Chapter 4

The Nature of Attractor Basins in Dissipative Systems

A phenomenon of considerable importance in the study of dissipative dynamical systems is the occurrence of more than one attractor for a fixed set of system parameters. This behaviour is termed multistability, and refers to the existence of two or more asymptotic states in which the dynamics can be periodic, quasiperiodic, or chaotic. (When there are only two possible states, this is referred to as bistability; this is commonly encountered in systems displaying hysteresis.) Multistability is known to occur in a variety of areas such as neuroscience [140, 141], chemical reaction dynamics [94], optics [15], and condensed matter physics [129].

Multistability also depends on system parameters: as parameters are varied, the stability of different attractors change. Thus the number of attractors can vary with parameters and indeed in weakly dissipative systems [37, 38], it can be arranged to have an arbitrary number of attractors! Each of these attractors has a basin of attraction, and thus one question that naturally arises is the following: How does the volume of the basin of a given attractor vary as it is created? And further, how does this change with the emergence of new attractors?

These questions, which we address in this Chapter, are of practical relevance, for instance in the issue of pattern-selection, which occurs when a weakly dissipative interaction is applied to a spatially-extended system [29]. Stabilizing or controlling such systems with small perturbations in terms of noise or external forcing [37] are important goals. Multistability is pertinent here since the choice of which specific attractor the system asymptotically goes to is based both on initial conditions, and this is influenced by, statistically, on the
relative volume of the basin of the given attractor.

Earlier studies have addressed some aspects of this problem. In one approach, starting from Hamiltonian systems where there are no attractors and the motion is either on KAM tori or on chaotic orbits [87], the creation of different attractors as damping is introduced was studied by Schmidt and Wang [142] in the standard map system. They examined the bifurcation of periodic orbits as a function of the nonlinearity to observe the emergence of multiple coexisting attractors. Similarly, Feudel et al [38] studied the qualitative behaviour of a single mechanical rotor with weak damping, and found that for small dissipation the system can have an arbitrary number of attractors. They further showed that the basins of these attractors were closely interwoven so that the phase space had an intricate and complex structure.

Most attractors in multistable system with weak dissipation are periodic in nature: chaotic attractors are rarely found close to the conservative limit [37]. The actual number of possible attractors is related to the number of periodic orbits, and scales as a function of the damping, while the basin of chaotic attractors typically shrink. It is also known that this behaviour is fragile: multistability can be suppressed by weak perturbations [121, 122, 28] which offer the possibility of control [41, 120, 156].

Basin size evolution between dissipative and conservative limits has also been studied systematically in the Hénon map [42, 134] along paths in parameter space. Volumes decrease exponentially [42] as the damping approaches zero. Along parameter paths delimiting stability regions for trajectories of different periods, basin volume shrinks rapidly independent of period. The basin volume of the periodic attractors increases with the unfolding of the bifurcation cascade along different saddle-node bifurcations lines.

In these earlier studies of multistable systems it was shown that the basin volume varies in one of two typical ways. Along a characteristic period-doubling path to chaos volumes have a Gaussian profile [134] as a function of the parameter for periodic attractors. Along paths in the parameter space where a strange attractor disappears, the variation is exponential [37]. Here I have studied the basin size scaling at the saddle-node bifurcations and its dependence on the control parameters. The attractor which appears in a saddle-node bifurcation, has a power-law scaling of the basin volume.

The basic features of multistable systems have been studied here through the specific examples of the Hénon and Ikeda maps. We observe that periodic attractors appear with a power-law scaling in the relative basin volume and the scaling exponent depends on the system parameters. The integrated basin volume for attractors of short periods (1, 2, and
3, say) increases with the damping strength while the originally complex structure of the multistable system with many coexisting higher period orbits shows a exponential decay in the basin volume of higher period orbits with increasing damping strength. These appear to be general properties of multistable systems.

4.1 Bifurcations and Attractor basins

In conservative systems there can be no attractors. In Hamiltonian systems the motion is either regular or chaotic [87]. In the former case, the motion is typically quasiperiodic and lies on the surface of tori in the phase space. Chaotic dynamics lies in the so-called homoclinic or heteroclinic "tangle", formed by the intersections of the stable and unstable manifolds of unstable fixed points [87, 152]. In systems with 2 freedoms, the dynamics can be examined on the Poincaré surface of section, and then tori appear as closed curves or sets of closed curves (termed islands), while chaotic motion appear as unconnected but space-filling points. The same can be observed in mappings corresponding to Hamiltonian dynamics (as for example the standard mapping [27]).

The toroidal motions—the islands—are associated with marginally stable periodic orbits, and the eigenvalues of the Jacobian matrix are equal to one in absolute value. The large primary islands are surrounded by smaller secondary islands which in turn are surrounded by even smaller islands and so on. There is, naturally, an infinite number of periodic orbits [87].

When damping is introduced in such a system, it is clear that these features will change. The marginally stable periodic orbits now become periodic attractors, and the eigenvalues of the Jacobian are smaller than one in absolute value [37]. However, not every periodic orbit becomes an attractor: it turns out that only a finite number can be located. The number of attractors in weakly dissipative systems depends on the nonlinearity as well as on the damping parameters. This can be systematically investigated by starting with a conservative system, where there are no attractors, and adding dissipation [37] as discussed in the following two examples.
4.1.1 The Hénon map

The Hénon mapping [64],

\[ \begin{align*}
    x_{n+1} &= A - x_n^2 - (1 - \nu)y_n \\
    y_{n+1} &= x_n
\end{align*} \tag{4.1} \]

is a suitable system where this systematic study can be carried out. The two parameters \( A \) and \( \nu \) govern nonlinearity (\( A \) can also be considered a forcing term) and damping. As \( \nu \) varies between 0 and 1, the system goes from the conservative limit to the strongly damped case when it reduces to the 1-d logistic map. In the Hamiltonian limit of zero damping, infinitely many stable but nonattracting periodic orbits exist with regular motion around these stable periodic orbits on invariant KAM surfaces. Interpreted with these islands of stability in phase space are chaotic regions [87].

For low damping, the dynamical behaviour is dominated by the appearance and disappearance of periodic attractors of different periods leading to a very complex bifurcation diagram which is strongly dependent on the parameters. Effectively, all the stable periodic orbits become attracting sinks as the KAM tori are destroyed and persistent chaotic motion disappears [88]. Although it is likely that there are an infinite number of periodic orbits in the Hamiltonian limit, only finitely many coexisting periodic sinks can be located when the damping is nonzero since the higher-order periodic orbits typically lose their stability with increasing damping for a given value of forcing [38]. Nonetheless, the number of attractors can in principle be arbitrarily high if the damping is chosen small enough.

At fixed \( \nu \), \( A \) controls the bifurcation sequence for each individual attractor. Most of the periodic attractors appear via a saddle-node bifurcation, and lose their stability through period-doubling bifurcations. The bifurcation diagram at \( \nu = 0.01 \) is shown in Fig. 4.1 and it can be seen that there are regions in parameter space with a large number of coexisting periodic attractors. Regardless of whether an attractor is simple or strange, its basin of attraction can be complex with possibly fractal boundaries. In particular, for multistable systems, the basins of the coexisting attractors frequently exhibit an extreme type of fractal structure which is characterized by non-zero uncertainty exponents [49, 51].

For small damping, it is known that the higher-order periodic attractors in the Hénon map have very small basins of attraction: Indeed they are difficult to detect for this reason [37]. To determine the number of detectable coexisting periodic orbits, we iterate \( 10^4 \) initial conditions on a grid of \( 1200 \times 1800 \) in the rectangle \( [-2, 2] \times [-3, 3] \). The accuracy with which each periodic orbit \( P_N \) of period \( N \) is given by \( |x(N) - x(0)| \leq 10^{-8} \).
number of attractors obtained thus depends on the resolution of the initial conditions in the state space. A finer resolution in the state space can reveal more of the attractors of high periods, though the total volume of phase space occupied by the basins of the different families of attractors remains approximately constant.

The basins of the lower order periodic attractors are significantly larger than the basins of high-order periodic attractors. Shown in Fig. 4.2(a-b) are the basins of period-1 and 3 attractors for $A = 1.0$ and $\nu = 0.01, 0.1$ in dark and light shading. The basins of all other attractors is shown in white. Similarly the basin of period-1 and period-5 attractors with other coexisting attractors is shown in Fig. 4.2(c-d) for $A = 2.6$ and $\nu = 0.01, 0.1$. There is an attractor at infinity also because of the quadratic term $ax^2$. 

Figure 4.1: Bifurcation diagram for the Hénon map with $\nu = 0.01$. 

![Bifurcation diagram for the Hénon map with $\nu = 0.01$.]
Figure 4.2: (a-b) Basin of attraction of coexisting period-1 (black) and period-3 (gray) attractors with $A = 1.0$ and $\nu = 0.01, 0.1$ respectively and (c-d) Basin of attraction of coexisting period-1 (black) and period-5 (gray) attractors with $A = 2.6$ and $\nu = 0.01, 0.1$ respectively for Hénon map.
4.1.2 Ikeda Map

For another example of a multistable system, we choose the Ikeda map which has also been studied extensively,

\[ z_{n+1} = I + (1 - \nu)z_n \exp(ik - \frac{ip}{1 + |z_n|^2}). \]  

The variable \( z_n \) is a complex quantity, \( z_n = x_n + iy_n \) so that the mapping is two-dimensional. The Ikeda map [70] models the evolution of an electric field inside a nonlinear optical ring cavity, namely a laser system, and \( z_n \) is related to the amplitude and phase of the \( n^{th} \) laser pulse exiting the cavity. The parameter \( I \) is related to the laser input amplitude and can thus thought of as the forcing, the damping \( \nu \) is related to the reflection properties of the
Figure 4.4: Basin of attraction of coexisting period-1, period-3, period-4, and period-5 attractors with $A = 0.45$ and $\nu = 0.01$ for Ikeda map.
mirrors in the cavity and the parameter \( k \) is the empty cavity detuning while \( p \) measures the detuning due to a nonlinear dielectric medium. As in many previous studies we fix the parameters \( k \) and \( p \) to 0.4 and 3.5 respectively, and obtain the bifurcation diagram at \( \nu = 0.01 \) as a function of forcing \( I \) is shown in Fig. 4.3. It is clearly possible to locate several intervals in \( I, \nu \) parameter space corresponding to multistable behaviour.

The many coexisting attractors have a very complex basin structure, some idea of which can be got from Fig. 4.4(a-d) where the period-1, 3, 4 and 5 attractors and their corresponding basins are shown for parameter values \( I = 0.45 \) and \( \nu = 0.01, k = 0.4 \) and \( p = 3.5 \). In this system, there is also an attractor at infinity, and there are points in this region of the phase plane that are attracted to it.

### 4.2 Basin size scaling

Given the nature of these (and most other typical) dynamical systems, it is not an easy matter to compute the absolute volume or area of the basin of attraction of a specific attractor. The relative volume can be estimated computationally, by measuring the relative number of initial points that go to a specific attractor, either by choosing a grid of points in phase space, or via a Monte Carlo method. This provides an upper-bound to the relative basin volume.

We first focus on those attractors created through saddle-node bifurcations in the two systems introduced above.

#### 4.2.1 The Hénon map

Consider the bifurcation diagram of Hénon map shown in Fig. 4.1 for \( \nu = 0.01 \), and a period \( p \) attractor \( A_p \), created through a saddle-node bifurcation at parameter \( A = A_c \). We compute the relative volume of the attractor basin \( f_p^A \) as a function of the damping \( \nu \) between 0.01 and 0.3. This is shown in Fig. 4.5 where it can be seen that the basin volume becomes nonzero at a critical value of the nonlinearity, \( A_c \). The critical value shifts towards lower \( A \) for \( p = 3 \) and to larger values for \( p = 5 \) as the damping is increased. The attractor basins grows as a power-law [147, 148],

\[
f_p^A \sim (A - A_c)^7.
\]  

This dependence is similar to the manner in which the attractor basin geometry depends on parameters at saddle–node bifurcations; recall that a variety of behaviours are also
Figure 4.5: The relative basin volume of period-3 and period-5 attractors for different damping parameter values for Hénon map.

characterized by power laws [149]. In the present case, the scaling exponent is obtained only numerically and the scaling of \( f \) versus \((A - A_c)\) with \(a\) the scaling exponent \(\gamma\) for damping \(\nu = 0.01\) for period-3 and period-5 is shown in Fig. 4.6(a). The dependence of the scaling exponent \(\gamma\) on the damping strength \(\nu\) is also studied. In Fig. 4.7), the scaling exponent \(\gamma\) is shown as a function of the damping \(\nu\) for the period-3 and period-5 attractors. The exponent decays with damping and vanishes for very high damping.

What about the higher period attractors and their basins? It turns out that these tend to have vanishingly small basins of attraction, so it is more relevant to examine the
Figure 4.6: The scaling of relative basin volume $f$ of period-3 and period-5 attractors with the scaling exponent $\gamma$ for damping $\nu = 0.01$ for (a) Hénon and (b) Ikeda maps.
Figure 4.7: The scaling exponent $\gamma$ as a function of damping for period-3 and period-5 attractors for Hénon map.

integrated basin volume and study this as a function of the damping,

$$f_p = \int_0^4 dA f_p^A.$$  \hspace{1cm} (4.4)

This quantity provides another measure of the relative importance of a given attractor, and is also evaluated numerically. The basin volume of different periodic attractors is summed for different forcing strength $A$ with $\delta A = 0.01$ between 0 and 4. For low periods, $p = 1, 2, 3$ the basin volume $f_p$ increases linearly with damping while for all higher periods, i.e. $p = 5, 7, etc$, it decreases exponentially, as shown in the Fig. 4.8. It is numerically verified that the basin volume of all the periodic attractors with period greater than 4
Figure 4.8: The basin volume integrated over the nonlinearity parameter as a function of damping for (a) period-1, period-2, and period-3 and (b) period-5, and period-7 attractors for Hénon map.

shows a exponential decay with damping. The behaviour of the period-4 attractor basin volume shows a crossover from the linear to exponential behaviour in the system.

The integrated basin volume has physical relevance for systems with time-dependent control parameters. In that case the particular state of the multistable system will shows some change in its properties, i.e. the basin volume, with time which can be summed over and used as a performance parameter or to characterize or / and to control the system with noise or external forcing.
Figure 4.9: The relative basin volume of period-3 and period-5 attractors for different damping parameter values $\nu = 0.005, 0.01, 0.015, \text{ and } 0.02$ for Ikeda map.

### 4.2.2 Ikeda Map

Some idea of the generality of the above results is provided by examination of the Ikeda map. Considering the creation of an attractor through a saddle-node bifurcation corresponding to Fig. 4.3 for different damping strength $\nu = 0.005, 0.01, 0.015, \text{ and } 0.02$, in Fig. 4.9 it can be seen that the basin volume of attractor $p$, $f_p^*$ here also becomes nonzero at a critical value of the nonlinearity, $I_c$, which shifts towards larger values for $p = 3$ and for $p = 5$ as the damping is increased. The attractors appear abruptly, and their basin volume grows as a power-law [147, 148]. The scaling of $f$ versus $(I - I_c)$ with a the scaling exponent $\gamma$ for damping $\nu = 0.01$ for period-3 and Period-5 is shown in Fig. 4.6(b).
Figure 4.10: The basin volume integrated over the nonlinearity parameter as a function of damping for (a) period-1, period-3, and period-4 and (b) period-5, and period-7 attractors for Ikeda map.

The dependence of scaling exponent $\gamma$ on the damping strength $\nu$ is also studied for period-3 and period-5. The scaling exponent $\gamma$ decays with damping and vanishes even at low damping strength, i.e. $\nu \approx 0.02$ as compared with the results of Hénon map where the multistability is present for wide range of parameter space.

We also compute the total basin volume of low period attractors summed over the nonlinearity as a function of the damping for Ikeda map. The basin volume of different periodic attractors is summed for different forcing strength $A$ with $\delta I = 0.001$ between 0.2 and 1.2. For low periods, $p = 1, 2, 3$ the basin volume $f_p$ shows a quadratic increase with damping for a small region near zero damping limit as compared with a linear behaviour in the case of Hénon map for a larger range of damping while for all higher periods, i.e.
\[ p = 5, 7, \text{etc}, \] it decreases exponentially, as shown in the Fig. 4.10.

Earlier studies \cite{37} have shown that the basins of chaotic attractors in the bounded region of the state space shrink exponentially as the damping approaches zero, and the basin volume of periodic attractors is well approximated by Gaussian profiles, independent of period \cite{134}. Our results for the Hénon and Ikeda maps shows that the low-period attractors basin volume shrink linearly, while the basin volume (which is relatively small) of high-period attractors increases exponentially as the damping vanishes. We expect that this result will hold for all multistable systems.

4.3 Discussion

In this Chapter, we have studied attractor basin evolution as parameters vary in systems that display multistability. For the weakly dissipative system, there can be very large numbers of coexisting attractors, most of which are periodic orbits. The size of the basin of attraction depends (inversely) on the period, and as parameters vary, the basins show typical patterns of variation.

The models that we have examined are the Hénon and Ikeda systems. In weakly dissipative multistable systems, when an attractor is created at a saddle-node bifurcation, its basin volume grows as a power in the parameter and the scaling exponent is a function both of the period \( p \) as well as the damping strength \( \nu \). These results are quite general and we expect to have a similar behaviour for all multistable systems. One can expect that the power-law growth of basin volume is related to the manner in which lengths scale in dissipative nonlinear systems, but the theory of this scaling has not been fully clarified so far. If these vary as power-laws, it is clear that volumes will also vary as a power-law. Work on this aspect of the problem is currently under way.

The influence of noise on dynamical systems led to several interesting phenomena including noise-induced chaos-order transitions, attractor hopping etc. \cite{45}. The first investigation of the role of noise on a system exhibiting multistability was performed in the study of jumps between the metastable states induced by noise \cite{6}. The noise-induced selection of attractors in coupled systems is also studied \cite{77}. It has been found for multistable systems that for a certain noise level some attractors gain a higher probability to be observed than others \cite{82}. The mechanism of preference of attractors is highly relevant for experimental observations of multistable systems, since measurements, which are inevitably contaminated with noise, might reveal a much smaller number of stable states.
than the noise free system actually possesses.

With extremely sensitive dependence on control parameters and initial conditions, multistable system pose a challenge in the context of chaos control. The present studies of the relative basin volumes of coexisting attractors will help in the study of synchronization, coherence and other related phenomena relating to complex networks.