^r \bar{A}^r \bar{\bar{A}}^{n-r}. \tag{3.3}\]

If \(A\) is a field operator then it can be expressed as a function of creation and
annihilation operators \(a^\dagger\) and \(a\) and consequently further simplification of (3.2)
is possible by using the identity

\[\langle : (A(a,a^\dagger))^k : \rangle_{|\text{poissonian}\rangle} = \langle (A(a,a^\dagger))^k \rangle_{|\text{poissonian}\rangle} \tag{3.4}\]

where, the notation \( : (A(a,a^\dagger))^k : \) is simply a binomial expansion in which
powers of the \(a^\dagger\) are always kept to the left of the powers of \(a\). Here it would be
interesting to note that (3.4) helps us to show that the Glauber Sudarshan \(P\)
function is negative for condition (3.2). It is clear from (3.3) that the problem
of finding out the \(n\)th order moment of the operator \(A\) essentially reduces to a
problem of operator ordering (normal ordering) of \(A^r\). Here, we observed the
lowest order nonclassicality for \(n = 2\). In this particular case \((n = 2)\) we obtain
the condition of squeezing of electric field, if \(A = X = \frac{1}{\sqrt{2}}(a + a^\dagger)\) and obtain
the condition of antibunching if \(A = N = a^\dagger a\). Now if we need to generalize
the idea of these well known lower order nonclassical effects we have to find out
normal ordered form of \(X^r\) and \(N^r\).
3.2 Simplified condition for higher order squeezing

To obtain the condition for higher order squeezing (HOS), we need to use the following operator ordering theorem introduced by Pathak in [91]:

**Theorem 1**: If any two bosonic annihilation and creation operators \( a \) and \( a^{\dagger} \) satisfy the commutation relation \((1.17)\). Then for any integral values of \( m \)

\[
(a^{\dagger} + a)^m_{\text{N}} = \sum_{r=0}^{m} t_{2r} mC_{2r} : (a^{\dagger} + a)^{m-2r} : \quad (3.5)
\]

with

\[
t_{2r} = \frac{2r!}{2^r(r)!} = (2r - 1)!! = 2^r \left( \frac{1}{2} \right)_r \quad (3.6)
\]

where, the subscript \( N \) stands for the normal ordering\(^1\), \((x)_r\) is conventional Pochhammer symbol defined in (1.49) and \( n!! \) is double factorial, which is defined as

\[
n!! = \begin{cases} 
  n(n-2) \ldots 5.3.1 & \text{for } n > 0 \text{ odd} \\
  n(n-2) \ldots 6.4.2 & \text{for } n > 0 \text{ even} \\
  1 & \text{for } n = -1, 0
\end{cases} \quad (3.7)
\]

This theorem of normal ordering is not restricted to the ordering of annihilation and creation operator, rather it is valid for any arbitrary operator \( E^+ \) and its conjugate \( E^- \) which satisfy,

\[
[E^+, E^-] = C. \quad (3.8)
\]

This is so because \((3.8)\) can easily be reduced to the form of \((1.17)\) as \( \left[ \frac{E^+}{\sqrt{C}}, \frac{E^-}{\sqrt{C}} \right] = 1 \). Apart from theorem 1 we need the following identity to proceed further:

\(^1\)One can write \( f(a, a^{\dagger}) \) in such a way that all powers of \( a^{\dagger} \) always appear to the left of all powers of \( a \). Then \( f(a, a^{\dagger}) \) is said to be normal ordered.
\[ nC_r \cdot C_j = \frac{n!}{(n-r)!} \frac{r!}{(r-j)!} j! = \frac{n! \cdot (n-j)!}{((n-j)-(r-j))! (r-j)!} = nC_j \cdot n-jC_{r-j}. \] (3.9)

Using (3.3), (3.5), (3.6) and (3.9), the \( n \)th order moment of \( \Delta E = E - \bar{E} \) (where the quadrature variable \( E = (a + a^\dagger) \)) can be expressed as

\[
\langle (\Delta E)^n \rangle = \sum_{r=0}^{n} nC_r (-1)^r E^r \bar{E}^{n-r} = \sum_{r=0}^{n} nC_r (-1)^r \sum_{i=0}^{\frac{n}{2}} t_{2i} rC_{2i} (a^\dagger + a)^{r-2i} \langle a^\dagger + a \rangle^{n-r-2i} + \sum_{i=0}^{\frac{n}{2}} t_{2i} nC_{2i} \langle \Delta E \rangle^{n-2i}.
\] (3.10)

Now, if we follow (3.2), and define \( n \)th order squeezed state as a quantum mechanical state in which \( n \)th order moment \( \langle (\Delta E)^n \rangle \) is shorter than its Poissonian state value then the condition for \( n \)th order squeezing reduces to

\[
\langle (\Delta E)^n \rangle < t_n = (n - 1)!!
\] (3.11)

which can be alternatively written as

\[
\sum_{i=0}^{\frac{n}{2}} t_{2i} nC_{2i} \langle \Delta E \rangle^{n-2i} < 0.
\] (3.12)

or,

\[
\langle (\Delta E)^n \rangle = \sum_{r=0}^{n} \sum_{i=0}^{\frac{n}{2}} \sum_{k=0}^{r-2i} (-1)^r t_{2i} r^{-2i} C_k C_r C_{2i} (a^\dagger + a)^{n-r} \langle a^\dagger k a^{-2i-k} \rangle < (n-1)!!.
\] (3.13)

Conditions (3.11) and (3.12) coincide exactly with the definition of Hong Mandel squeezing, reported in earlier works\(^2\) [11, 14] and the equivalent condition

\(^2\)If we choose \( E_1 = E^+ + E^- \) in analogy with Hong and Mandel [11] then (3.11) reduces to

\[
\langle (\Delta E)^n_1 \rangle < t_n C_{\frac{n}{2}} = (n - 1)!! C_{\frac{n}{2}}.
\]
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(3.13) considerably simplifies the calculation of HOS. Now instead of $E$ if we calculate the $n$th order moment for usual quadrature variable $X$ defined as $X = \frac{1}{\sqrt{2}}(a + a^\dagger)$, then we obtain

\[
\langle (\Delta X)^n \rangle < \frac{1}{2^\frac{n}{2}} t_n = \frac{1}{2^\frac{n}{2}} (n - 1)!! = \left(\frac{1}{2}\right)^\frac{n}{2}
\]  

(3.14)

or,

\[
\sum_{r=0}^{n} \sum_{i=0}^{\frac{n}{2}} \sum_{k=0}^{r-2i} (-1)^r \frac{1}{2^\frac{n}{2}} t_{2i} r^{r-2i} C_k^n C_r^r C_{2i}^r \langle a^\dagger + a \rangle^{n-r} \langle a^\dagger k a^{r-2i-k} \rangle < \left(\frac{1}{2}\right)^\frac{n}{2}.
\]  

(3.15)

Starting from the generalized notion of higher order nonclassicality (3.2) we have obtained a closed form expression of Hong-Mandel squeezing with the help of Theorem 1. The use of Theorem 1 not only simplifies the condition but significantly reduces the calculational difficulties. To be precise, to study the possibility of HOS for an arbitrary quantum state $|\psi\rangle$ we just need to calculate $\langle a^\dagger + a \rangle$ and $\langle a^\dagger k a^{r-2i-k} \rangle$. Calculation of these expectation values is simple. For example, if we can expand the arbitrary state $|\psi\rangle$ in the number state basis as

\[
|\psi\rangle = \sum_{j=0}^{N} C_j |j\rangle,
\]  

(3.16)

then we can easily obtain,

\[
\langle \psi | a^\dagger k a^{r-2i-k} \langle \psi \rangle = \sum_{j=0}^{N - \text{Max}[k, r-2i-k]} C_{j+k}^r C_{j+r-2i-k} \frac{1}{j!} \frac{1}{(j + k + r - 2i)! (j + k)!} \left(\frac{1}{2}\right)^\frac{n}{2}.
\]  

(3.17)

where $n$ is even. This is the generalized expression obtained in [11] by using some other tricks.
where $\text{Max}$ yields the largest element from the list in its argument and

$$
\langle a^\dagger + a \rangle = \sum_{m=0}^{N-1} \sqrt{(m + 1 \, (C_m C_{m+1}^* + C_{m+1}^* C_m)}.
$$

(3.18)

Therefore,

$$
\langle (\Delta X)^n \rangle = \sum_{r=0}^{n} \sum_{i=0}^{r-2} \sum_{k=0}^{r-2i} (-1)^r \frac{1}{2^r} t_{2i} \frac{r-2i}{C_k \, n^r \, C_{2i}} \\
\times \left( \sum_{m=0}^{N-1} \sqrt{(m + 1 \, (C_m C_{m+1}^* + C_{m+1}^* C_m)} \right)^{n-r} \\
\times \sum_{j=0}^{N-\text{Max}[k, r-2i-k]} C_{j+k}^* C_{j+r-2i-k} \left( (j + k + r - 2i)! (j + k) ! \right)^{\frac{1}{2}}.
$$

(3.19)

In general, if we know the effect of $a^s$ on the state $|\Psi\rangle$ and the orthogonality conditions $\langle \Psi'|\Psi \rangle$ then we can easily find out $\langle (\Delta X)^n \rangle$. Further, since (3.19) is a $C$-number equation, analytical tools like MAPLE and MATHEMATICA can also be used to study the possibilities of observing higher order squeezing (or higher order nonclassicality in general). This point will be more clear in Chapter 4, where we will provide specific examples. Here we would like to note that we can normalize (3.14) and rewrite the condition of HOS as

$$
S_{HM}(n) = \frac{\langle (\Delta X)^n - \left( \frac{1}{2} \right)^n \rangle}{\left( \frac{1}{2} \right)^n} < 0,
$$

(3.20)

where the subscript $HM$ stand for Hong-Mandel.

### 3.2.1 Brandt-Greenberg operators and k-photon coherent state

The k-photon coherent state was introduced by D’Arino and coworkers by using Brandt-Greenberg multi-photon operators [65] $A_k$ and $A_k^\dagger$, which are defined as

$$
A_k^\dagger = \left[ \begin{array}{c} N \\ k \end{array} \right] \frac{N - k}{N} a^\dagger_k,
$$

(3.21)

$$
A_k = (A_k^\dagger)^\dagger,
$$

(3.22)
where the function \([x]\) is defined as the greatest integer less or equal to \(x\); \(a^\dagger\) and \(a\) are the usual bosonic creation and annihilation operator and \(N = a^\dagger a\) is the number operator. This particular form of Brandt-Greenberg operators is also used in the work of Buzek and Jex [20] in which they have studied the amplitude \(k\)th power squeezing of the \(k\)-photon coherent states. These operators satisfy the commutation relation analogous to (1.17), i.e. they satisfy,

\[
[A_k, A_k^\dagger] = 1. \tag{3.23}
\]

If any operator and its Hermitian conjugate satisfies this kind of commutation relation then it has to satisfy the operator ordering theorem 1 and consequently we will be able to define Hong-Mandel squeezing in terms that particular operator (in a modified Fock space). For example, if we define the quadrature variables \(X_{1k}\) and \(X_{2k}\) as

\[
X_{1k} = A_k + A_k^\dagger \\
X_{2k} = A_k - A_k^\dagger \tag{3.24}
\]

then we can define the condition for \(n\)th order Hong-Mandel squeezing as

\[
\sum_{r=0}^{n} \sum_{i=0}^{\frac{r}{2}} \sum_{k=0}^{r-2i} (-1)^r \frac{1}{2^2} t_{2i}^r C_k C_i^n C_r C_{2i} (A^\dagger + A)^{n-r} \langle A^\dagger k A r - 2i - k \rangle < \left( \frac{1}{2} \right)^{\frac{n}{2}}. \tag{3.25}
\]

This provides an extended notion of Hong-Mandel squeezing in a modified Hilbert space.

### 3.3 Higher order subpoissonian photon statistics

In analogy to the procedure followed to derive the Hong-Mandel higher order squeezing condition from the generalized expression (3.2) of higher order nonclassicality, we wish to study the nonclassicality associated with \(A(a, a^\dagger) = N = a^\dagger a\). As we have already discussed, for this purpose we will require op-
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Operator ordered form of $N^r$. Since the operator ordered expansion of $N^r$ will not contain any off-diagonal term so it is justified to assume that the normal ordered form of $(N)^r$ can be given as

$$N^r = \sum_{i=1}^{r} C_{r,i} : N^i : = \sum_{i=1}^{r} C_{r,i} a^i a^i.$$  \hspace{1cm} (3.26)

From this equation it is clear that $C_{r,1} = C_{r,r} = 1$ and we can write $N^{r+1}$ as

$$N^{r+1} = \sum_{i=1}^{r+1} C_{r+1,i} a^i a^i = \sum_{i=1}^{r} C_{r,i} a^i a^i a = \sum_{i=1}^{r} (C_{r,i} a^{i+1} a^{i+1} + iC_{r,i} a^i a^i)$$  \hspace{1cm} (3.27)

where the operator ordering identity, $a^i a^j = a^j a^i + i a^i a^{j-1}$ is used. Now, we can obtain a closed form normal ordered expansion of $N^r$ provided we know the solution of the recurrence relation:

$$C_{r+1,i} = iC_{r,i} + C_{r,i-1}$$  \hspace{1cm} (3.28)

with $C_{r,0} = 0$ and $C_{r,1} = 1$. One can easily identify (3.28) as the famous recurrence relation of Stirling number of second kind [94]. Thus we can write

$$N^r = \sum_{k=1}^{r} S_2(r,k) a^k a^k = \sum_{k=1}^{r} S_2(r,k) : N^k : = \sum_{k=1}^{r} S_2(r,k) N^{(k)},$$ \hspace{1cm} (3.29)

where $S_2(r,k)$ is the Stirling number of second kind $N^{(k)} = a^k a^k$ is the kth factorial moment of the number operator $N$. Now using (3.2), (3.3) and (3.29) we can obtain the condition of higher order subpoissonian photon statistics as

$$\langle (\Delta N)^n \rangle = \sum_{r=0}^{n} S_2(r,k) a^r a^r = \sum_{r=0}^{n} S_2(r,k) a^r a^r < \langle (\Delta N)^n \rangle_{\text{poissonain}}$$

or,

$$d_h(n-1) = \sum_{r=0}^{n} \sum_{k=1}^{r} S_2(r,k) a^r a^r < \langle (\Delta N)^n \rangle_{\text{poissonain}} < 0.$$  \hspace{1cm} (3.30)
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3.3.1 Relation between the criteria of HOA and HOSPS

The criterion of HOA is expressed in terms of higher order factorial moments of number operator as shown in (2.5). There exist several criterion for the same which are essentially equivalent. Here we would like to investigate how are they related to the criterion of HOSPS. To do so let us combine the condition for HOSPS (3.30) and that of HOA (2.5) to yield

\[ d_h(n-1) = \sum_{r=0}^{n} \sum_{k=1}^{r} S_2(r,k) n^r C_r(-1)^r \left[ \langle N^{(k)} \rangle - \langle N \rangle^k \right] \langle N \rangle^{n-r} = \sum_{r=0}^{n} \sum_{k=1}^{r} S_2(r,k) n^r C_r(-1)^r d(k-1) \langle N \rangle^{n-r} < 0. \]

\[ (3.31) \]

Above relation connects the condition of HOA (2.5) and that of HOSPS (3.30) but does not provide any conclusion about the mutual satisfiability. Physically it is expected from the analogy of lower order phenomenon that all states that show HOA should show HOSPS but the reverse should not be true. We have not succeed in showing that analytically but we can establish that with the help of simple density matrix of the form \( \frac{1}{2}(|a\rangle\langle a| + |b\rangle\langle b|) \), where \( |a\rangle \) and \( |b\rangle \) are Fock states. The results are shown in the Table 3.1.

All the criterion related to HOA and HOSPS essentially lead to same kind of nonclassicality which belong to the class of strong nonclassicality according

<table>
<thead>
<tr>
<th>Density matrix</th>
<th>Antibunching</th>
<th>SPS</th>
<th>HOA ((l = 3))</th>
<th>HOSPS ((n = 4))</th>
<th>Conclusions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2}(</td>
<td>3\rangle\langle 3</td>
<td>+</td>
<td>8\rangle\langle 8</td>
<td>) )</td>
<td>No</td>
</tr>
<tr>
<td>( \frac{1}{2}(</td>
<td>4\rangle\langle 4</td>
<td>+</td>
<td>10\rangle\langle 10</td>
<td>) )</td>
<td>No</td>
</tr>
</tbody>
</table>

Table 3.1: HOA and HOSPS are not the manifestation of the same phenomenon and consequently it is incorrect to use the condition of HOA as a test of HOSPS.

The negativity of \( d_h(n - 1) \) will mean \((n - 1)th\) order subpoissonian photon statistics. This condition is equivalent to the condition of HOSPS obtained by Mishra and Prakash [29].
to the classification scheme of Arvind et al [95]. The Table 3.1 shows that HOA and HOSPS may be present in a system in absence of corresponding lower order phenomenon. It also shows that HOA and HOSPS are not the same phenomenon. To be precise, HOSPS can be present in a system even in absence of HOA. Thus it is not proper to consider the condition of HOA as the condition of HOSPS. In [17] Duc has recently used criterion of HOA to study possibilities of observing HOSPS in photon added coherent state. Incorrect choice of criterion may yield incorrect conclusions so we need to be very careful before choosing a criterion of higher order nonclassicality.

3.4 Conclusions

The criteria of HOSPS and Hong Mandel type of higher order squeezing are derived from a single framework. Using that framework and operator ordering theorem a simpler form of the Hong-Mandel higher order squeezing criterion is derived and generalized for the multi-photon Bose operators of Brandt and Greenberg. The relation between HOA, HOSPS and HOS is investigated in detail and certain interesting observations in this regard have been reported. For example, it is shown that the lower order antibunching, HOA and HOSPS appear in novel regimes (i.e. they may or may not appear simultaneously as shown in Table 3.1). But in literature, HOA and HOSPS have been used as synonymous [17]. Our observations establish that it is incorrect to use the condition of HOA as a test of HOSPS. In the present chapter, attempts have been made to understand the mutual relationship between different higher order nonclassical states. The effort is successful to provide an insight into the mutual relations between the well known nonclassical states and opens up a possibility of similar work in broader class of nonclassical states. The simpler framework for the study of possibilities of observing HOS and HOSPS, provided in this chapter will be used to explore the possibilities of observing HOS and HOSPS in intermediate states in the next chapter. The simpler framework is also expected
to be useful in the future work to study the possibilities of observing HOS and HOSPS in other quantum states too.