CHAPTER 3

ELASTIC-PLASTIC TRANSITION OF TRANSVERSELY ISOTROPIC THICK - WALLED ROTATING CYLINDER UNDER INTERNAL PRESSURE

3.1 INTRODUCTION

The constantly increasing industrial demand for axisymmetrical cylindrical and spherical components has concentrated the attention of designers and scientists on this particular area of activity. The progressive, worldwide scarcity of materials, combined with their consequently higher cost, makes it increasingly less attractive to confine design to the customary elastic regime only. For an ideal plastic material without strain hardening, the stress distribution in solid rotating cylinder has been given by Nadai [5]. The addition of a central hole and consideration of a rigid plastic material have been discussed by Davis and Connelly [99]. Rimrott [100] considered the hollow cylinder in the fully plastic state, using large strain theory and von Mises yield condition. Hodge and Balaban [101] gave elastic - plastic solution of a rotating cylinder.

Thick-walled cylinders of circular cross-section are used commonly either as pressure vessels intended for storage in industrial gases or as media for transportation of high pressurized fluids. A thick-walled cylinder is also widely used as a structural component in oil refineries, power industries, atomic power plants, etc. Problem of thick-walled cylinder have been analyzed by many authors [102 - 107] under different conditions. Development of the theory for thick cylinders is concerned with sections remote from the ends since distribution of the stresses around the joints makes analysis at the ends particularly
complex. For central sections the applied pressure system which is normally applied to thick cylinders is symmetrical and all points on an annular element of the cylinder wall will be displaced by the same amount. This amount depending on the radius of the element. Consequently there can be no shearing stress set up on the transverse planes and stresses on such planes are principal stresses. Similarly, since the radial shape of the cylinder is maintained there are no shears on radial and tangential planes and again stresses on such planes are principal stresses.

In this chapter, elastic-plastic stresses for transversely isotropic thick-walled rotating cylinder under internal pressure have been obtained by using transition theory [42-44, 59, 92-95, 108-109]. Results obtained have been discussed numerically and depicted graphically.

### 3.2 GOVERNING EQUATIONS

Consider a thick-walled circular cylinder of internal and external radii ‘a’ and ‘b’ respectively, subjected to internal pressure ‘p’ applied at the internal surface and rotating with an angular speed ‘ω’. The stress state in the wall is essentially triaxial and initial analysis gives the principal stresses as axial, tangential and radial.

The displacement components and generalized components of strain are same as in equation (2.1) and (2.3) respectively.

The stress-strain relations for transversely isotropic materials are

\[
T_{rr} = C_{11}e_{rr} + (C_{11} - 2C_{66})e_{\theta\theta} + C_{13}e_{zz},
\]

\[
T_{\theta\theta} = (C_{11} - 2C_{66})e_{rr} + C_{11}e_{\theta\theta} + C_{13}e_{zz},
\]

\[
T_{zz} = C_{13}e_{rr} + C_{13}e_{\theta\theta} + C_{33}e_{zz},
\]

\[
T_{rz} = T_{\theta\theta} = T_{r\theta} = 0,
\]

(3.1)

where \(C_{ij}\)'s are material constants.

Using equations (2.3) in equations (3.1), the stress-strain relations can be written in the form

\[
T_{rr} = (C_{11} / n)\left[1 - (\beta + r\beta')^n\right] + \left[(C_{11} - 2C_{66}) / n\left[1 - \beta^n\right]\right] + C_{13}e_{zz},
\]

\[
T_{\theta\theta} = \left[(C_{11} - 2C_{66}) / n\left[1 - (\beta + r\beta')^n\right]\right] + \left[(C_{11} / n)\left[1 - \beta^n\right]\right] + C_{13}e_{zz},
\]

\[
T_{zz} = \left(C_{13} / n\right)\left[1 - (\beta + r\beta')^n\right] + \left(C_{13} / n\right)\left[1 - \beta^n\right] + C_{33}e_{zz},
\]

\[
T_{rz} = T_{\theta\theta} = T_{r\theta} = 0.
\]

(3.2)

Substituting equation (3.2) in (2.6), we get a non-linear differential equation in \(\beta\) as
\[ nPC_{11} \beta^{n+1} (1+P)^{n-1} \frac{dP}{d\beta} = -nP C_{11} \beta^n (1+P)^n - (C_{11} - 2C_{66}) nP \beta^n + 2C_{66} \left[1 - \beta^n (1+P)^n\right], \]
\[ -2C_{66} (1-\beta^n) + \rho nr^2 \omega^2 \]

(3.3)

where \( r\beta' = \beta P \).

The transitional points of \( P \) in equation (3.3) are \( P \to -1 \) and \( P \to \pm \infty \).

The boundary conditions are
\[ T_{rr} = -p \quad \text{at} \quad r = a, \]
and
\[ T_{rr} = 0 \quad \text{at} \quad r = b. \]

(3.4)

The resultant force normally applied to the ends of cylinder is
\[ 2\pi \int_a^b rT_{rr} \, dr = \pi a^2 p. \]

(3.5)

### 3.3 SOLUTION THROUGH PRINCIPAL STRESSES

It has been shown [69, 95-98, 110-111] that the asymptotic solution through the principal stress leads from elastic to plastic state at the transition point \( P \to \pm \infty \). For finding the transitional stresses at the transition point \( P \to \pm \infty \), we define the transition function \( R \) as
\[ R = 2(C_{11} - C_{66}) + nC_{13} e_{zz} - nT_{rr} \]
\[ = \beta^n \left[ (C_{11} - 2C_{66}) + C_{11} (1+P)^n \right]. \]

(3.6)

Taking the logarithmic differentiation of equation (3.6) with respect to \( 'r' \) and using the asymptotic value \( P \to \pm \infty \), we get the transition function \( R \) on integration as
\[ R = A_1 r^{-C_1}, \]

(3.7)

where \( A_1 \) is a constant of integration and \( C_1 = 2C_{66} / C_{11} \).

Using equation (3.7) in equation (3.6), we have
\[ T_{rr} = C_3 - (A_1 / n) r^{-C_1}, \]

(3.8)

where
\[ C_3 = \left[ 2(C_{11} - C_{66}) + nC_{13} e_{zz} \right] / n. \]

Using boundary condition (3.4) in equation (3.8), the integration constants are
\[ A_1 = nb^{C_1} \left[ \frac{P}{(b/a)^{C_1} - 1} \right], \quad C_3 = \left[ \frac{P}{(b/a)^{C_1} - 1} \right]. \]

(3.9)

Substituting the value of \( A_1 \) and \( C_3 \) in equation (3.8), we get
\[ T_{rr} = \left[ \frac{p}{(b/a)^{C_1} - 1} \right] \left[ 1 - \left( \frac{b}{r} \right)^{C_1} \right]. \] \tag{3.10}

Using equation (3.10) in equation (2.6), we have
\[ T_{\theta\theta} = \left[ \frac{p}{(b/a)^{C_1} - 1} \right] \left[ 1 - (1 - C_1) \left( \frac{b}{r} \right)^{C_1} \right] + \rho r^2 \omega^2. \] \tag{3.11}

The axial stress is obtained from equation (2.5) as
\[ T_{zz} = \frac{C_{13}}{2(C_{11} - C_{66})} \left[ T_{rr} + T_{\theta\theta} \right] + \left[ \frac{C_{33}(C_{11} - C_{66}) - C_{13}^2}{C_{11} - C_{66}} \right] e_{zz}. \] \tag{3.12}

Applying the end condition (3.5) in equation (3.12), the axial strain is given by
\[ e_{zz} = \frac{(C_{11} - C_{66})}{C_{33}(C_{11} - C_{66}) - C_{13}^2} \left[ \frac{a^2 p}{b^2 - a^2} - \frac{a^2 C_{13} p}{(C_{11} - C_{66})(b^2 - a^2)} - \frac{C_{13}}{4(C_{11} - C_{66})} \rho \omega^2 (b^2 + a^2) \right]. \] \tag{3.13}

Substituting equation (3.13) in equation (3.12), we get
\[
\begin{align*}
T_{zz} &= \frac{C_{13}}{C_{11}(2 - C_1)} \left[ \left( \frac{p}{(b/a)^{C_1} - 1} \right) \left( 2 - \left( 2 - C_1 \left( \frac{b}{r} \right)^{C_1} \right) \right) + \frac{a^2 p}{b^2 - a^2} - \frac{2a^2 pC_{13}}{C_{11}(2 - C_1)(b^2 - a^2)} \right] \\
&+ \frac{C_{13} \rho r^2 \omega^2}{C_{11}(2 - C_1)} - \frac{C_{13} \rho \omega^2 (b^2 + a^2)}{2C_{11}(2 - C_1)}.
\end{align*}
\] \tag{3.14}

From equation (3.10) and (3.11), we obtained the stress difference as
\[ T_{\theta\theta} - T_{rr} = \left[ \frac{p}{(b/a)^{C_1} - 1} \right] C_1 \left( \frac{b}{r} \right)^{C_1} + \rho r^2 \omega^2. \] \tag{3.15}

It is found that the value of \[T_{\theta\theta} - T_{rr}\] is maximum at \( r = a \), which means yielding of the cylinder will take place at the internal surface. Therefore, we have
\[
|T_{\theta\theta} - T_{rr}|_{r=a} = \left[ \frac{p}{(b/a)^{C_1} - 1} \right] C_1 \left( \frac{b}{a} \right)^{C_1} + \rho a^2 \omega^2 \equiv Y, \text{say.} \] \tag{3.16}

The pressure required for initial yielding is given by
\[ P_i = \frac{p}{Y} = \left( 1 - \left( \frac{\rho a^2 \omega^2}{Y} \right) \right) \left[ \left( \frac{b}{a} \right)^{C_1} - 1 \right]. \] \tag{3.17}
Using equation (3.17) in equation (3.10), (3.11) and (3.14), we get transitional stresses as

\[
\sigma_r = \frac{T_{rr}}{Y} = \left[ \frac{P}{(b/a)^{C_1} - 1} \right] \left[ 1 - \left( \frac{b}{r} \right)^{C_1} \right],
\]

\[
\sigma_\theta = \frac{T_{\theta\theta}}{Y} = \left[ \frac{P}{(b/a)^{C_1} - 1} \right] \left[ 1 - \left( C_1 \left( \frac{b}{r} \right)^{C_1} \right) \right] + \frac{\rho r^2 \omega^2}{Y},
\]

\[
\sigma_z = \frac{T_{zz}}{Y} = \frac{C_{13}}{C_{11}(2-C_1)} \left[ \frac{P}{(b/a)^{C_1} - 1} \right] \left[ 2 - (2-C_1 \left( \frac{b}{r} \right)^{C_1} \right] + \frac{a^2 P}{b^2-a^2} + \frac{C_{13}}{C_{11}(2-C_1)} \frac{\rho r^2 \omega^2}{Y} \\
- \frac{2a^2 C_{13} P}{C_{11}(2-C_1)(b^2-a^2)} - \frac{C_{13}}{2C_{11}(2-C_1)} \frac{\rho \omega^2 b^2}{Y} - \frac{C_{13}}{2C_{11}(2-C_1)} \frac{\rho \omega^2 a^2}{Y}.
\]

(3.18)

Equations (3.18) give elastic - plastic transitional stresses in thick - walled rotating cylinder under internal pressure.

For fully plastic state \((C_1 \rightarrow 0)\), equation (3.16) becomes

\[
|T_{\theta\theta} - T_{rr}|_{r=b} = \frac{P}{\log(b/a)} + \rho \frac{b^2 \omega^2}{Y^*} = Y^*, \text{ say}.
\]

(3.19)

The pressure required for fully plastic state is obtained from equation (3.19) as

\[
P_f = \frac{P}{Y^*} = \left( 1 - \left( \rho \frac{b^2 \omega^2}{Y^*} \right) \right) \log(b/a).
\]

(3.20)

From equation (3.18), we get plastic stresses as

\[
\sigma_r^* = \frac{T_{rr}^*}{Y^*} = \left( 1 - \rho \frac{b^2 \omega^2}{Y^*} \right) \log \left( \frac{r}{b} \right),
\]

\[
\sigma_\theta^* = \frac{T_{\theta\theta}^*}{Y^*} = \left( 1 - \rho \frac{b^2 \omega^2}{Y^*} \right) \left( \log \left( \frac{r}{b} \right) \right) + \rho \frac{r^2 \omega^2}{Y^*},
\]

\[
\sigma_z^* = \frac{T_{zz}^*}{Y^*} = \frac{C_{13}}{2C_{11}} \left[ \left( 1 - \rho \frac{b^2 \omega^2}{Y^*} \right) \left( 1 + 2 \log \left( \frac{r}{b} \right) \right) \right] + \frac{C_{13}}{2C_{11}} \rho \frac{r^2 \omega^2}{Y^*} \left( 1 - \rho \frac{b^2 \omega^2}{Y^*} \right) \log \left( \frac{b}{a} \right)
\]

\[
- \frac{a^2 C_{13}}{C_{11}(b^2-a^2)} \left[ \left( 1 - \rho \frac{b^2 \omega^2}{Y^*} \right) \log \left( \frac{r}{b} \right) \right] + \frac{C_{13}}{2C_{11}} \rho \frac{r^2 \omega^2}{Y^*} - \frac{C_{13}}{4C_{11}} \rho \frac{\omega^2}{Y^*} \left( b^2 + a^2 \right).
\]

(3.21)
3.4 ISOTROPIC MATERIAL

For an isotropic material

\[
C_{11} = C_{22} = C_{33} , \\
C_{12} = C_{21} = C_{13} = C_{31} = C_{23} = C_{32} = C_{11} - 2C_{66} .
\]

In terms of Lamé’s constants \( \lambda \) and \( \mu \), these can be written as

\[
C_{11} = \lambda + 2\mu , \\
C_{12} = \lambda , \quad C_{66} = \frac{1}{2}(C_{11} - C_{12}) = \mu.
\]

Hence

\[
C_i = \frac{2C_{66}}{C_{11}} = \frac{2\mu}{\lambda + 2\mu} = C .
\]

From equation (3.15), we have

\[
T_{\theta\theta} - T_{rr} = \left[ \frac{p}{(b/a)^c - 1} \right] C(b/r)^c + \rho r^2 \omega^2 . \tag{3.22}
\]

It is found that the value of \( |T_{\theta\theta} - T_{rr}| \) is maximum at \( r = a \), which means that yielding of the cylinder will take place at the internal surface.

\[
|T_{\theta\theta} - T_{rr}|_{r=a} = \left[ \frac{p}{(b/a)^c - 1} \right] C \left( \frac{b}{a} \right)^c + \rho a^2 \omega^2 \equiv Y , \text{ say.} \tag{3.23}
\]

The pressure required for initial yielding is given by

\[
P_i = \frac{p}{Y} = \left( \frac{1 - \left( \rho a^2 \omega^2 / Y \right)}{C(b/a)^c} \right) \left( \frac{b}{a} \right)^c - 1 . \tag{3.24}
\]

By using equation (3.24), we get the elastic-plastic transitional stresses as

\[
\sigma_r = \frac{T_{rr}}{Y} = \left[ \frac{P_i}{(b/a)^c - 1} \right] \left[ 1 - (b/r)^c \right] ,
\]

\[
\sigma_\theta = \frac{T_{\theta\theta}}{Y} = \left[ \frac{P_i}{(b/a)^c - 1} \right] \left[ 1 - (1 - C)(b/r)^c \right] + \frac{\rho r^2 \omega^2}{Y} ,
\]

\[
\sigma_z = \frac{T_{zz}}{Y} = \left( \frac{1 - C}{2 - C} \right) \left[ \frac{P_i}{(b/a)^c - 1} \right] \left[ 2 - (2 - C)(b/r)^c \right] + \frac{P_i}{(b/a)^2 - 1} \\
+ \left( \frac{1}{2} \right) \left[ \frac{1 - C}{2 - C} \right] \left( \frac{\rho b^2 \omega^2}{Y} \right) \left[ 2(r/b)^2 - 1 - (a/b)^2 \right] - \frac{2P_i}{(b/a)^2 - 1} \left( \frac{1 - C}{2 - C} \right) . \tag{3.25}
\]
The pressure required for fully plastic state is given as
\[ P_f = \frac{P}{Y^*} = \left(1 - \rho b^2 \omega^2 / Y^*\right) \log(b/a). \] (3.26)

For fully plastic state \((C \to 0)\), the stresses in non-dimensional form are
\[ \sigma_r^* = \left[1 - \rho b^2 \omega^2 / Y^*\right] \log(r / b), \]
\[ \sigma_\theta^* = \left[1 - \rho b^2 \omega^2 / Y^*\right] \left[1 + \log(r / b)\right] + \rho r^2 \omega^2 / Y^*, \]
\[ \sigma_z^* = \frac{1}{2} \left[\frac{1 - \rho b^2 \omega^2}{Y^*} \left[1 + 2 \log(r / b)\right] + \frac{1}{2} \frac{\rho r^2 \omega^2}{Y^*} - \frac{\rho b^2 \omega^2}{Y^*} \left[1 + (a/b)^2\right]\right]. \] (3.27)

These equations are same as obtained by Pankaj [112].

### 3.5 NUMERICAL ILLUSTRATION AND DISCUSSION

As a numerical illustration, elastic constants \(C_i^*\)'s have been given in table 2.1 for transversely isotropic material (Mg) and isotropic material (Brass). The pressure required for initial yielding \(P_i\) and fully plasticity \(P_f\) at different angular speeds has been given in table 3.1. It is observed that the percentage increase in pressure required for initial yielding to become fully plasticity decreases with the increase in radii ratio. When the speed of rotation increases from \(\Omega^2 = 2\) to \(\Omega^2 = 3\), there is significant change in percentage required from initial yielding to fully plastic state. The percentage change from initial yielding to fully plasticity increases with the increase in radii ratio. From table 3.1, it has been observed that a thick-walled circular cylinder made of transversely isotropic material, requires high percentage increase in pressure to become fully plastic as compared to isotropic material from its initial yielding and this percentage goes on increasing with the increase in angular speed.

It has been observed from figure 3.1 that a thick-walled circular cylinder made of isotropic material having radii ratio \((a/b = 0.2)\) yields at high pressure as compared to cylinder made of transversely isotropic material whereas for a thick-walled circular cylinder having radii ratio \((a/b = 0.5)\) yields at low pressure. With the increase in angular speed, it has been observed that less pressure is required for initial yielding at radii ratio \((a/b = 0.2)\) for transversely isotropic material as compared to isotropic material. Curves for transitional stresses have been drawn in figure 3.2 –figure 3.4. For the thick-walled circular cylinder without rotation, it has been observed that circumferential stress is maximum at the external surface. Circumferential stress increases with the increase in rotation and radii ratio. With the
increase in radii ratio the circumferential stress increases. In figure 3.5 – figure 3.7, curves have been drawn between stresses and radii ratio \( R = r/b \) for fully plastic state and the same trends are seen for fully plastic state. It has been observed that for fully plastic state, circumferential stress is maximum at external surface.

### 3.6 CONCLUSION

A thick-walled circular cylinder made of isotropic material yields at the internal surface at a high pressure as compared to cylinder made of transversely isotropic material. With the increase in angular speed, less pressure is required for initial yielding at the internal surface for transversely isotropic material as compared to isotropic material, while percentage increase in pressure required from initial yielding to fully plastic state is high for cylinder made of transversely isotropic material as compared to cylinder made of isotropic material. Thus, it can be concluded that circular cylinder under internal pressure made of isotropic material is on the safer side of the design as compared to the circular cylinder under internal pressure made of transversely isotropic material.

<table>
<thead>
<tr>
<th>( \Omega )</th>
<th>( P )</th>
<th>( a/b = 0.2 )</th>
<th>( a/b = 0.3 )</th>
<th>( a/b = 0.4 )</th>
<th>( a/b = 0.5 )</th>
<th>( a/b = 0.2 )</th>
<th>( a/b = 0.3 )</th>
<th>( a/b = 0.4 )</th>
<th>( a/b = 0.5 )</th>
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<tbody>
<tr>
<td>0</td>
<td>( P_i )</td>
<td>1.06</td>
<td>0.98</td>
<td>0.72</td>
<td>0.57</td>
<td>52</td>
<td>36</td>
<td>28</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>( P_f )</td>
<td>1.61</td>
<td>1.2</td>
<td>0.92</td>
<td>0.69</td>
<td>1.61</td>
<td>1.2</td>
<td>0.92</td>
<td>0.69</td>
</tr>
<tr>
<td>2</td>
<td>( P_i )</td>
<td>0.98</td>
<td>0.72</td>
<td>0.49</td>
<td>0.29</td>
<td>63</td>
<td>67</td>
<td>84</td>
<td>141</td>
</tr>
<tr>
<td></td>
<td>( P_f )</td>
<td>1.61</td>
<td>1.2</td>
<td>0.92</td>
<td>0.69</td>
<td>1.61</td>
<td>1.2</td>
<td>0.92</td>
<td>0.69</td>
</tr>
<tr>
<td>3</td>
<td>( P_i )</td>
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<td>0.64</td>
<td>0.37</td>
<td>0.14</td>
<td>244</td>
<td>275</td>
<td>386</td>
<td>900</td>
</tr>
<tr>
<td></td>
<td>( P_f )</td>
<td>3.22</td>
<td>2.41</td>
<td>1.83</td>
<td>1.39</td>
<td>3.22</td>
<td>2.41</td>
<td>1.83</td>
<td>1.39</td>
</tr>
</tbody>
</table>

Table 3.1: Pressure required for initial yielding and fully plastic state at different angular speeds.
Figure 3.1: Pressure required for initial yielding at the internal surface of the cylinder at different angular speeds.

Figure 3.2: Transitional stresses for a thick-walled cylinder under internal pressure at angular speed $\Omega^2 = 0$. 
Figure 3.3: Transitional stresses for a thick-walled cylinder under internal pressure at angular speed $\Omega^2 = 2$.

Figure 3.4: Transitional stresses for a thick-walled cylinder under internal pressure at angular speed $\Omega^2 = 3$. 
Figure 3.5: Fully - plastic stresses for a thick - walled cylinder under internal pressure at angular speed $\Omega^2 = 0$.

Figure 3.6: Fully - plastic stresses for a thick - walled cylinder under internal pressure at angular speed $\Omega^2 = 2$.
Figure 3.7: Fully-plastic stresses for a thick-walled cylinder under internal pressure at angular speed $\Omega^2 = 3$. 