CHAPTER 1

INTRODUCTION

1.1 HISTORICAL OUTLINE

Research in solid mechanics is essentially not only for basic understanding of mechanical phenomenon but also to advance engineering methodology in most of areas throughout mechanical and structural technology. Advances in the subject are central to assure safety, reliability and economy in the design of devices, structures and complete systems and hence to the continued development of power generation technology such as fusion, nuclear and gas turbine power, aerospace and surface transportation technology, earthquake resistant designs, offshore structures, orthopedic devices, material processing and manufacturing technologies.

The mathematical theory of elasticity is occupied with an to calculate the state of strain, or relative displacement, within a solid body which is subject to the action of an equilibrating system of forces or is in a state of slight internal relative motion and with the endeavours to obtain results which shall be practically important in applications to architecture, engineering and all other useful arts in which the material of construction is solid.

The first mathematician to consider the nature of the resistance of solids to rupture was Galileo (1638). Although he treated solids as inelastic, not being in possession of any law connecting the displacements produced with the forces or of any physical hypothesis capable of yielding such a law, yet his enquiries gave the direction which was subsequently followed by many investigators. In the history of the theory started by the questions of Galileo, undoubtedly the two great landmarks are the discovery of Hooke’s law in 1660 and the formulation of the general equations by Navier in 1821. Hooke’s law provided the necessary experimental foundation for the theory. When the general equations had been obtained, all
questions of the small strain of elastic bodies were reduced to a matter of mathematical formulation.

Hooke gave in 1678 the famous law of proportionality of stress and strain which bears his name, in the words “Ut tensio sic vis; that is, power of any spring is in the same proportion with the tension thereof”. By “spring” Hooke means, as he proceeds to explain, any “spring body”, and by “tension” what he should now call “extension” or more generally “strain”. This law he discovered in 1660, but did not publish until 1676 and then only under the form of the anagram, ceiiinossstuu. This law forms the basis of the mathematical theory of elasticity and we shall hereafter consider its generalization, and its range of validity in the light of modern experimental researches.

During the first period in the history of our science (1638-1820), while the various investigations of special problems were being made, there was a cause at work which led to wide generalizations. This cause was physical speculation concerning the constitution of bodies. In the eighteenth century the Newtonian conception of material bodies, as made up of small parts which act upon each other by means of central forces, displaced the cartesian conception of a plenum pervaded by “vortices”. Newton regarded his “molecules” as possessed of finite sizes and definite shapes but his successors gradually simplified them into material points.

Navier was the first to investigate the general equations of equilibrium and vibration of elastic solids. He set out from the Newtonian conception of the constitution of the bodies and assumed that the elastic reactions arise from variations in the intermolecular forces which result from changes in the molecular configuration. He regarded the molecules as material points and assumed that the force between two molecules, whose distance is slightly increased, is proportional to the product of the increment of the distance and some function of the initial distance.

Consideration of plastic state of matter is today of interest to many branches of science and engineering. The scientific study of the plasticity of metals justly is regarded as beginning in 1864. In that year Tresca published a preliminary account of experiments on punching and extrusion, which led him to state that a metal yielded plastically when the maximum shear stress attained a critical value. Tresca’s yield criterion was applied by Saint-Venant to determine the stresses in a partly plastic cylinder subjected to torsion or bending (1870) and in a completely plastic tube expanded by internal pressure (1872). Levy (1871), adopting Saint-Venant conception of an ideal plastic material, proposed three-dimensional relations between
stress and rate of plastic strain. In 1900, Guest investigated the yielding of hollow tubes under combined axial tension and internal pressure and obtained results broadly in agreement with the maximum shear stress criterion (1913) which was interpreted by Hencky some years afterwards as implying that yielding occurred when the elastic shear-strain energy reached a critical value. In 1923, Nadai investigated both, theoretically and experimentally, the plastic zone in a twisted prismatic bar of arbitrary contour. Since then the subject has been studied intensively.

Many modern developments have resulted in the criterion of new engineering constructions and devices which operate under either elevated or low temperature conditions. For many years the influence of elevated temperature on material properties has been considered in the design of steam turbines, automobiles parts, oil refinery and other chemical equipment. In recent years high speed aircraft, gas turbines missiles, rockets and nuclear reactors have been among the developments in which high temperature exist. In these applications the influence of high temperature on the material properties is an important design consideration. The extent of this influence depends upon many factors including the material, loading conditions and state of stress. In addition, elevated temperatures produce creep or “time-flow” in materials. Creep can be defined as the time-dependent deformation produced in solids subjected to stress. For many materials, including most metals and alloys, elevated temperature must be applied in order for creep to be produced. The first recorded experiments relating to creep appeared in 1830’s in connection with suspension bridges, measuring instruments and steam engines. After a long time, Andrade (1910) gave the concepts of primary, secondary and tertiary creep in case of uniaxial creep tests with constant load or stress. In 1929, Norton discovered the exponential law, which applies to many metals, i.e.

\[ \dot{\varepsilon} = \frac{d\varepsilon}{dt} = k\sigma^n \]  

(1.1)

where \( \dot{\varepsilon} \) is strain rate, \( \sigma \) is stress and \( k \) and \( n \) are constants.

In around 1930’s, Bailey [1] had shown that creep (time-dependent plasticity) deformation of structural metals takes place under constant volume and a superimposed hydrostatic pressure does not influence creep deformation. From these facts and the assumption of isotropy, Odquist (1934) deduced constitutive relations for secondary creep under triaxial stresses, which have the same form as von-Mises equation for time-dependent plasticity. A lot of progress has been made theoretically as well as experimentally to define certain aspects of the
involved mechanisms of creep, yet no complete theory to explain this complex creep phenomenon is available.

1.2 ELASTIC-PLASTIC AND CREEP PHENOMENON
The “theory of elasticity” deals with the study of stress, strain and displacement in an elastic body under the influence of external forces. All structural materials possess to a certain extent the property of elasticity, that is, if external forces producing deformation of a structure do not exceed a certain limit, the deformation disappears with the removal of forces. For various engineering disciplines the purpose of studying elasticity is to analyze the stress and displacements of structural and machine elements in the elastic range and thereby to check the sufficiency of their strength, stiffness and stability.

“Theory of plasticity” is the name given to the mathematical study of stress and strain in plastic deformed solids. This follows the well-established precedent set up by the “theory of elasticity”. The relation of plastic and elastic properties of metals to crystal structures and cohesive forces belongs to the subject now known as ‘metal physics’. The theory of plasticity takes at its starting point, certain experimental observations of the macroscopic behavior of a plastic solid in uniform states of combined stress. The task of the theory is twofold: first, to construct explicit relations between stress and strain agreeing with the observations as closely and as universally as needed and second, to develop mathematical techniques for calculating non-uniform distributions of stress and strain in bodies permanently distorted in any way. Unlike elastic solids in which the state of strain depends only on the final state of stress, the deformation that occurs in plastic solid is determined by the complete history of loading. The plasticity problem is therefore essentially incremental in nature, the final distortion of the solid being obtained as the sum total of the incremental distortions following the strain path.

The situation in which material deforms continuously to load for a prolonged period of time, usually at elevated temperature is called ‘creep’ and so a constant load test is called a ‘creep test’. The conventional stress (load divided by initial cross-section) is called a ‘creep-stress’. The gradual strain is called ‘creep-strain’, the strain-time curve one obtains is called ‘creep-curve’ and the slope of the curve is called the ‘creep-rate’. Creep strains may be elastic, plastic or a combination of these and might occur too slowly to detect or too rapidly to follow. Even in such cases, however, there is evidence that creep strain is always time-dependent. For many materials including most metals and alloys, elevated temperatures may be applied in order for creep to be produced. Changes in temperature induce thermal expansion and have
ill-defined effects upon the material behavior. However, in various non-metallic materials such as plastics, wood and concrete, creep occurs at normal temperatures.

The theory of elasticity and plasticity describe the mechanics of deformation of most engineering solids. Both the theories as applied to metals and alloys are based on experimental studies of the relation between stress and strain in a polycrystalline aggregate under simple loading conditions. Thus they are phenomenological nature on the macroscopic scale and, as yet, owe little knowledge to structure of metals. However, in order to understand the limitations so imposed on the theories, the engineers, with the main interest in design and manufacture must have some knowledge of the structure of metals. The validity of the predictions of any mathematical model of deformation and stress-distribution, no matter how carefully constructed depends ultimately on one’s ability to access the mechanical properties and characteristic of the material to be used and to apply this information in the actual calculations. For this reason, testing material properties is a matter of utmost importance for engineers. No design can possibly be regarded as successful unless it is firmly based on good understanding of the potential of the material and its consequent ability to respond satisfactorily to the proposed system of loading. It is important to realize that any mechanical testing technique - no matter how appropriate - can provide information only about the average material properties to be generally expected but cannot provide an explanation of why a material behaves in a certain manner. It does not even necessarily follow that data obtained in a test on a specific material will be totally reliable when the material apparently the same, but from a different batch, is used in a manufacturing process. We can only expect that in the absence of any unknown internal defects, i.e., metallic and non-metallic inclusions, alloy will behave in the way suggested by the test results. The basic method for bringing out the behavior of structural materials is the ‘tensile test’.

The detailed stress-strain behaviour is defined in figure 1.1. For the first part of the test, it will be observed that Hooke’s law is obeyed, i.e. the material behaves elastically and stress is proportional to strain, giving the straight-line graph indicated. Some point A is eventually reached, however, when the linear nature of the graph ceases and this point is termed as a ‘limit of proportionality’. For a short period beyond this point the material may still be elastic in the sense that deformations are completely recovered when load is removed (i.e. strain returns to zero) but Hooke’s law does not apply. The limiting point B for this condition is termed the ‘elastic limit’. For most practical purposes it can often be assumed that points A and B are coincident. Beyond the elastic limit plastic deformation occurs and strains are not
totally recoverable. There will thus be some permanent deformation when load is removed. After the points \( C \), termed the upper limit point and \( D \), the lower yield point, relatively rapid increases in strain occur without correspondingly increases in load or stress. The graph thus becomes much more shallow and covers a much greater portion of the strain axis than the elastic range of the material. The capacity of a material to allow these large plastic deformations is a measure of the so called ductility of the material.

**Figure 1.1:** Typical tensile test curve for mild-steel (Stress-strain).

Beyond the yield point some increase in load is required to take the strain to point \( E \) on the graph. Between \( D \) and \( E \) the material is said to be in the elastic - plastic state. Some of the section remaining elastic and hence contributing to recovery of the original dimensions if load is removed, the remainder being plastic. Beyond \( E \) the cross-sectional area of the bar begins to reduce rapidly over a relatively small length of the bar and the bar is said to neck. This necking takes place whilst the load reduces, and fracture of the bar finally occurs at the point \( F \).

In experimental curve like that of figure 1.1 the strain is usually defined as \( e = \left( \frac{l-l_0}{l_0} \right) \), where \( l \) is the length of the deformed specimen originally of length \( l_0 \). On occasion for experiments
involving large strains, the experimental results are plotted using the measures \( e = \log \left( \frac{l}{l_0} \right) \).

For \( e \leq 1 \) the two measures virtually coincide, as for example in the initial elastic region.

A completely characteristic feature of metal’s stress-strain curve is that the slope in the plastic region is much smaller than in the elastic region. The stress-strain curve is characteristic of the material and depends not only on the chemical constitution but also on heat-treatment and methods of fracture. The strain at fracture is used as a measure of ductility, i.e., if the fracture strain is ‘large’, the material is said to be ductile and if it is ‘small’, it is used to be brittle. The terms ‘large’ and ‘small’ are usually related to the strain at yield.

The stress-strain behaviour that we have seen, namely, the initial elasticity followed by plasticity to fracture, does not depend upon the time. As soon as the stress changes, strain changes. Elastic and plastic behaviour are thus said to be instantaneous. However, they depend on the rate of straining. As increase in the rate of straining of the material, irrespective of the method of testing, increases the strain-hardening effect and therefore raises the level of yield stress. That is, if the tensile test is performed at different strain rates from very slow to very fast, the typical effect on the stress-strain curve is shown in figure 1.2. This shows that an increase in strain rate causes an increase in elastic modulus, yield stress, fracture stress and decrease in ductility. The material properties at high strain rate depend not only on the strain actually imposed but also to a greater degree on the strain hardening effects and therefore on the thermal control which will influence the movement of dislocations and the rate of their flow. Usually an increase in the test temperature decreases both the yield and elastic modulus. Thus, the temperature also influences the fracture ductility.

At high temperatures, we observe that plastic strain, under the effect of a relatively small stress, grows with time. Creep is the gradual increase of the plastic strains in a material with time at constant load. Particularly at elevated temperature some materials are susceptible to this phenomenon and even under the constant load mentioned strains can increase continually until fracture. This form of fracture is particularly relevant to turbine blades, nuclear reactors, furnaces, rocket motors, etc.

The general form of the strain verses time graph or creep curve is shown in figure 1.3 for two typical operating conditions. In each case the curve can be considered to exhibit four principal features:

1. An initial strain, due to initial application of load. In most cases this is an elastic strain.
2. A primary creep region, during which the creep rate (slope of the graph) diminishes.
(3) A **secondary creep** region, when the creep rate is sensibly constant.

(4) A **tertiary creep** region, during which the creep rate accelerates to final fracture.

**Figure 1.2:** Effect of strain-rate on tensile test.

**Figure 1.3:** Creep Curve
In general, strain-time curve are of the shape as in figure 1.3 where four stages in the creep curve can be distinguished. At $t = 0$, the curve $OA$ shows an instantaneous response $\varepsilon_0$, which depending on the magnitude of stress, could be elastic or elasto-plastic. The portion $AB$ is characterized by a relatively high creep strain rate, which decreases with time as a result of strain hardening, is called primary creep, transient creep or logarithmic creep. This portion of the curve is attained at both low and high temperatures. In fact, it can occur in the absence of thermal activation. The portion $BC$, which is linear, corresponds to constant creep strain rate or maximum creep rate where the effect of strain-hardening is balanced by an annealing influence and is called the secondary creep. At some point, another transition occurs and creep rate again accelerates and an upward curving portion $CD$ is attained. This final stage, which is called tertiary creep, represents a region where creep rate continues to increase and where reduction in cross-sectional area is accompanied by an increase in stress finally resulting in fracture. For the lower stresses and temperature the final stage of creep is not observable during the usual time covered by creep tests. Materials differ in the arrangements of these regimes; some have hardly any secondary while other has hardly any tertiary creep and so on. The curve also shows that for every material, rupture eventually occurs.

We usually associate creep with high temperatures, but whether the temperature required is high or not, really depends upon the material. Potentially, creep can take place at any temperature above absolute zero. In various non-metallic materials such as concrete, wood and natural high polymers, creep occurs at normal temperature. For typical structural materials like Magnesium (Mg), Lead (Pb) and its alloys, creep occurs at or below room temperature whereas in other materials such as metals, creep occurs at high temperature.

This creep phenomenon of metals is known for over a decade. But it is only recently that any progress could be made to formulate a realistic picture of the mechanism for creep. It was the complexity of the process which is responsible for slow progress of past in developing a satisfactory understanding of the theory of creep. A lot of progress has been made theoretically as well as experimentally to define certain major aspects of the involved mechanism of creep, yet no complete theory to explain this complex creep phenomenon is available.

When metals are severely deformed in a particular direction, as in rolling, drawing, stretching or forging, the mechanical properties may be anisotropic on a macroscopic scale. Single crystals are usually highly anisotropic with respect to plastic deformation. The fabrication
procedures used to produce metals in this section at most invariably give rise to anisotropic mechanical properties.

The anisotropic behaviour of most cubic metals can be handled by defining the yield criterion experimentally. The Hill’s theory [2], does not satisfactorily account for the mechanical anisotropy of any of the anisotropic metals except that of materials exhibiting planar isotropy. It does not allow the presence of a Bauschinger effect and applied only to anisotropic materials of restricted orthotropic symmetry. Anisotropic and the Bauschinger effect are generally found together, but they arise from different causes and the Bauschinger effects can be removed by a mid annealing while the preferred orientation is retained. However, due to recent aerospace and commercial applications, that is, helicopter, rotor blade, compressors, fly wheel, automobile structures etc., anisotropic non-homogeneous materials having variable thickness are effectively utilized. Thus there is an obvious need for further research in this area.

A number of review papers, books and proceedings of symposia have been published with macroscopic elastic, plastic and creep behaviour of anisotropic materials. The books published include those by Hill [2], Love [3], Prescott [4], Nadai [5], Marsden and Hughes [6], Bridgman [7], Hoffmann and Sachs [8], Swainger [9], Green [10], Sokolinikoff [11], Finnie and Heller [12], Goodier and Hodge [13], Lubahn and Felger [14], Hearmon [15], Johnson and Meller [16], Marin [17], Goodier and Timoshenko [18], Odquist [19], Parkus [20], Kraus [21], Boyle and Spence [22], Chakrabarty [23], Nabarro and Villiers [24], Penny and Marriott [25], Han and Reddy [26], Altenbach and Skrzypek [27], Ganczarski and Skrzypek [28], Hetnarski and Ignaczak [29] and Sadd [30].

1.3 FINITE DEFORMATION

A good deal of work has been done on the theory of finite deformation. It has been applied to various problems [31-35] which cannot be dealt with the classical theory of small deformation (infinitesimal). It has qualitatively predicted a yield point and has combined into one, the two rival hypothesis of elastic failure, i.e., principal maximum stress hypothesis and maximum shear stress or maximum principal stress difference hypothesis. It has proved the existence of Bauschinger effect by giving the result that yield stress in compression can be several times than in tension. Its application gives axial stresses in cylinders subjected to large torsional shafts, which are neglected in the classical theory.
Many technically important problems in elasticity, including those of buckling and stability call for a consideration of finite deformation, that is, deformation in which the displacements together with derivatives are no longer small. To define definite finite deformation in a continuous medium, there are two methods, namely, the Lagrangian and the Eulerian. The strain components can be described by the co-ordinates of a typical particle either in the strained state as the independent variables or in the unstrained state as the independent variables. The former is known as the Eulerian description and the later is the Lagrangian description. Many investigators have adopted the Lagrangian viewpoint for reasons of mathematical convenience whereas the importance of using the Eulerian method has been stressed by Seth [31-33], Murnaghan [34] and many other authors. Seth [31] has quoted in this connection:

“Like the body stress equations, these (strain components) should be referred to the actual position of a point $P$ of the material in the strained condition, and not to position of the point considered before strain. The importance of this point, overlooked by various authors, cannot be exaggerated. Apparently, Filon and Coker, were the first to notice it and to stress its importance”.

In case of infinitesimal deformation, the Eulerian and the Lagrangian view points coalesce, and there is no need for the distinction between the two.

Let us consider an aggregate of particles in a continuous medium that lie along the curve $C_0$ in the undeformed state and let after deformation it lies along curve $C$. Let the co-ordinates of a particle lying on curve $C_0$ (before deformation) be denoted by $(a_1, a_2, a_3)$ and let the co-ordinates of the same particle after deformation (now lying on the curve $C$) be $(x_1, x_2, x_3)$.

Then the elements $dS_0$ and $dS$ of the arc of the curves $C_0$ and $C$ respectively are given by

$$dS_0^2 = da_1^2 + da_2^2 + da_3^2 = da_i da_i , \quad (i = 1, 2, 3) \tag{1.2}$$

and

$$dS^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_i dx_i , \quad (i = 1, 2, 3) \tag{1.3}$$

In Eulerian description $x_i$ are taken as independent variable and equations of transformation are of the form $a_i = a_i(x_1, x_2, x_3)$. We can write

$$da_i = a_{i,j} dx_j . \quad (i, j = 1, 2, 3) \tag{1.4}$$

Substituting equation (1.4) in equation (1.2) and (1.3), we have

$$dS_0^2 = da_i da_i = a_{i,j} a_{i,k} dx_j dx_k , \quad (i, j, k = 1, 2, 3) \tag{1.5}$$
and \[ dS^2 = dx_i dx_j = \delta_{jk} dx_j dx_k \, . \] (1.6)

The necessary and sufficient condition for the transformation \( a_i = a_i(x_1, x_2, x_3) \) to be one of rigid body motion is that \( dS^2 \) and \( dS_0^2 \) should be equal for all curves \( C_0 \). Hence we take the difference \( dS^2 - dS_0^2 \) as a measure of strain and write
\[ dS^2 - dS_0^2 = 2e_{jk}^4 dx_j dx_k \, . \] (1.7)

The strain tensor \( e_{jk}^4 \) was introduced by Cauchy for infinitesimal strains and by Almansi and Hamel for finite strains and is known as Almansi strain tensor. From the expression (1.5) and (1.6), we have
\[ dS^2 - dS_0^2 = (\delta_{jk} - a_{i,j} a_{i,k}) dx_j dx_k \, . \] (1.8)

Therefore comparing equations (1.7) and (1.8), we get
\[ 2e_{jk}^4 = \delta_{jk} - a_{i,j} a_{i,k} \, . \] (1.9)

Writing the strains \( e_{jk}^4 \) in terms of the displacement components \( u_i = x_i - a_i \), we get
\[ 2e_{jk}^4 = u_{j,k} + u_{k,j} - u_{i,j} u_{i,k} \, , \] (1.10)

where the function \( e_{jk}^4 \) are called the Eulerian strain components. If Lagrangian co-ordinates are used then \( a_i \) are taken as independent variable and equations of transformation are of the form \( x_j = x_j(a_1, a_2, a_3) \). We can write
\[ dx_i = x_{i,j} da_j \, . \] (i, j = 1, 2, 3) (1.11)

Thus the elements \( dS_0 \) and \( dS \) of the arc of the curve \( C_0 \) and \( C \) are given by
\[ dS_0^2 = da_i da_i = \delta_{jk} da_j da_k \, , \] (1.12)

and
\[ dS^2 = dx_i dx_i = x_{i,j} x_{i,k} dx_j dx_k \, . \] (1.13)

The Lagrangian components of strain \( \varepsilon_{jk} \) are defined as
\[ dS^2 - dS_0^2 = 2\varepsilon_{jk} da_j da_k \, . \] (1.14)

Also from equations (1.12) and (1.13), we have
\[ dS^2 - dS_0^2 = (x_{i,j} x_{i,k} - \delta_{jk}) da_j da_k \, . \] (1.15)

Comparing equations (1.14) and (1.15), we have
\[ 2\varepsilon_{jk} = x_{i,j} x_{i,k} - \delta_{jk} \, . \] (1.16)

Expressing \( \varepsilon_{jk} \) in terms of displacement components \( u_i = x_i - a_i \), we get
\[ 2\varepsilon_{jk} = u_{j,k} + u_{k,j} + u_{i,j}u_{i,k}. \]  

(1.17)

To show the fact that the differentiation in equation (1.10) is carried out with respect to the variable \( x_i \) while equation (1.17) is carried out with respect to the variable \( a_i \), the typical expressions for \( e^A_{jk} \) and \( \varepsilon_{jk} \) in an unbridged form can be written as

\[
e^A_{xx} = \frac{\partial u}{\partial x} - \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right],
\]

\[
\varepsilon_{aa} = \frac{\partial u}{\partial a} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial a} \right)^2 + \left( \frac{\partial v}{\partial a} \right)^2 + \left( \frac{\partial w}{\partial a} \right)^2 \right],
\]

and

\[
2e^A_{yy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} - \left[ \frac{\partial u}{\partial y} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial w}{\partial y} \right],
\]

\[
2e^A_{xy} = \frac{\partial u}{\partial a} + \frac{\partial v}{\partial a} - \left[ \frac{\partial u}{\partial a} \frac{\partial u}{\partial a} + \frac{\partial v}{\partial a} \frac{\partial v}{\partial a} + \frac{\partial w}{\partial a} \frac{\partial w}{\partial a} \right],
\]

where \( e^A_{xx} \) and \( \varepsilon_{aa} \) represent the extension of vectors originally parallel to the co-ordinate axes, while \( e^A_{ab}, e^A_{ij} \) represent shear or change of angle between vectors originally at the right angles.

For small deformations, that is, if the displacements and the derivatives are small, then the two representations, namely Eulerian and Lagrangian, become the same and hence it is immaterial which system is employed. In this case, we may neglect the non-linear terms in the partial derivatives in equation (1.10) and (1.17) and reduces both set of formula to

\[
e^A_{jk} = \eta_{jk} = \frac{1}{2} \left[ u_{j,k} + u_{k,j} \right] \quad (j, k = 1,2,3)
\]

which exists in the case of infinitesimal transformations.

The strain components (1.10) in cylindrical polar co-ordinates are given by

\[
e^{A}_{rr} = \frac{\partial u}{\partial r} - \frac{1}{2} \left[ \left( \frac{\partial u}{\partial r} \right)^2 + \left( \frac{\partial v}{\partial r} \right)^2 + \left( \frac{\partial w}{\partial r} \right)^2 - v^2 \right],
\]

\[
e^{A}_{\theta\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{r - 1}{2r^2} \left[ \left( \frac{\partial u}{\partial \theta} \right)^2 + \left( \frac{\partial v}{\partial \theta} \right)^2 + \left( \frac{\partial w}{\partial \theta} \right)^2 - v^2 \right] - \frac{u^2}{r} \frac{\partial u}{\partial \theta} + \frac{u^2 + u r^2}{r} \frac{\partial v}{\partial \theta} + u^2 \frac{\partial w}{\partial \theta} + u^2 + r^2 v^2 \right],
\]

\[
e^{A}_{zz} = \frac{\partial w}{\partial z} - \frac{1}{2} \left[ \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right],
\]
\[ e_{rr}^A = \frac{1}{2} \left( \frac{\partial u}{\partial r} + \frac{\partial v}{\partial r} \right) - \frac{1}{2r} \left[ \frac{\partial u}{\partial \theta} + \frac{\partial (rv)}{\partial \theta} \right] - \frac{1}{2r} \left[ \frac{\partial v}{\partial \theta} + \frac{\partial (ru)}{\partial \theta} \right] + \frac{r}{2} \left( \frac{\partial \nu}{\partial \theta} \right) + \frac{ru}{2} \left( \frac{\partial \theta}{\partial \theta} \right), \]
\[ e_{\theta \theta}^A = \frac{1}{2} \left( \frac{\partial \nu}{\partial \theta} + \frac{\partial u}{\partial \theta} \right) - \frac{1}{2r} \left[ \frac{\partial \nu}{\partial \theta} + \frac{\partial (ru)}{\partial \theta} \right] + \frac{ru}{2} \left( \frac{\partial \theta}{\partial \theta} \right), \]
\[ e_{zz}^A = \frac{1}{2} \left( \frac{\partial \nu}{\partial z} + \frac{\partial w}{\partial z} \right) - \frac{1}{2r} \left[ \frac{\partial \nu}{\partial z} + \frac{\partial (rz)}{\partial z} \right] + \frac{rz}{2} \left( \frac{\partial \theta}{\partial z} \right), \]
\[ e_{zr}^A = \frac{1}{2} \left( \frac{\partial \nu}{\partial r} + \frac{\partial w}{\partial r} \right) - \frac{1}{2r} \left[ \frac{\partial \nu}{\partial \theta} + \frac{\partial (ru)}{\partial \theta} \right] + \frac{ru}{2} \left( \frac{\partial \theta}{\partial z} \right), \]
\[ (1.18) \]

where \( u, v, w \) and \( e_{rr}^A, e_{\theta \theta}^A, e_{zz}^A, e_{zr}^A, e_{\theta r}^A \) are the physical components of the displacement \( u_i \) and strain tensor \( e_{ij}^A \) respectively.

For cylinder/ disk problems, the distribution of stresses is symmetrical about the axis of revolution and hence the stresses and strains are independent of polar angle \( \theta \). The displacement components in cylindrical polar co-ordinates are given by [36]
\[ u = r(1 - \beta), \quad v = 0 \quad \text{and} \quad w = dz \]
\[ (1.19) \]
where \( \beta \) is a function of \( r = \sqrt{x^2 + y^2} \) only and \( d \) is a constant.

Substituting equation (1.19) in (1.18), the finite components of strain are
\[ e_{rr}^A = \frac{1}{2} \left[ 1 - (r\beta' + \beta)^2 \right], \]
\[ e_{\theta \theta}^A = \frac{1}{2} \left[ 1 - \beta^2 \right], \]
\[ e_{zz}^A = \frac{1}{2} \left[ 1 - (1 - d)^2 \right], \]
\[ e_{zr}^A = e_{\theta r}^A = e_{\theta \theta}^A = 0, \]
\[ (1.20) \]
where \( \beta' = \frac{d\beta}{dr} \).

### 1.4 TRANSITION

The deformation theory of plasticity does not satisfy the condition of continuity of the relationship between stress and deformation on transition from loading to unloading. In analyzing this and other theories of plasticity we may encounter similar circumstances in which certain consequences of the theory appear to be contradictory or physically unacceptable. In the classical theory of plasticity, the material region is assumed to be divided into an elastic and plastic region, which is separated by a yield surface depending on the symmetry and other physical considerations. In other words perfect elasticity and ideal
plasticity are two extreme properties of the material and the use of ad-hoc rule like yield condition amounts to divide the two extreme properties by a sharp line, which is physically not possible. In the plastic region Prandtl - Reuss or von-Mises equations are used with a yield condition and the boundary condition that the normal stress components should be continuous on the yield surface. This linear theory has given rise to a vast amount of important results, which have been corroborated by experiments more exactly, then one may expect. However, its main drawbacks are the following assumptions:

(i) Even though the material at a point has yielded, the material at a neighbouring point still remains elastic.

(ii) An yield surface of the assumed type separating the elastic and plastic regions exists.

(iii) For any given material there is a function of the three principal stresses which always has a value when yielding begins regardless of the stress state.

(iv) The same functional relationship applies to all materials although the numerical value of the function is different for different materials.

Elastic - plastic transition is obtained in current literature with the help of semi-empirical yield condition like that of Tresca or von-Mises. The stresses are obtained from the elastic solution and then substituted in the yield condition to get the transition surface. The possibility of treating it as a transition or turning point phenomenon in finite deformation has not been explored. When plastic state tends to set in, the stress-strain relation undergoes a change. This must be reflected in our equations. The linear classical theory does not take this phenomena into account. Tresca pointed out that a transition state, which he called mid-zone, exists when a material passes from elastic-zone to plastic-zone and this is later on supported by Todhunter and Pearson [37]. In the transition state whole of the material participates and not simply a selected region or a line as assumed by classical theories. This is expressed by Sir Lawerence and supported by Bragg. When he pointed out that an element of a crystal embedded in the matrix shall be pictured as being strained together with the whole mass of the metal and hence slip must take place by a whole intra-atomic distance. A recent numerical study on flow and deformation theories in plasticity was undertaken to see as to what extent a continuous approximation involving the idea of transition to an elastic-plastic material in terms of the stress-strain law, would lead to a satisfactory convergent solution. The results obtained showed excellent approximations and convergence to elastic - perfectly plastic solution.

The demand of high speed technology in transportation, communication and energy conservation have forced us to take serious notice of non-linearity. But some of us still find it
difficult to get rid of a century old habit of analytical and experimental research in continuum mechanics, which leisurely leaned very heavily on linearization. If a medium $A$ changes into $B$ through a transition state $T$, $A$ and $B$ may be almost linear but $T$ is non-linear. Since this non-linearity is difficult to investigate, workers have taken to the artifice of replacing it by singular, non-differentiable or discontinuous surfaces. This piece-wise treatment necessitates the use of ad-hoc and semi-empirical laws; which may or may not exist. Linearizing non-linear problems by perturbation, boundary layers and other techniques do not provide satisfactory explanation for some important characteristics of non-linearity. As a result a number of important physical effects do not get adequate scientific expression. There is hardly any information, which we do not know about linear fields. Their existence, uniqueness and stability are well established. Nature does not always confirm to our abstract concepts of linearity, smoothness, symmetry, identity, homomorphism and isotropy. It is true that physical phenomena tend to behave linearly in course of time, which may be millions of years but the demands of modern technology want to compress years into fraction of a second. This transition, which frequently occurs in nature, has to be tackled. In fact all linear-disciplinary fields, which are so important in modern research, give rise to important transition problems. But both the macro and micro analysts have devoted very little attention to them due to non-linearity involved in the analytic treatment.

There are four different ways of treating transition fields. Firstly, they are asymptotic in character and hence they should be associated with some singularities or criticalities of the differential system describing them. If the singularities are not obvious in one plane, it should be possible by continuous mapping to recognize them in some other plane. Secondly, transition field may be interpreted as asymptotic subspace obtained from the intersection of two spaces representing different media. Thirdly, from the group theoretic point of view, all continuous deformations form a symmetric group. At transition, the nature of this group changes. For example, an elastic body, which belongs to the orthogonal group, becomes unin-molecular on becoming plastic. Lastly, from the macroscopic point of view, one can imagine that at transition the macro-element breaks down, with the result that the corresponding transformation matrix becomes singular.

In general, the material from elastic state can go over into:

(i) Plastic state, or
(ii) Creep state, or
(iii) First to plastic then to creep or vice-versa, under external loading system.
When the material under experiment goes from primary state of creep to secondary or tertiary state, transition takes place. A plastic or a creep state is a transition state from initially elastic one. All these cases of transition can be expected to occur when some functions of elasticity of the medium takes on critical values. These functions are called transition functions. These may be either principal stresses or principal stress differences or stress invariants or any suitable combination of these. These critical (asymptotic) values have to be determined at the transition points of the differential equation describing the medium. If a number of transition states occur at the same point, the transition function will have different limiting values and the point will be a multiple one, each branch of which will then correspond to different state, whereas the classical treatment uses different constitutive equations for each state involved. Transition is a natural phenomenon and there is hardly any branch of science or technology in which we do not come across transition from one state to another. Elasticity - plasticity, visco-elastic, creep, fatigue, relaxation are well known examples. At present, they are generally treating as discontinuous. For elastic - plastic deformations, the ideal model has to assume some type of yield condition. For creep, not only the yield condition but also a number of creep strain laws have to be assumed. All these assumptions leave the mind a little dissatisfied. More scientific advancements should attempt to reduce these assumptions. This has motivated us to pay some attention to the treatment of transition problems in the domain of solid mechanics.

1.4.1 IDENTIFICATION OF THE TRANSITION STATE

When a material at a point has yielded, it is more reasonable to expect that the material at the neighbouring points are on their way to yield, rather than assume that they remain in the elastic state as completely opposed to the plastic state of the nearby particle. As the plastic yielding of a material is a consequence of collapse of its internal or macroscopic structure, the plastic yielding will be complete or partial depending on the existing physical conditions. This leads us again to the recognition of two material states: a transition state and a plastic state.

There are different ways to explain how transition may occur from one state to another state:

(i) At transition, the differential system defining the elastic state should attain some criticality.

(ii) The complete breakdown of the macroscopic structure at transition should correspond to the degeneracy of the material (spatial) strain ellipsoid. This means that the length of at least one of the axes of the strain ellipsoid should be zero or infinity.
(iii) If we consider the plastic state as an image of the elastic state under the
transformation \( x^k = x^k \left( X^k \right) \),
where \((X, Y, Z), (x, y, z)\) are the co-ordinates of a point in the undeformed and deformed state,
respectively. Then at the transition, the Jacobian of the transformation should be zero or
infinity. This means when transition occurs, one-to-one correspondence between the two
states no longer holds.

1.4.2 GENERAL TREATMENT OF TRANSITION THEORY CORRESPONDING TO
THE VANISHING OF THE JACOBIAN TRANSFORMATION

Consider the transformation
\[
  x^k = x^k \left( X^k \right),
\]
which maps the metric space \( A \) into metric space \( B \). If we identify \( B \) as the plastic state and \( A \)
as the elastic state, then the isomorphism of the transformation is presumably destroyed, since
this process is irreversible and the material may change from elastic to the plastic, creep or
fatigue state. Hence, the Jacobian of transformation will therefore correspond to the transition
state from \( A \) to \( B \).

When a continuum changes from a state \( A \) to another state \( B \), as has been explained before, the
invariants of the stress and strain tensors undergo some kind of a constraint. This constraint
should be obtainable from the condition, namely the vanishing of Jacobian of transformation,
since the latter corresponds to the transition state.

If \( u, v \) and \( w \) are the displacements along the rectangular Cartesian coordinate axes, then
\[
  X = x - u, \quad Y = y - v, \quad Z = z - w,
\]
where \((X, Y, Z), (x, y, z)\) are the co-ordinates of a point in the undeformed and deformed state,
respectively. Hence the Jacobian
\[
  J = \frac{\partial (X, Y, Z)}{\partial (x, y, z)} = \left| \begin{array}{ccc}
  \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial z} \\
  \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z} \\
  \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z}
\end{array} \right| = \left| \begin{array}{ccc}
  \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} & -\frac{\partial w}{\partial x} \\
  -\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\
  -\frac{\partial u}{\partial z} & -\frac{\partial v}{\partial z} & 1-\frac{\partial w}{\partial z}
\end{array} \right| \quad (1.21)
\]
is referred to the deformed state and
to the undeformed state.

From (1.21), we have

\[
J^2 = \begin{vmatrix}
1 - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} & 1 - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} & 1 - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \\
-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} & -\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} & -\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \\
-\frac{\partial u}{\partial z} & -\frac{\partial u}{\partial z} & -\frac{\partial u}{\partial z}
\end{vmatrix}
= \begin{vmatrix}
1 - 2e_{xx} & -2e_{xy} & -2e_{xz} \\
-2e_{yx} & 1 - 2e_{yy} & -2e_{yz} \\
-2e_{zx} & -2e_{zy} & 1 - 2e_{zz}
\end{vmatrix},
\]

(1.23)

where \( 1 - 2e_{xx} = -2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \),

and \( 1 - 2e_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial v}{\partial y} \), etc.

From (1.23), if \( J = 0 \), which is a transition condition corresponding to asymptotically large extensions, we have

\[
1 - 2J_1' + 4J_2' - 8J_3' = 0,
\]

(1.24)

where

\[
J_1' = e_{xx} + e_{yy} + e_{zz},
\]

\[
J_2' = e_{xx}e_{yy} + e_{xy}e_{xz} + e_{xx}e_{zz} - e_{xy}e_{yx} - e_{xz}e_{zx} - e_{yz}e_{zy},
\]

\[
J_3' = e_{xx}e_{yy}e_{zz} - e_{xx}e_{yx}e_{zy} - e_{xx}e_{yy}e_{zx} + e_{xy}e_{zx}e_{yy}e_{xz} - e_{xy}e_{yx}e_{zy} - e_{xy}e_{yx}e_{zy} - e_{xz}e_{zx}e_{yy}.
\]

In short, we can also write it as

\[
J_1' = \frac{1}{1!} \delta^k_l e^k_l,
\]

\[
J_2' = \frac{1}{2!} \delta^{kmn}_{ln} e^k_m e^n_l,
\]

\[
J_3' = \frac{1}{3!} \delta^{knp}_{lnm} e^k_l e^m_n e^n_p.
\]

The symbol \( \delta^{kmn}_{ln} \) and \( \delta^{knp}_{lnm} \) are generalized Kronecker deltas.
Referring to the principal axes of the strain ellipsoid in the deformed state, we have from equation (1.24)
\[8J_3 - 4J_2 + 2J_1 = 1,\]  
(1.25)
where
\[J_1 = e_{11} + e_{22} + e_{33},\]
\[J_2 = e_{11}e_{22} + e_{22}e_{33} + e_{33}e_{11},\]
\[J_3 = e_{11}e_{22}e_{33},\]
(1.26)
where \(e_{11}, e_{22}, e_{33}\) are the principal strains.

Relations similar to (1.25) should hold at transition for any type of medium i.e. isotropic, anisotropic, homogeneous or heterogeneous and should come out from the transition condition independent of the constitutive equations and the momentum equations. The strain (stress) invariants are, in general, independent of each other; a functional relation exists between them only at the transition state because of the constraint.

Now from the constitutive equations (isotropic material)
\[\tau_{y} = \lambda \delta_{y} e_{\alpha\alpha} + 2\mu e_{y},\]
we have
\[e_{11} = \frac{1}{E} \left[ \tau_{11} - \sigma(\tau_{22} + \tau_{33}) \right],\]
\[e_{22} = \frac{1}{E} \left[ \tau_{22} - \sigma(\tau_{11} + \tau_{33}) \right],\]
\[e_{33} = \frac{1}{E} \left[ \tau_{33} - \sigma(\tau_{22} + \tau_{11}) \right],\]
where \(e_{\alpha}\) are principal strains.

Hence
\[J_1 = \left( \frac{1 - 2\sigma}{E} \right) I_1,\]
\[J_2 = \frac{1}{E^2} \left[ (1 + \sigma)^2 I_2 - \sigma(2 - \sigma)I_1^2 \right],\]
\[J_3 = \frac{1}{E^3} \left[ (1 + \sigma)^3 I_3 - \sigma(1 + \sigma)^2 I_1I_2 - \sigma^2 I_1^3 \right],\]
where the \(I_k\) 's are stress invariants, \(J_k\) 's are strain invariants, \(E\) is Young’s modulus of elasticity and \(\sigma\) is Poisson’s ratio.
Now from (1.25), we have
\[
\frac{8}{E^3} \left[ (1 + \sigma)^3 I_3 - \sigma(1 + \sigma)^2 I_1, I_2 - \sigma^2 I_1 \right] - \frac{4}{E^2} \left[ (1 + \sigma)^2 I_2 - \sigma(2 + \sigma)I_1 \right] + \frac{2}{E} \left[ (1 - 2\sigma)I_1 \right] = 1
\]
(1.27)

The relation (1.27) should hold at transition. Now (1.27) can be written in a more simpler form using the following notation:
\[
I_2 = (\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 = 2I_1^2 - 6I_2.
\]

Taking \( T_\mu = \frac{\tau_\mu}{E}, \) \( K_1 = \frac{I_1}{E}, \) \( K_2 = \frac{I_2}{E^2}, \) \( K_3 = \frac{I_3}{E^3} \) and \( K'_2 = \frac{I'_2}{E^2}, \)
(1.27) becomes
\[
8 \left[ (1 + \sigma)^3 K_3 + \frac{1}{6} \sigma(1 + \sigma)^2 K_1 K'_2 - \frac{1}{3} \sigma(\sigma^2 - \sigma + 1)K'_3 \right]
+ \frac{4}{3} \left[ \frac{1}{2} (1 + \sigma)^2 K'_2 - (1 - 2\sigma)K_1^2 \right] + 2\left[ (1 - 2\sigma)K'_1 \right] = 1
\]
(1.28)

The invariant relation (1.28) among the stress invariants should hold good at transition state. The condition involves elastic effect. For the fully plastic state \( \sigma \rightarrow \frac{1}{2} \) (condition of incompressibility) we have from (1.28)
\[
3K'_3 + 2 \left( 27K_3 + \frac{3}{2} K_1 K'_2 - K'_1 \right) = 2.
\]
(1.29)

Equation (1.29) could be taken as the most general yield condition for all types of media irrespective of their properties.

Rewriting (1.29), again we get
\[
3\left[ (T_{11} - T_{22})^2 + (T_{22} - T_{33})^2 + (T_{33} - T_{11})^2 \right]
+ 2\left[ (2T_{11} - T_{22} - T_{33}) (2T_{22} - T_{11} - T_{33}) (2T_{33} - T_{22} - T_{11}) \right] = 2
\]
(1.30)

The general form of the yield condition (1.30) given above has been obtained independent of the equations of equilibrium. Equation (1.30) can alternatively be written as follows:
\[
L_1^2 + L_2^2 + L_3^2 + 2L_1 L_2 L_3 = 2,
\]
(1.31)
or
\[
\left( T_{11}^d \right)^2 + \left( T_{22}^d \right)^2 + \left( T_{33}^d \right)^2 + 6T_{11}^d T_{22}^d T_{33}^d = \frac{2}{9},
\]
(1.32)

where \( L_i = (2T_{11} - T_{22} - T_{33}) \), etc. and \( T_{\mu}^d \) are the deviatoric stress tensors in the non-dimensional form.
It is interesting to note that (1.30) reduces to Hencky - von Mises yield condition or Tresca’s yield condition in some of the following special cases.

(I) PLANE STRAIN

In the case of plane strain, $e_{33} = 0$ so that $J_3 = 0$. From (1.25) we get

$$4J_2 - 2J_1 + 1 = 0$$

and in the limiting incompressible case ($\sigma \to 1/2$), we get from (1.29)

$$K_2' = \frac{2}{3}$$

or equivalently

$$\left(\tau_{11} - \tau_{22}\right)^2 + \left(\tau_{22} - \tau_{33}\right)^2 + \left(\tau_{33} - \tau_{11}\right)^2 = \frac{8}{3}y^2$$

(1.33)

which is exactly of the same form as von-Mises yield criterion, i.e.

$$\left(\tau_{11} - \tau_{22}\right)^2 + \left(\tau_{22} - \tau_{33}\right)^2 + \left(\tau_{33} - \tau_{11}\right)^2 = 2y^2$$

(1.34)

where $E = 2y$ and $y$ is the yield stress in tension.

We notice here that the constant $2y^2$ is replaced by $\frac{8}{3}y^2$. This is to be expected as (1.33) is a particular form of (1.29).

(II) PLANE STRESS

In this case, suppose $T_{33} = 0$. Then the yield condition (1.30) reduces to

$$\left(T_{11} + T_{22} + 1\right)\left(2T_{11} - T_{22} - 1\right)\left(2T_{22} - T_{11} - 1\right) = 0.$$  

(1.35)

This yields the following three conditions

$$T_{11} + T_{22} + 1 = 0, \text{ or } \ T_{out} = -1, \text{ or } \ \tau_{out} = -2y$$

$$2T_{11} - T_{22} - 1 = 0 \text{ or } \ T_{11}^{d} = \frac{1}{3}, \text{ or } \ \tau_{11}^{d} = \frac{2}{3}y$$

$$2T_{22} - T_{11} - 1 = 0, \text{ or } \ T_{22}^{d} = \frac{1}{3}, \text{ or } \ \tau_{22}^{d} = \frac{2}{3}y$$

The last two conditions are exactly Tresca’s yield condition. It is interesting to note that in a number of these cases the yield takes place through a deviatoric principal stress taking on a maximum value.
The yield condition (1.29), which is an invariant relation, could be regarded as general yield condition for all types of materials, irrespective of their properties. While von Mises or Tresca’s yield condition does not take into account the distinction between the yield stress in tension and yield stress in compression, as (1.29) does and hence includes Baushinger’s effect.

1.5 GENERALIZED STRAIN MEASURE

The response of real materials to the external loading, in general, is non-linear in character. The division of deformation into different types arose from a desire for linearization of engineering problems. In large deformations, which involve plastic flow, creep and fatigue, the current treatment requires a number of ad-hoc and semi-empirical assumptions. The adoption of these semi-empirical laws has complicated the problem without evolving any simple concept governing them. One source of the troubles is the use of classical measures of deformation produced in a medium even when we know from experiments that non-linearity is a characteristic of such deformed media. Abstract measure theory has been highly developed; even then it has not been employed in the problems of non-linear mechanics suitably. In classical mechanics, ordinary measures have been found sufficient and there arises no need for their extension. The equation of equilibrium and the concept of stresses are well defined, only the measures of deformation are flexible. A continuum approach necessarily means the introduction of non-linear measures. Deformation fields associated with irreversible phenomenon such as elastic-plastic deformation, creep, relaxation, fatigue and fracture, etc. have been known to be non-linear in character as revealed by extensive experimental studies. The use of classical measures of deformation is totally insufficient to deal with transitions and hence the corresponding constitutive equation of the medium has to be made complicated. From the above analysis it appears that we should construct generalized measure of deformation to resolve the difficulty of using classical measures to explain natural phenomenon in continuum mechanics.

Rabotnov has pointed out difficulties, particularly the ambiguities in the interpretation of the experimental data and that involved in the choice of suitable constitutive equations for various states of creep described in the figure 1.3. Two parameters characterize each of these states, one for the measure and other for the irreversibility. Since the creep strain rate depends upon the stress and temperature at a structural state, it is not the total strain but the total rate of creep strain, which is significant. Therefore, it is expected that a generalized measure concept
in which the two parameters are experimentally determined may give a better insight into creep behaviour.

The nature of the strain-rate described in four stages of creep - elastic, transient, secondary and rupture in the figure 1.3 is different in each case. As the deformation is non-linear and there arises the need for generalized strain rate measure, which can be used in all the stages. Seth [38] has defined the generalized principal strain measure as

\[ e_{ii} = \frac{1}{n} \left[ 1 - \frac{\int_0^1 (1 - 2e_{ii}^d)^{\frac{n}{2}} \, de_{ii}^d}{n} \frac{1}{n} \left[ 1 - \left( 1 - 2e_{ii}^d \right)^{\frac{n}{2}} \right] \right], \]  

(1.36)

where \( n \) is the measure and \( e_{ii}^d \) is principal Almansi finite strain components. In Cartesian framework we can readily write down the generalized measure in terms of any other measures.

For uni-axial case, it is given by Seth [38] as

\[ e = \frac{1}{n} \left[ 1 - \left( \frac{l_0}{l} \right)^n \right], \]  

(1.37)

where \( l_0 \) and \( l \) are initial and strained lengths of the rod respectively. For \( n = 0, 1, 2, -1, -2 \), it gives the Hencky, Swainger, Almansi, Cauchy and Green measures respectively. Seth [39 - 40] has shown that the well-known creep strain law used in current literature such as Norton’s law, Kachanov law, Odquist law, Andrade’s law etc. can be derived from the generalized measure.

The generalized components of strain are obtained by using equation (1.20) in equation (1.36) as

\[ e_{rr} = \frac{1}{n} \left[ 1 - (\beta + r\beta')^n \right], \]

\[ e_{\theta\theta} = \frac{1}{n} \left[ 1 - \beta^n \right], \]

\[ e_{zz} = \frac{1}{2} \left[ 1 - (d')^n \right], \]

\[ e_{r\theta} = e_{\theta r} = e_{zr} = 0, \]

where \( r\beta' = \beta P \), \( \beta' = \frac{d\beta}{dr} \) and \( n \) is the measure.

The generalized strain measure not only gives the well known strain measures as special cases, but it is also be used to find the creep stresses when it is combined with the transition
point analysis of the governing differential equations. Seth has shown that the transition point analysis does not require the assumption of incompressibility, creep strain law and yield condition as used in classical theory. He and others have successfully applied transition theory [41-45] it to a large number of problems [46-58]. Showing that the asymptotic solution of the governing differential equations at the transition point gives the results which are obtained by assuming yield criteria when they exist. The most important contribution to be made by generalized measure is that it makes the semi-empirical laws and jump conditions unnecessary. If such laws exist, they come out with the analytic treatment as a particular case. Thus, an important function of non-linear measure is to explain transition without assuming conditions to match the two solutions at transition.

In this thesis, an attempt has been made to analyze some problems of technical importance on the basis of transition concept and generalized strain measure.

### 1.6 CONSTITUTIVE EQUATIONS

Equations characterizing the individual material and its response to applied loads are called constitutive equations. The macroscopic behavior is described by these equations resulting from the internal constitution of the material. Materials in the solid state behave in such a complex way that when entire range of possible temperatures and deformations is considered, it is not feasible to write down one equation or set of equations to describe accurately a real material over its entire range of behaviour. Instead separate equations are formulated to describe the various kinds of ideal materials response, each of which is a mathematical formulation designed to approximate physical observations of a real material’s response over a suitably restricted range. The classical equations were introduced separately to meet specific needs and made as simple as possible to describe many physical situations. Some of the ideas involved in formulating simple equations for such ideal material are illustrated below.

#### (I) ELASTIC STATE

A material is said to be elastic when a material recovers its original form completely upon removal of forces causing the deformation, and there is one to one relationship between the state of stress and state of strain for a given temperature. It is assumed to obey Hooke’s law, which for uniaxial stress situation states that extension produced is proportional to the applied force. Under triaxial loading, classical elasticity theory assumes generalized Hooke’s law expressing each stress component as a linear combination of all the strains, that is
where $T_{ij}$ and $e_{ij}$ are the stress and strain tensor respectively. These nine equations contain a total of 81 coefficients $C_{ijkl}$, but not all the coefficients are independent. The symmetry of $T_{ij}$ and $e_{ij}$ reduces the number of independent coefficients to 36. The number of independent constants is further reduced by reason of their symmetry properties. If we assume the existence of strain energy density $U$ when the system is adiabatic or isothermal, the number of elastic constants is reduced from 36 to 21. For monoclinic materials (materials having one plane of elastic symmetry), there are only 13 independent elastic constants. In case of orthotropic materials (materials which possess three orthogonal planes of elastic symmetry) the number of elastic constants is further reduced to 9. For transversely isotropic materials (single response in one direction and different in other two directions), there are only five independent elastic constants. Finally, there are only two independent elastic constants known as Lame’s parameters for isotropic elastic materials (materials in which elastic properties are independent of orientation of axes).

A material in which the elastic properties depend on the orientation of the sample is said to be anisotropic. If the anisotropic elasticity of the material is to be fully-described, a total of 21 independent parameters are needed in $6 \times 6$ compliance matrix $[C]$ as shown in matrix below

$$
[C] = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
\text{Symmetry} & C_{44} & C_{45} & C_{46} & C_{56} & C_{66} \\
\end{bmatrix}
$$

For transversely isotropic materials, the constitutive equations in matrix form are

$$\begin{align*}
T_1 &= \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
0 & C_{22} & C_{23} & 0 & 0 & 0 \\
0 & 0 & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66} \\
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5 \\
e_6 \\
\end{bmatrix}
\end{align*}$$

or in cylindrical polar co-ordinates, these can be written as
Since 1950, the theory of elasticity for anisotropic bodies has been continually developed and enriched with new investigations. Thus, the general theory has been placed on a rigorous scientific basis and a number of laws have been established, with the result that this theory, first worked out by B. de Saint-Venant and P.V. Bekhterev, has been revived. Of great importance is the development and construction of many entirely new anisotropic materials, which possess a number of advantages over those previously known (e.g. glass-fiber reinforced plastics). Thus, over a quarter of century this branch of science has made great progress, both in a theoretical and a purely practical way, i.e., in constructing new anisotropic materials.

The class of materials obeying the generalized Hooke’s law may be quite wide, and hence it is necessary to give a classification reflecting their distinguishing features. As far as the elastic properties are concerned, all bodies may be divided on the one hand into homogeneous and non-homogeneous and on the other hand into isotropic and anisotropic. A homogeneous body, with regards to the elastic properties means one whose elastic properties are the same at different points. A non-homogeneous body has different elastic properties at different points. If the elastic characteristics, for example the elastic moduli, vary from point to point in a continuous manner, the non-homogeneity may also be termed continuous. However, if the elastic characteristic undergo discontinuities in passing from point to point then the non-homogeneity is said to be discontinuous.

In modern structures the members are made not only of materials that are usually considered as homogeneous and isotropic in design, but also of anisotropic materials, which show a sharp difference of elastic properties in different directions. An example of such materials is provided by natural wood: it is well known that the elastic modulus of wood in tension parallel to the grain is considerably greater than the corresponding modulus in tension.

\[
T_{rr} = C_{11} e_{rr} + C_{12} e_{\theta\theta} + C_{15} e_{zz},
\]

\[
T_{\theta\theta} = C_{21} e_{rr} + C_{22} e_{\theta\theta} + C_{23} e_{zz},
\]

\[
T_{zz} = C_{31} e_{rr} + C_{32} e_{\theta\theta} + C_{33} e_{zz},
\]

\[
T_{\theta\theta} = 2C_{44} e_{\theta\theta}, \quad T_{zz} = 2C_{55} e_{zz}, \quad T_{rr} = 2C_{66} e_{rr}, \quad (1.39)
\]

where

\[
C_{12} = (C_{11} - 2C_{66}) = C_{21},
\]

\[
C_{13} = C_{31} = C_{32} = C_{23},
\]

\[
C_{55} = C_{44}, \quad C_{11} = C_{22}, \quad C_{33}.
\]
perpendicular to the grain, and that its elastic constants depend on the direction in relation to the wood fibers. Synthetic materials used in aircraft construction, such as delta wood, aircraft plywood, fabric laminate are anisotropic (and non-homogeneous) materials. Crystals and some rocks exhibit anisotropic behaviour. Anisotropy of concrete has been investigated by many authors. Beside, materials having anisotropy depend on the internal structure. Modern structures elements with so-called artificial anisotropy are also used.

To design anisotropic members undergoing elastic strains, it is necessary to know how to determine the stresses and strains in anisotropic bodies theoretically, i.e. to solve problems of the theory of elasticity for anisotropic bodies. In the case of an anisotropic homogeneous body the number of independent elastic constants may be considerably larger, i.e. 21 in the general case of anisotropy. To solve problems of the distribution of stress and strain in an anisotropic body, it is necessary to proceed from the equations of the theory of elasticity with the consideration of elastic properties in different directions and accordingly contain more than two elastic constants.

At present the theory of elasticity for isotropic bodies is comprehensive and thoroughly developed. The theory of elasticity for anisotropic bodies is not so well developed, but in this field, too, a great body of material has been accumulated in the form of papers published in various periodicals.

For elastically isotropic material, which has same elastic properties in all directions, we have just two independent elastic constants, i.e.

\[ C_{11} = C_{22} = C_{33}, \]
\[ C_{12} = C_{21} = C_{13} = C_{31} = C_{23} = C_{32} = C_{11} - 2C_{66}. \]  

(1.40)

In terms of Lame’s constant \( \lambda \) and \( \mu \), these can be written as

\[ C_{12} = \lambda, \quad C_{66} = \frac{1}{2}(C_{11} - C_{12}) = \mu \quad \text{and} \quad C_{11} = \lambda + 2\mu. \]  

(1.41)

If we substitute equation (1.40) and (1.41) in equation (1.39), the result may be written in the form

\[ T_{ij} = \lambda \delta_{ij} I_1 + 2\mu e_{ij}, \]  

(1.42)

where \( \delta_{ij} \) is Kronecker delta and \( I_1 = e_{ii} \) is the first strain invariant.

The coefficients of the constitutive equations discussed above specifying the relationship between stress and strain for the material in general depend on the temperature. But we
usually assume that the dependence is sufficiently small such that the coefficient may be treated as constants during the deformation.

Even though we neglect the variation of the elastic constants with temperature, we may be compelled to take account of the thermal expansion of the material, which often produces dimensional changes as large as those produced by the applied forces. If these dimensional changes are prevented by support constraints or surrounded material, thermal stresses are induced in addition to the stresses related to the strains according to the elastic constitutive equations.

The thermo elastic constitutive equations for transversely isotropic material are given by

\[
\begin{align*}
T_{rr} &= C_{11} e_{rr} + \left( C_{11} - 2 C_{66} \right) e_{\theta\theta} + C_{13} e_{zz} - \beta_1 T, \\
T_{\theta\theta} &= \left( C_{11} - 2 C_{66} \right) e_{rr} + C_{11} e_{\theta\theta} + C_{13} e_{zz} - \beta_2 T, \\
T_{r\theta} &= C_{13} e_{rr} + C_{13} e_{\theta\theta} + C_{33} e_{zz} - \beta_2 T, \\
T_{rz} &= T_{r\theta} = T_{\theta z} = 0,
\end{align*}
\]

(1.43)

where

\[
\beta_1 = C_{11} \alpha_1 + 2 C_{12} \alpha_2, \quad \beta_2 = C_{12} \alpha_1 + (C_{22} + C_{33}) \alpha_2
\]

and \( \alpha_i \) are the thermal expansion coefficients.

The thermo elastic constitutive equations for isotropic material are given by

\[
T_{ij} = \lambda \delta_{ij} I_1 + 2 \mu e_{ij} - \xi T \delta_{ij},
\]

(1.44)

where \( \xi = \alpha \left( 3 \lambda + 2 \mu \right) \) and \( T \) is the temperature. For the special case of steady heat flow, we have

\[
\nabla^2 T = T_{i\ell} = 0.
\]

(1.45)

(II) PLASTIC STATE

As metals obey Hooke’s law only in a certain range of small strain and when a metal is strained beyond an elastic limit, Hooke’s law is no longer valid, the behaviour of metals beyond their elastic limit is rather complicated as discussed in section 1.2. For analysis of continuum stress and strain distributions, a constitutive theory of plasticity must satisfy the yield condition under combined stresses because of the reason that uniaxial condition \( |T| = Y \) is inadequate when there is more than one stress component.

For an ideal plastic solid obeying von Mises yield criterion and flow rule, the following simplifications are taken:
(i) The plastic strain assumes incompressibility and the plastic strain deviation tensor is the same as the plastic strain tensor.

(ii) The material is elastic and obeys Hooke’s law as long as the second invariant $J_2 = \frac{1}{2} T_i T_j$ of the stress deviation tensor is less than a constant $K^2$. In other words, no change in plastic strain can occur

$$e_j^\nu = 0 \quad \text{when} \quad J_2 < K^2.$$  \hspace{1cm} (1.46)

(iii) Yielding can occur (elastic limit is reached) only when

$$J_2 = K^2.$$ \hspace{1cm} (1.47)

When the yielding condition $J_2 - K^2 = 0$ prevails, the rate of change of plastic strain is proportional to the stress deviation,

$$e_j^\nu = \frac{1}{\mu} J_{ij} ; \quad \mu > 0,$$ \hspace{1cm} (1.48)

where $\mu$ is a positive factor of proportionality, which has the dimensions of the coefficients of viscosity of a fluid. Any stress-state corresponding to $J_2 > K^2$ cannot be realized in the material.

The above set of laws contains two essential parts: (i) the yielding criterion and (ii) stress-strain relations in the elastic-plastic regions. In these specifications, the yielding condition is based on the second invariant of stress deviation tensor. Such a yielding criterion was first proposed by von Mises. The constant $K$ can be identified with the yield stress in simple shear while for a work hardening material, $K$ will be allowed to change with strain history.

Tresca’s yield condition is sometimes used instead of von Mises yield condition. Tresca’s criterion stipulates that maximum shear stress must have the constant value $K$ during plastic flow. To express Tresca’s idea analytically, it is the simplest to use the principal stresses $(T_1, T_2, T_3)$. If it is known that $T_1 \geq T_2 \geq T_3$ then Tresca’s yielding condition is

$$f \equiv T_1 - T_3 - 2K = 0.$$ \hspace{1cm} (1.49)

However $f$ in this form is not analytic. It violates the rule in such a manner that principal axes labeled as 1, 2 and 3 should not affect the form of yield function. To obey this rule, we observe that Tresca’s condition states that during plastic flow one of the differences $|T_1 - T_2|, |T_2 - T_3|, |T_3 - T_1|$ has the value $2K$. Hence we may write

$$f \equiv \left| \left( T_1 - T_2 \right)^2 - 4K^2 \right| \quad \text{or} \quad \left| \left( T_2 - T_3 \right)^2 - 4K^2 \right| \quad \text{or} \quad \left| \left( T_3 - T_1 \right)^2 - 4K^2 \right| = 0.$$ \hspace{1cm} (1.50)
This equation is symmetric with respect to the principal stresses.

In the above treatments, the ideal theories of elasticity and plasticity are dealt with separately and then linked together through a semi-empirical law, called the yield condition. Such a law may or may not exist. What actually happens is that when a medium starts to yield, a constraint is placed on the invariant of the strain tensor of the field such that they satisfy functional relation of the form

$$f(I_1, I_2, I_3) = 0,$$  \hspace{1cm} (1.51)

where $I_1, I_2$ and $I_3$ are the first, second and third invariants of the strain tensor.

The form of $f$ should be determined from the condition that the modulus of transformation takes on a singular value like zero or infinity and not by any ad-hoc considerations. In current treatment, it is argued that the incompressibility of the material makes $I_1$ vanish and since $I_3$ can be taken to be very small, equation (1.51) can be reduce to the form

$$I_2 = a \text{ constant,}$$ \hspace{1cm} (1.52)

which is known as the Huber-von Mises yield condition. If two of the principal stresses are equal or one is the arithmetic mean of the other two, condition (1.52) reduces to the Tresca yield condition (1.50). But it is clear that equation (1.52) cannot account for the Bauschinger effect, for which $I_1$ must appear in the yield condition. Seth [59] has expressed the yield stress $Y$ in simple tension when the material is in the fully plastic state as

$$Y = \frac{E}{n},$$ \hspace{1cm} (1.53)

where $E$ is the response coefficient in the transition range. It is also concluded that the yield stress in compression is twice that in tension and the general form of yield condition also contains the Bauschinger effect.

The stress-strain relations for an elastic-perfectly plastic solid were first proposed by Prandtl (1924) for the case of plane strain deformation. The general form of the equations was given by Reuss (1930). Reuss assumed that the plastic strain increment, denoted by a superscript ‘p’ in the following equations, is at any instant proportional to the instantaneous stress deviation $J_{ij}$ and shear stresses. Thus

$$\frac{d\epsilon_{11}^p}{J_{11}} = \frac{d\epsilon_{22}^p}{J_{22}} = \frac{d\epsilon_{33}^p}{J_{33}} = \frac{d\epsilon_{12}^p}{J_{12}} = \frac{d\epsilon_{23}^p}{J_{23}} = \frac{d\epsilon_{31}^p}{J_{31}} = d\lambda$$

or

$$d\epsilon_{ij}^p = J_{ij} d\lambda, \hspace{1cm} (i, j = 1, 2, 3)$$ \hspace{1cm} (1.54)
where $d\lambda$ is an instantaneous non negative constant of proportionality which may vary throughout a straining program.

The equations state that a small increment of plastic strain depends upon the current deviatoric stress, and not on increment. Also the principal axes of stress and plastic strain increments coincide. The equations are only a statement about the ratio of the plastic strain increments in the various $x, y, z$ directions. It gives no direct information about their absolute magnitude.

The total strain increment is the sum of the elastic strain increment (denoted by a superscript ‘$e$’) and the plastic strain increment. Thus

$$de_{ij} = de_{ij}^p + de_{ij}^e$$

$$= J_{ij} d\lambda + \left\{ \frac{dJ_{ij}}{2G} + \frac{1 - 2\nu}{E} \delta_{ij} dT_{kk} \right\},$$  \hspace{1cm} (1.55)

where $\nu$ is the Poisson’s ratio.

Since the plastic strain causes no change of plastic volume, we may write the condition of incompressibility in terms of the principal or normal strains as

$$de_{11}^p + de_{22}^p + de_{33}^p = 0$$

or

$$de_{ii}^p = 0$$  \hspace{1cm} (1.56)

Equation (1.54) then gives

$$\frac{de_{11}^p - de_{22}^p}{T_{11} - T_{22}} = \frac{de_{22}^p - de_{33}^p}{T_{22} - T_{33}} = \frac{de_{33}^p - de_{11}^p}{T_{33} - T_{11}} = d\lambda. $$  \hspace{1cm} (1.57)

Equation (1.57) states that Mohr circles of stress and plastic strain increment are similar.

Equation (1.54) can be rewritten in terms of normal stresses, giving rise to equations of the form

$$de_{ii}^p = \frac{2}{3} d\lambda \left[ T_{ii} - \frac{1}{2} (T_{22} - T_{33}) \right].$$  \hspace{1cm} (1.58)

Equation (1.55) thus consists of three equations of the type

$$de_{11}^p = \frac{2}{3} d\lambda \left[ T_{11} - \frac{1}{2} (T_{22} - T_{33}) \right] + \frac{dT_{11} - \nu (dT_{22} + dT_{33})}{E}$$  \hspace{1cm} (1.59)

and three of the type

$$de_{23} = T_{23} d\lambda + \frac{dT_{23}}{2G}. $$  \hspace{1cm} (1.60)
Finally, on examining equation (1.55), it will be seen that the volumetric and deviatoric strain increments can be separated in the expression for the total strain increment. Including the von-Mises yield criterion, the Prandtl-Reuss equations may then be written as

\[
d e^i_o = J_o d\lambda + \frac{d J_o}{2G} ,
\]

\[
d e^i = \left(1 - 2\nu\right) \frac{d T^i_o}{E} ,
\]

\[
J_o J^o = 2K^2 .
\] (1.61)

These equations for elastic-plastic solid are usually difficult to handle in a real problem and, in consequence, there are relatively few solutions.

In problems of large plastic flow, the elastic strains may often be neglected altogether. The material is then considered as being a perfectly plastic solid. When the stresses are below the yield point, no straining takes place, and the total strain increment are identical. Levy and von-Mises proposed stress-strain relations for such type of material. By presenting the relations between stress and strain, we have not followed the historical development of the field. At the present time, it seems more logical to consider the Levy-Mises equations as a special form of the Prandtl-Reuss equations. However, it was Saint-Venant (1870) who first proposed that the principal axes of strain increment coincided with the axes of principal stress. The general relationship between strain increment and the reduced stresses was first introduced by Levy (1871) and independently by von-Mises (1913). These equations are now known as Levy-Mises equations and may be written as

\[
\frac{de_1}{J_1} = \frac{de_2}{J_2} = \frac{de_3}{J_3} = \frac{de_{12}}{J_{12}} = \frac{de_{23}}{J_{23}} = \frac{de_{31}}{J_{31}} = d\lambda .
\] (1.62)

The superscript ‘\(p\)’ of equation (1.54) may be dropped, since the total strain increments are identical. Further, the Mohr circles of stress and strain increment are identical. In terms of total stresses, the Levy-Mises relation has three equations of type

\[
d e_{11} = \frac{2}{3} d\lambda \left[T_{11} - (T_{22} + T_{33})\right] ,
\] (1.63)

and three of that type

\[
d e_{23} = T_{23} d\lambda .
\] (1.64)

Since the elastic strains are not taken into account, the Levy-Mises relations obviously cannot be used to obtain information about “Elastic Spring-back” or residual stresses. For that more complex Prandtl-Reuss equation must be used.
Creep problems are complex as compared to plasticity problems. Laboratory creep tests with complex stress conditions present technical difficulties and the experiments must be performed very carefully if the results are to be reasonably reliable. Therefore, the available experiment data is slender and does not provide a reliable basis for a creep theory that is capable of describing the behaviour of materials under complex stresses. Moreover, tests can only be made with plane stresses and we have no information about creep performance with stresses on the three axes.

Like plasticity theory, the theory of creep under complex stress is based on certain speculative considerations, which are only partially confirmed experimentally. There are many ways in which the theory can be extended to varying stressed states. In real objects the nature of stressed state usually varies comparatively little with time, and therefore, the different theories lead to different results. For steady state of creep, Odquist [19] has formulated the constitutive equations by considering the rate of strain energy function $\dot{W}$ with von-Mises yield criterion. It relates the rate of steady state of creep to the second invariant of the stress deviator tensor in the following form

$$\dot{e}_{ij} = \frac{\partial \dot{W}}{\partial J_{ij}} = \frac{3}{2} \left( \frac{\sigma_e}{\sigma} \right)^{n-1} \frac{J_{ij}}{\sigma_e},$$

(1.65)

where $\dot{e}_{ij}, J_{ij}, \sigma_e$ are the strain rate tensor, stress deviator tensor and effective stress respectively, $\sigma_e$ and $n$ are material constants.

Stress-strain relations in this form are mostly used to analyze the creep problems based on the following hypothesis:

(i) The material is incompressible.

(ii) Creep rate is independent of superimposed hydrostatic pressure.

(iii) Existence of flow potential with von Mises yield condition.

(iv) Norton’s law holds in the special case, i.e. for uniaxial case.

An alternate approach to the problem of multi-axial stationary creep is made possible by Wahl [60-61] who considered maximum shear stress (Tresca) as a stress invariant together with the associated flow rule for the body relations.

It has already been pointed in section 1.4 that transition state exists when a material goes from elastic state to plastic and then to creep state. In classical treatment different constitutive equations are used for each state, based on some hypothesis, which simplifies the problem to
some extent. Firstly, the deformations are assumed to be small to make infinitesimal strain theory applicable. Secondly, the constitutive equations of the material are simplified by assuming incompressibility of the material and in some cases without this assumption, it is not even possible to find the solution of the problem in closed form. By using Seth’s transition theory, it has been shown that the same constitutive equations are used for different states, though the elastic constants have different meanings in each state.

Equations of equilibrium in cylindrical polar co-ordinates for a body, having variable thickness $h$ in the radial direction are given by

$$\frac{\partial (hT_{rr})}{\partial r} + \frac{h}{r} \left( \frac{\partial T_{r\theta}}{\partial \theta} \right) + \frac{h}{r} \left( \frac{\partial T_{r\theta}}{\partial r} \right) + h(T_{r\theta} - T_{\theta\theta}) + hF_r = 0,$$

$$\frac{\partial (hT_{r\theta})}{\partial r} + \frac{h}{r} \left( \frac{\partial T_{r\theta}}{\partial \theta} \right) + \frac{2h}{r} T_{r\theta} + hF_\theta = 0,$$

$$\frac{\partial (hT_{rz})}{\partial r} + \frac{h}{r} \left( \frac{\partial T_{r\theta}}{\partial r} \right) + h \left( \frac{\partial T_{rz}}{\partial z} \right) + \frac{2h}{r} T_{rz} + hF_z = 0,$$

(1.66)

where $F_r$, $F_\theta$, $F_z$ are the body forces along $r, \theta, z$ direction.

For a body having constant thickness, equations (1.66) becomes

$$\frac{\partial (T_{rr})}{\partial r} + \frac{1}{r} \left( \frac{\partial T_{r\theta}}{\partial \theta} \right) + \frac{1}{r} \left( \frac{\partial T_{r\theta}}{\partial r} \right) + \frac{1}{r} (T_{r\theta} - T_{\theta\theta}) + F_r = 0,$$

$$\frac{\partial (T_{r\theta})}{\partial r} + \frac{1}{r} \left( \frac{\partial T_{r\theta}}{\partial \theta} \right) + \frac{1}{r} \left( \frac{\partial T_{r\theta}}{\partial r} \right) + \frac{2}{r} T_{r\theta} + F_\theta = 0,$$

$$\frac{\partial (T_{rz})}{\partial r} + \frac{1}{r} \left( \frac{\partial T_{r\theta}}{\partial r} \right) + \frac{1}{r} \left( \frac{\partial T_{r\theta}}{\partial z} \right) + \frac{2}{r} T_{rz} + F_z = 0.$$

(1.67)

For rotating disks and rotating cylinders, the axis of rotation is taken along $z$-axis and hence the body force is the centrifugal force with components

$$F_r = \rho \omega^2 r; \quad F_\theta = 0; \quad F_z = 0,$$

where $\rho$ is the density of the material of the disk.