CHAPTER 9

CREEP DEFORMATION OF A THIN ROTATING DISK OF EXPONENTIALLY VARYING THICKNESS WITH INCLUSION

9.1 INTRODUCTION

This chapter is concerned with the analysis of a thin rotating disk made of isotropic material with exponentially varying thickness. Rotating disk and cylinders play an important role in machine design. As a matter of fact the stresses in disks and cylinders depend on the angular velocity with which they rotate. These bodies become partially or wholly plastic when the stresses generated satisfy Tresca’s yield condition. There are many applications of such type of rotating disks, such as in turbines, rotors, flywheels and with the advent of computers, disk drives. Naturally, with all these applications and interest, there has been much research in this field. The problem of the stresses in thin rotating circular disk of variable thickness made of isotropic material were discussed by Martin (1923), Suhara and Yaskawa (1929), Sen (1935), Malkin (1935), Bisshopp (1944) and others. The stress distribution produced by a constant angular velocity in an elastic-plastic annular disk has been known for a long time. The analysis of stress distribution in circular disk rotating at high speed is important for a better understanding of the behaviour and optimum design of structures. Many authors use the incompressibility of the material in the theory of creep as the starting point for calculating stresses. The condition of incompressibility in the problems of creep deformations is one of the most important assumptions which simplify the problem. It is well
known that there are many materials which show compressibility effect in creep deformation. The classical theory does not account for these effects. In this study, creep stresses and strain rates for a thin rotating disk with exponentially variable thickness have been obtained using transition theory [42-44, 59, 92-95, 124-125]. Results obtained have been discussed numerically and depicted graphically.

**9.2 GOVERNING EQUATIONS**

We consider a thin disk of constant density with central bore of radius ‘a’ and external radius ‘b’. The disk is rotating with angular speed ‘ω’ about an axis perpendicular to its plane and passed through the centre of the disk. The thickness of the disk is assumed to be constant and is taken to be sufficiently small so that it is effectively in a state of plane stress, that is, the axial stress $T_{zz}$ is zero. The disk is assumed to be symmetric with respect to the mid plane. The governing equations are same as that of section 4.2 with the boundary condition as

$$u = 0 \quad \text{at} \quad r = a,$$

and

$$T_{rr} = 0 \quad \text{at} \quad r = b. \quad (9.1)$$

**9.3 SOLUTION THROUGH PRINCIPAL STRESS DIFFERENCE**

For finding the creep stresses, the transition function $R$ is calculated through principal stress difference at the transition point $P \to -1$, which leads to creep state. The transition function $R$ is defined as

$$R = T_{rr} - T_{00} = \frac{2\mu}{n} \beta^n \left[ 1 - (P + 1)^n \right]. \quad (9.2)$$

Taking the logarithmic differentiation of equation (9.2) with respect to ‘$r$’, we get

$$\frac{d}{dr} \log R = -\frac{nP \left[ 1 - (P + 1)^n - \beta (P + 1)^{n-1} \frac{dP}{d\beta} \right]}{\beta^n \left[ 1 - (P + 1)^n \right]}.$$

Substituting the value of $dP/d\beta$ from equation (4.6) in equation (9.3) and taking asymptotic value $P \to -1$, the equation (9.3) becomes

$$\frac{d}{dr} \log R = -\frac{1}{r(2-C)} \left[ n(3-2C) + \frac{nP\omega^2 r^{n+2}}{2\mu D^n} + 1 + (1-C)k \left( \frac{r}{b} \right)^k - \frac{k(3-2C)}{D} r^p \left( \frac{r}{b} \right)^k \right]. \quad (9.4)$$

Here asymptotic value of $\beta$ as $P \to -1$ is $D/r$, where $D$ is a constant.

On integration of the above equation with respect to ‘$r$’, one gets
where \( A \) is constant of integration, which can be determined by the boundary condition. Here

\[
F_1 = -\frac{n(3-2C)+1}{(2-C)}, \quad F_2 = -\frac{n\rho \omega^2}{2\mu D''(2-C)(n+2)}, \quad F_3 = -\frac{(1-C)}{(2-C)b^k}, \quad F_4 = \frac{K(3-2C)}{(2-C)b^k D''(n+k)}.
\]

From equation (9.2) and (9.5), the transition function \( R \) is calculated as

\[
R = T_{rr} - T_{r\theta} = Ar^{\frac{1}{n}} \exp[F_3 r^{n+2} + F_3 r^K + F_4 r^{n+k}].
\] (9.6)

Substituting equation (9.6) in (4.5), we get

\[
T_{rr} = -\frac{A}{h_0} \int r^{\frac{1}{n}} e^{\frac{r}{b}} \exp[F_3 r^{n+2} + F_3 r^K + F_4 r^{n+k}] dr - \rho \omega^2 \int r e^{\frac{r}{b}} \exp[F_3 r^{n+2} + F_3 r^K + F_4 r^{n+k}] dr + B \frac{1}{h_0} e^{\frac{r}{b}}.
\] (9.7)

where \( B \) is constant of integration which can be determined by the boundary condition. Using boundary conditions (9.1) in equation (9.7), we get

\[
T_{rr} = A e^{\frac{r}{b}} \int r^{\frac{1}{n}} e^{\frac{r}{b}} r^{\frac{1}{n}-1} \exp[F_3 r^{n+2} + F_3 r^K + F_4 r^{n+k}] dr + \rho \omega^2 \int r e^{\frac{r}{b}} dr.
\] (9.8)

Solving equations (9.6) and (9.8), the circumferential stress is

\[
T_{r\theta} = A [\int r^{\frac{1}{n}} e^{\frac{r}{b}} r^{\frac{1}{n}-1} \exp[F_3 r^{n+2} + F_3 r^K + F_4 r^{n+k}] dr - r^{\frac{1}{n}} \exp[F_3 r^{n+2} + F_3 r^K + F_4 r^{n+k}]] - e^{\frac{r}{b}} \rho \omega^2 \int r e^{\frac{r}{b}} dr
\] (9.9)

Using equations (9.2) and (9.6) and taking asymptotic value \( P \rightarrow -1 \), we get

\[
\beta = \left[ \frac{n(3-2C)}{E(2-C)} Ar^{\frac{1}{n}} \exp[F_3 r^{n+2} + F_3 r^K + F_4 r^{n+k}] \right]^{\frac{1}{n}}.
\] (9.10)

The radial displacement is calculated by using equations (9.10) and (2.1) as

\[
u = r - \left[ \frac{n(3-2C)}{E(2-C)} Ar^{\frac{1}{n}} \exp[F_3 r^{n+2} + F_3 r^K + F_4 r^{n+k}] \right]^{\frac{1}{n}}.
\] (9.11)

From the boundary condition (9.1) and equation (9.11), we get the constant of integration as

\[
A = \frac{E(2-C)}{n(3-2C) a^{\frac{1}{n}} \exp[F_3 a^{n+2} + F_3 a^K + F_4 a^{n+k}]}
\]

Substituting the value of constant of integration in equations (9.8), (9.9) and (9.11), we get
The following non-dimensional components are introduced as

\[
R = \frac{r}{b}, \quad \sigma_r = \frac{T_r}{E}, \quad \sigma_\theta = \frac{T_\theta}{E}, \quad \Omega^2 = \frac{\rho \omega^2 b^2}{E}, \quad \bar{u} = \frac{u}{b}.
\]

Creep stresses and displacement in non-dimensional form becomes

\[
\sigma_r = \frac{(2-C) e^{Fr} \int_r^b e^{-Fr} R^{Fr-1} \exp[F_2 R a^{n+2} + F_3 R K + F_4 R a^{n+K}] dR}{n(3-2C)R_0^{Fr} \exp[F_2 a^{n+2} + F_3 a K + F_4 a^{n+K}]} + \Omega^2 e^{Fr} \int_r^b R e^{-Fr} dR,
\]

\[
\sigma_\theta = \frac{(2-C) e^{Fr} \int_r^b e^{-Fr} R^{Fr-1} \exp[F_2 R a^{n+2} + F_3 R K + F_4 R a^{n+K}] dR}{n(3-2C)R_0^{Fr} \exp[F_2 a^{n+2} + F_3 a K + F_4 a^{n+K}]} - R^{Fr} \exp[F_2 R a^{n+2} + F_3 R K + F_4 R a^{n+K}]\]

\[
+ \Omega^2 \int_r^b R e^{-Fr} dR.
\]

\[
\bar{u} = \frac{u}{b} = R - R \left[ \frac{R^{Fr} \exp[F_2 R a^{n+2} + F_3 R K + F_4 R a^{n+K}]}{R_0^{Fr} \exp[F_2 a^{n+2} + F_3 a K + F_4 a^{n+K}]} \right]^{\frac{1}{n}},
\]

where \( F_2' = \left[ \frac{n \Omega^2 (3-2C)b^n}{D^n (2-C)^2 (n+2)} \right] \),

\[
F_3' = \left[ \frac{(1-C)}{(2-C)} \right],
\]

\[
F_4' = \left[ \frac{K (3-2C)b^n}{(2-C)D^n (n+K)} \right].
\]
For a disk made of incompressible material \((C \to 0)\), equations (9.15), (9.16) and (9.17) becomes

\[
\sigma_r = 2e^{R^2} \int_R^{R^2} e^{-R^2} R^{F_1} \exp[F_2 R^{n+2} + F_3 R^K + F_4 R^{n+K}] dR + \Omega^2 e^{R^2} \int_R^{R^2} R e^{-R^2} dR,
\]

\[
(9.18)
\]

\[
\sigma_\theta = \left[ 2 \left[ e^{R^2} \int_R^{R^2} e^{-R^2} R^{F_1} \exp[F_2 R^{n+2} + F_3 R^K + F_4 R^{n+K}] dR \right] - R^{F_1} \exp[F_2 R^{n+2} + F_3 R^K + F_4 R^{n+K}] \right] \frac{1}{3nR_0^2} \int_R^{R^2} R e^{-R^2} dR,
\]

\[
+ e^{R^2} \Omega^2 \int_R^{R^2} R e^{-R^2} dR
\]

\[
(9.19)
\]

\[
\bar{u} = R - R \left[ R^{F_1} \exp[F_2 R^{n+2} + F_3 R^K + F_4 R^{n+K}] \right] \frac{1}{3nR_0^2} \int_R^{R^2} R e^{-R^2} dR
\]

\[
(9.20)
\]

### 9.4 STRAIN RATES

When creep sets in, strain should be replaced by strain rate. The stress-strain relation becomes

\[
\dot{\epsilon}_{ij} = \left( \frac{1 + \nu}{E} \right) T_{ij} - \frac{\nu}{E} \delta_{ij} \Theta,
\]

\[
(9.21)
\]

where \(\dot{\epsilon}_{ij}\) is the strain rate tensor with respect to flow parameter \(t\) and \(\Theta = T_{11} + T_{22} + T_{33}\).

Differentiating (2.3) with respect to \(t\), one gets

\[
\dot{\beta}_{oo} = -\beta^{n-1} \dot{\beta},
\]

\[
(9.22)
\]

For SWAINGER measure \((n = 1)\) one has from equation (9.22)

\[
\dot{\beta}_{oo} = -\dot{\beta},
\]

\[
(9.23)
\]

The transition value of equation (9.2) at \(P \to -1\) gives

\[
\beta = \left[ \frac{n(3 - 2C)}{2 - C} \right]^{\frac{1}{n}} (\sigma_r - \sigma_\theta)^{\frac{1}{n}}
\]

\[
(9.24)
\]

Using equations (9.22), (9.23) and (9.24) in equation (9.21)
\[ \dot{e}_{rr} = \left[ \frac{n(\sigma_r - \sigma_\theta)(3 - 2C)}{(2 - C)} \right]^{-\frac{1}{n-1}} (\sigma_r - \gamma \sigma_\theta), \]
\[ \dot{e}_{\theta\theta} = \left[ \frac{n(\sigma_r - \sigma_\theta)(3 - 2C)}{(2 - C)} \right]^{-\frac{1}{n-1}} (\sigma_\theta - \gamma \sigma_r), \]
\[ \dot{e}_{zz} = \left[ \frac{n(\sigma_r - \sigma_\theta)(3 - 2C)}{(2 - C)} \right]^{-\frac{1}{n-1}} \{\gamma (\sigma_r + \sigma_\theta)\}. \]  
(9.25)

For incompressible material \((C \to 0)\), equation (9.25) becomes
\[ \dot{e}_{rr} = \left[ \frac{3n(\sigma_r - \sigma_\theta)}{2} \right]^{-\frac{1}{n-1}} \left( \frac{2\sigma_r - \sigma_\theta}{2} \right), \]
\[ \dot{e}_{\theta\theta} = \left[ \frac{3n(\sigma_r - \sigma_\theta)}{2} \right]^{-\frac{1}{n-1}} \left( \frac{2\sigma_\theta - \sigma_r}{2} \right), \]
\[ \dot{e}_{zz} = \left[ \frac{3n(\sigma_r - \sigma_\theta)}{2} \right]^{-\frac{1}{n-1}} \left( \frac{1}{2}(\sigma_r + \sigma_\theta) \right). \]  
(9.26)

### 9.5 NUMERICAL ILLUSTRATION AND DISCUSSION

For calculating the stresses, strain rates and displacement based on the above analysis, the following values of measure \((n)\), constant \((D)\), compressibility \((C)\) and angular speed \((\Omega^2)\) have been taken as:
\[ \Omega^2 = \frac{\rho \omega^2 b^2}{E} = 50, 75 \quad C = 0, 0.25, 0.5 \]
\[ n = 1/5, 1/7, 1/9 \text{ (i.e. } N = 5, 7, 9) \quad D = 1. \]

The results are verified with those available in literature for constant thickness, i.e., \(k = 0\) [132]. From figures 9.1 – 9.3, it is seen that radial and circumferential stresses are maximum at the internal surface. It has been observed from figure 9.1, for incompressible material with rotating disk that the circumferential stress is going on increasing with the increase in measure \((N)\). The circumferential stress increases with the increase in angular speed. With the introduction of compressibility factor, there is small decrease in circumferential stress as compared to the disk made of incompressible material.

For the disk whose thickness decreases exponentially along radius, circumferential stress for compressible and incompressible material is again maximum at the internal surface. This
circumferential stress is going on increasing with the increase in angular speed. With the introduction of compressibility factor, circumferential stress is going on decreasing. For the disk whose thickness decreases exponentially along radius, circumferential stresses going on increasing. From the analysis of disk with exponentially varying thickness, it has been observed that if the thickness of the disk decreases exponentially from \( k = 2 \) to \( k = 4 \), circumferential stress is going on increasing.

In figures 9.4 – 9.6, curves have been drawn for creep strain rates and radii ratio \( R = r / b \). It is seen from the figures, that strain rates have maximum value at the internal surface. The radial and circumferential strain increases with the increase in angular speed. The radial and circumferential strain decreases with the increase in measure \( N \). It is seen that the deformation for a rotating disk whose thickness decreases exponentially along radius is much less as compared to the flat disk.

### 9.6 CONCLUSION

From the above analysis of stresses, it has been observed that circumferential stress is less for the compressible disk as compared to the incompressible disk. Also it is observed that circumferential stress is less for the disk whose thickness decreases exponentially along radius as compared to the flat disk. Therefore we can conclude that disk made of compressible material is on the safer side of the design as compared to the incompressible disk with exponentially variable thickness as well as to the flat disk (compressible and incompressible).
Figure 9.1: Creep stresses in a thin rotating disk along various radii ratio with compressibility ($C = 0, 0.25, 0.5$) and angular speed ($\Omega^2 = 50, 75$) with constant thickness ($k = 0$).
Figure 9.2: Creep stresses in a thin rotating disk along various radii ratio with compressibility ($C = 0, 0.25, 0.5$), angular speed ($\Omega^2 = 50, 75$) and variable thickness ($k = 2$).
Figure 9.3: Creep stresses in a thin rotating disk along various radii ratio with compressibility ($C = 0, 0.25, 0.5$), angular speed ($\Omega^2 = 50, 75$) and variable thickness ($k = 4$).
Figure 9.4: Strain rates distribution in a thin rotating disk along various radii ratio with compressibility $(C = 0, 0.25, 0.5)$, angular speed $(\Omega^2 = 50, 75)$ with constant thickness $(k = 0)$. 
Figure 9.5: Strain rates distribution in a thin rotating disk along various radii ratio with compressibility ($C = 0, 0.25, 0.5$), angular speed ($\Omega^2 = 50, 75$) with variable thickness ($k = 2$).
Figure 9.6: Strain rates distribution in a thin rotating disk along various radii ratio with compressibility $(C = 0, 0.25, 0.5)$, angular speed $(\Omega^2 = 50, 75)$ with variable thickness $(k = 4)$. 