CHAPTER I
PRELIMINARIES

Introduction: A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labeling were first introduced in the late 1960’s. Rosa [1967] defined a $\beta$-valuation of a graph $G$ with $e$ edges as an injection from the vertices of $G$ to the set \{0,1,2, ...,e\} such that when each edge $xy$ is assigned the label $|f(x) - f(y)|$ the resulting edge labels are distinct. $\beta$-valuations are functions that produce graceful labeling. However, the term graceful labeling was not used until Golomb [1977] studied such labeling later.

Sheppard [1976] has shown that there are exactly $q!$ gracefully labeled graphs with $q$ edges. Rosa [1967] has identified essentially three reasons why a graph fails to be graceful: (1) $G$ has “too many vertices” and “not enough edges,” (2) $G$ “has too many edges,” and (3) $G$ “has the wrong parity.” An infinite class of graphs that are not graceful for the second reason is given in [1986]. As an example of the third condition Rosa [1967] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles $C_{4n+1}$ and $C_{4n+2}$ are not graceful.

Acharya [1982] obtained that every graph can be embedded as an induced subgraph of a graceful graph and a connected graph can be embedded as an induced subgraph of a graceful connected graph.
Acharya, Rao, and Arumugam [2008] found: every triangle-free graph can be embedded as an induced subgraph of a triangle-free graceful graph; every planar graph can be embedded as an induced subgraph of a planar graceful graph; and every tree can be embedded as an induced subgraph of a graceful tree. These results demonstrate that there is no forbidden subgraph characterization of these particular kinds of graceful graphs.

The graceful labeling problem is to determine which graphs are graceful. When studying graceful labeling, we consider only finite graphs. For all notations in graph theory we follow Harary [2001].

While the graceful labeling of graphs is perceived to be a primarily theoretical subject in the field of graph theory and discrete mathematics, gracefully labeled graphs often serve as models in a wide range of applications. Such applications include coding theory, X-ray crystallography, Radar, Astronomy, Circuit design, Communication network addressing, Data base management, and models for constraint programming over finite domains.

Considering this point as a core, A. Solairaju and K. Chitra [2008] introduced a new concept of labeling called an Edge-Odd Graceful Labeling (EOGL).

They defined that a labeling is said to be an EOGL if it exists a bijection f from E to \{1, 3, ..., 2q-1\} so that the induced mapping f_ from E to \{0, 1, 2, ..., 2q-1\} given by \( f_\cdot(x) = \sum_{xy \in E} f(xy) \pmod{2q} \), the resulting labels are distinct. A graph G with p vertices and q edges is said to be an edge-odd graceful graph if it admits EOGL.
This chapter contains some basic definitions, preliminaries and a brief summary of results obtained on the gracefulness, edge – odd gracefulness and even edge gracefulness of undirected graphs and product graphs.

Section 1.1 – Basic concepts:

The first chapter is devoted for introduction. In this chapter some basic definitions like path, product graphs, circuit, wheel graph, web graph, n-squares, spanning tree, connected graph and graceful labeling are recalled.

**Definition 1.1.1 (Graph):** A graph \( G = (V, E) \) consists of a set of objects \( V = \{v_1, v_2, \ldots\} \) called vertices, and another set \( E = \{e_1, e_2, \ldots\} \) whose elements are called edges, such that each edge \( e_k \) is identified with an unordered pair \((v_i, v_j)\) of vertices. The vertices \( v_i, v_j \) associated with edge \( e_k \) are called the end vertices of \( k \). The vertices are represented as points and each edge as a line segment joining its end vertices.

If \( G \) has \( p \) vertex, \( q \) edge, then \( G \) is \((p, q)\) graph. \( \delta \) denotes the smallest degree of some vertex in \( G \). \( \Delta \) denotes the largest degree of some vertex in \( G \).

**Definition 1.1.2:** A graph is said to be finite if the order \( p \) (or \( q \)) is finite. A graph is called simple if there is no loop or multiple edges.
Definition 1.1.3 (Walk): It is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears more than once in a walk. A vertex however, may appear more than once.

Definition 1.1.4 (Closed walk): It is a walk in which beginning and ending vertices are the same.

Definition 1.1.5 (Path): An open walk in which no vertex appears more than once is called a path.

Definition 1.1.6 (Circuit): A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a circuit. $C_n$ is denoted as a circuit with ‘n’ vertices where $n \geq 1$.

Definition 1.1.7 (Connected Graph): It is a simple graph in which there is at least one path between any two vertices.

Definition 1.1.8 (Wheel): A wheel is a graph obtained from a cycle by adding a new vertex and edges joining it to all the vertices of the cycle; the new edges are called the spokes of the wheel.

Definition 1.1.9 (Cartesian product graph): The Cartesian product of $G$ and $H$, written $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting $(g, h)$ adjacent to $(g', h')$ if and only if (1) $g = g'$ and $hh' \in E(H)$, or (2) $h = h'$ and $gg' \in E(G)$. 
Definition 1.1.10 (Strong Product Graph): The strong product \(G \boxtimes H\) of two graphs \(G\) and \(H\) is defined by the vertex set \(V(G) \times V(H)\), and \((g, h)\) is adjacent to \((g', h')\) if \(g = g'\) and \(h\) adjacent to \(h'\) or \((g\) adjacent to \(g'\) and \(h = h'\)) or \((g\) adjacent to \(g'\) and \(h\) adj \(h'\)).

Definition 1.1.11 (Spanning tree): A spanning tree of a connected, undirected graph \(G\) is a tree that includes every vertex of \(G\).

Definition 1.1.12 (Generalized n-squares of type I) (1-nC4): n-C4 is a collection of \(n\) number of C4. 1-nC4 (\(n \geq 1\)) is a connected graph whose vertex set is \(\{v_1, v_2, v_3, v_4, \ldots, v_{4n}\}\) whose edge set is \(\{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{4n-1}v_{4n}, v_{4n}v_{4n-3}, v_2v_6, v_6v_{10}, \ldots, v_{4n-6}v_{4n-2}\}\).

Definition 1.1.13 (Generalized n-squares of type II) (2-nC4): n-C4 is a collection of \(n\) number of C4. 2-nC4 (\(n \geq 1\)) is a connected graph whose vertex set is \(\{v_1, v_2, v_3, v_4, \ldots, v_{4n}\}\) whose edge set is \(\{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{4n-1}v_{4n}, v_{4n}v_{4n-3}, v_2v_6, v_6v_{10}, \ldots, v_{4n-6}v_{4n-2}, v_3v_7, v_7v_{11}, \ldots, v_{4n-5}v_{4n-1}\}\).

Definition 1.1.14 (Generalized n-squares of type III) (3-nC4): n-C4 is a collection of \(n\) number of C4. 3-nC4 (\(n \geq 1\)) is a connected graph whose vertex set is \(\{v_1, v_2, v_3, v_4, \ldots, v_{4n}\}\) whose edge set is \(\{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{4n-1}v_{4n}, v_{4n}v_{4n-3}, v_2v_6, v_6v_{10}, \ldots, v_{4n-6}v_{4n-2}, v_3v_7, v_7v_{11}, \ldots, v_{4n-5}v_{4n-1}, v_4v_8, v_8v_{12}, \ldots, v_{4n-4}v_{4n}\}\).

Definition 1.1.15 (Generalized n-squares type IV) (4-nC4): n-C4 is a collection of \(n\) number of C4. 4-nC4 (\(n \geq 1\)) is a connected graph whose vertex set is \(\{v_1, v_2, v_3, v_4, \ldots, v_{4n}\}\) whose edge set is \(\{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{4n-1}v_{4n}, v_{4n}v_{4n-3}, v_2v_6, v_6v_{10}, \ldots, v_{4n-6}v_{4n-2}, v_3v_7, v_7v_{11}, \ldots, v_{4n-5}v_{4n-1}, v_4v_8, v_8v_{12}, \ldots, v_{4n-4}v_{4n}, v_1v_5, v_5v_9, \ldots, v_{4n-7}v_{4n-3}\}\).
Definition 1.1.16 (Web graph): It is a graph consisting of n copies of cycles \( C_n \) with corresponding vertices connected by spokes.

Definition 1.1.17: \((P_2 + N_n)^e\) is \((P_2 + N_n)\) merging by an edge and it is a connected graph such that every vertex of \((P_2 + N_n)\) is adjacent to every vertex of null graph \(N_n\) together with adjacent edges in \((P_2 + N_n)\).

Definition 1.1.18: \(P_k + N_n\) is a connected graph such that every vertex of \(P_k\) is adjacent to every vertex of null graph \(N_n\) together with adjacent edges in \(P_k\).

Definition 1.1.19: \(K_1 + P_n\) is a connected graph obtained from 1 copy of \(P_n\) (whose vertices are \(u_1, u_2, ..., u_{n-1}, u_n; u_1, v_2, v_3, ..., v_{n-1}, u_n\) as first copy of \(P_n\)) and a null vertex \(t(n_1)\) whose adjacency edges other than existing edges are \(tu_i\) (for every \(i = 1, 2, ..., n\); \(u_iv_i, i = 2, 3, ..., (n-1)\)).

Definition 1.1.20: \(K_1 + 2P_n\) is a connected graph obtained from 2 copies of \(P_n\) (whose vertices are \(u_1, u_2, ..., u_{n-1}, u_n; u_1, v_2, v_3, ..., v_{n-1}, u_n\) as first copy of \(P_n\); \(u_1, w_2, ..., w_{n-1}, u_n\) is second copy of \(P_n\)) and a null vertex \(t(n_1)\) whose adjacency edges other than existing edges are \(tu_i\) (for every \(i = 1, 2, ..., n\); \(u_iv_i, i = 2, 3, ..., (n-1)\); \(v_jw_j, j = 2, 3, ..., (n-1)\)).

Definition 1.1.21: \(K_1 + 3P_n\) is a connected graph obtained from 3 copies of \(P_n\) (whose vertices are \(u_1, u_2, ..., u_{n-1}, u_n; u_1, v_2, v_3, ..., v_{n-1}, u_n\) as first copy of \(P_n\); \(u_1, w_2, ..., w_{n-1}, u_n\) is second copy of \(P_n\); \(u_1, s_2, ..., s_{n-1}, u_n\) is third copy of \(P_n\)) and a null vertex \(t(n_1)\) whose adjacency edges other than existing edges are \(tu_i\) (for every \(i = 1, 2, ..., n\); \(u_iv_i, i = 2, 3, ..., (n-1)\); \(v_jw_j, j = 2, 3, ..., (n-1)\); \(w_kw_k, k = 2, 3, ..., (n-1)\)).
**Definition 1.1.22:** $K_{1+4P_n}$ is a connected graph obtained from 4 copies of $P_n$ (whose vertices are $u_1, u_2, \ldots, u_{n-1}, u_n$; $u_1, v_2, v_3, \ldots, v_{n-1}, u_n$ as first copy of $P_n$; $u_1, w_2, \ldots, w_{n-1}, u_n$ is second copy of $P_n$; $u_1, s_2, \ldots, s_{n-1}, u_n$ is third copy of $P_n$; $u_1, x_2, \ldots, x_{n-1}, u_n$ is fourth copy of $P_n$) and a null vertex $t(n_1)$ whose adjacency edges other than existing edges are $tu_i$ (for every $i = 1, 2, \ldots, n$); $u_iv_i$, $i = 2, 3, \ldots, (n-1)$; $v_jw_j$, $j = 2, 3, \ldots, (n-1)$; $w ks_k$, $k = 2, 3, \ldots, (n-1)$; $s l x_l$, $l = 2, 3, \ldots, (n-1))$.

**Definition 1.1.23:** $K_{1+5P_n}$ is a connected graph obtained from 4 copies of $P_n$ (whose vertices are $u_1, u_2, \ldots, u_{n-1}, u_n$; $u_1, v_2, v_3, \ldots, v_{n-1}, u_n$ as first copy of $P_n$; $u_1, w_2, \ldots, w_{n-1}, u_n$ is second copy of $P_n$; $u_1, s_2, \ldots, s_{n-1}, u_n$ is third copy of $P_n$; $u_1, x_2, \ldots, x_{n-1}, u_n$ is fourth copy of $P_n$; $u_1, y_2, \ldots, y_{n-1}, u_n$ is fifth copy of $P_n$) and a null vertex $t(n_1)$ whose adjacency edges other than existing edges are $tu_i$ (for every $i = 1, 2, \ldots, n$); $u_iv_i$, $i = 2, 3, \ldots, (n-1)$; $v_jw_j$, $j = 2, 3, \ldots, (n-1)$; $w ks_k$, $k = 2, 3, \ldots, (n-1)$; $s l x_l$, $l = 2, 3, \ldots, (n-1)$; $x_m y_m$, $m = 2, 3, \ldots, (n-1))$.

**Definition 1.1.24 (Even - edge graceful labeling):** A $(p, q)$ graph $G$ is said to have an even - edge graceful labeling, if there exists an injection from the edge set of $G$ to $\{1, 2, \ldots, 2q\}$ so that the induced mapping $f_+$ from the vertex set to $\{0, 1, 2, \ldots, 2k-2\}$ defined by $f_+(x) \equiv \sum (xy)/xy \in E \mod (2k)$ where $k= \max \{p, q\}$ are distinct and even.

**Definition 1.1.25 (Edge-graceful labeling):** Lo [1985] introduced the notion of edge - graceful graphs. A graph $G(V, E)$ is said to be edge - graceful if there exists a bijection $f$ from $E$ to $\{1, 2, \ldots, |E|\}$ such that the induced mapping $f_+$ from $V$ to $\{0, 1, \ldots, |V| - 1\}$ given by $f_+(x) = (f(xy)) \mod |V|$ taken over all edges $xy$ is a bijection.
Section 1.2 - Cartesian product of Path and Circuit:

**Introduction:** Gnanajothi [1991] proved that the following graphs are odd - graceful:

1. $P_n, C_n$ if and only if $n$ even; $K_{m,n}$;
2. combs $P_n \odot K_1$ (graphs obtained by joining a single pendant edge to each vertex of $P_n$);
3. books;
4. crowns $C_n \odot K_1$ (graphs obtained by joining a single pendant edge to each vertex of $C_n$) if and only if $n$ is even; the disjoint union of copies of $C_4$;
5. the one-point union of copies of $C_4$; $C_n \times K_2$ if and only if $n$ is even;
6. caterpillars; rooted trees of height 2; the graphs obtained from $P_n$ ($n \geq 3$) by adding exactly two leaves at each vertex of degree 2 of $P_n$;
7. the graphs obtained from $P_n \times P_2$ by deleting an edge that joins to end points of $P_3$ or by adjoining to each end vertex the path $P_4$.

She conjectures that all trees are odd-graceful and proves the conjecture for all trees of order up to 10.

Cartesian product graphs and related graphs are graceful. Prisms are graphs of the form $P_n \times C_m$. Jungreis and Reid [1992] obtained that $C_m \times P_n$ are graceful when $m$ and $n$ are even or when $m \equiv 0\text{(mod 4)}$. Gallian, Prout, and Winters [1992] found that all prisms $C_m \times P_2$ with a single vertex deleted or single edge deleted are graceful.
Kathiresan [1992] has shown that graphs obtained from ladders by sub dividing each step exactly once are graceful and that graphs obtained from ladders by sub dividing each step exactly once are graceful and that graphs obtained by appending an edge to each vertex of a ladder are graceful. Huang and Skeiena [1994] got that $C_m \times P_n$ is graceful for all $n$ when $m$ is even and for all $n$ with $3 \leq n \leq 12$ when $m$ is odd.

Eldergill [1997] generalized Gnanajothi’s result on stars by showing that the graphs obtained by joining one end point from each of any odd number of paths of equal length is odd - graceful. He also proved that one-point union of any number of copies of $C_6$ is odd - graceful.

Sekar [2002] investigated that the following graphs are odd - graceful: $C_m \odot P_n$ (the graph obtained by identifying an end point of $P_n$ with every vertex of $C_m$) where $n \geq 3$ and $m$ is even; $P_{a,b}$ when $a \geq 2$ and $b$ is odd; $P_{2,b}$ and $b \geq 2$; $P_{4,b}$ and $b \geq 2$; $P_{a,b}$ when $a$ and $b$ are even and $a \geq 4$ and $b \geq 4$; $P_{4r+1,4r+2}$; $P_{4r-1,4r}$; all $n$-polygonal snakes with $n$ even.

**Note:** $n$ denotes always a positive integer.

**Definition 1.2.1 Cartesian product graph:** The Cartesian product of $G$ and $H$, written as $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting $(g, h)$ adjacent to $(g', h')$ if and only if (1) $g = g'$ and $hh' \in E(H)$, or (2) $h = h'$ and $gg' \in E(G)$.

In this chapter, the edge - odd graceful labeling is obtained for the following graphs:

1. Cartesian Product of $P_2$ and $C_n$
2. Cartesian Product of $P_3$ and $C_n$
3. Cartesian Product of $P_4$ and $C_n
Example 1.2.2: The following graphs are edge - odd graceful due to various rules in graceful labeling.

(i). $P_2 \square C_n$, when $n$ is even and odd

(ii). $P_3 \square C_n$, $n$ is congruent to $(mod\ 6)$

(iii). $P_4 \square C_n$, $n$ is congruent to $(mod\ 6)$

(iv) $P_5 \square C_n$, $n$ is odd and congruent to $(mod\ 6)$; $n$ is even and congruent to $(mod\ 12)$

(v). $P_6 \square C_n$, $n$ is congruent to $(mod\ 4)$

(vi). $P_7 \square C_n$, $n$ is congruent to $(mod\ 4)$

Section 1.3 - Sum of Path and Null graph:

Introduction: A number of classes of graphs that are the join of graphs have been shown that to be graceful. Acharya [1982] obtained that if $G$ is a connected graph, then $G + \overline{K_n}$ is graceful. Bhat-Nayak and Gohkale [1986] found that $K_n + 2K_2$ is not graceful whereas Amutha and Kathiresan got that the graph obtained by attaching pendant edge to each vertex of $\overline{K_n} + 2K_2$ is graceful. Balakrishnan and Sampathkumar [1996] asked for which $m \geq 3$ is the graph $\overline{K_n} + mK_2$ graceful for all $n$. Ramirez-Alfosin [1999] investigated that if $G$ is graceful and $|V(G)| = |E(G)| = e$ and either 1 or $e$ is not a vertex label then $G + K_t$ is graceful for all $t$.

Sethuraman and e.tal [2001] analysed that the following graphs are graceful: the union of $n$ copies of $K_4$ with $i$ edges deleted for $1 \leq i \leq 5$ with one edge in common; $K_{1,m,n}$ with a pendant edge attached to each vertex is graceful.
Wu [2002] discussed that if \( G \) is a graceful graph with \( n \) edges and \((n + 1)\) vertices then the join of \( G \) and \( K_m \) and the join of \( G \) and any star are graceful. Jirimutu [2003] has shown that the graph obtained by attaching a pendant edge to every vertex of \( K_{m,n} \) is graceful.

Yousef [2003] proved that \( K_n + mK_2 \) is graceful if \( m \equiv 0 \) or \( 1 \pmod{4} \) and that \( K_n + mK_2 \) is not graceful if \( n \) is odd and \( m \equiv 2 \) or \( 3 \pmod{4} \). Redl [2003] obtained that the double cone \( C_n + K_2 \) is graceful for \( n = 3, 4, 5, 7, 8, 9, 11 \) and \( 12 \), but not graceful for \( n \equiv 2 \pmod{4} \).

The \( n \)-cone (also called the \( n \)-point suspension of \( C_m \)) \( C_m + K_n \) found to be graceful when \( m \equiv 0 \) or \( 3 \pmod{12} \) by Bhat -Nayak and Selvam [2003]. They also investigated that the following cones are graceful: \( C_4 + K_n, C_5 + K_2, C_7 + K_n, C_9 + K_2, C_{11} + K_n \) and \( C_{19} + K_n \).

Pan and Lu [2003] got that \( (P_2 + K_n) \cup K_{1,m} \) and \( (P_2 + K_n) \cup T_n \) are graceful. Barrientos [2005] also analysed: if \( G \) is a graceful graph of order \( m \) and size \( m - 1 \), then \( G \odot nK_1 \) and \( G + nK_1 \) are graceful;

If \( G \) is a graceful graph of order \( p \) and size \( q \) with \( q > p \), then \( (G \cup (q + 1 - p)K_1) \odot nK_1 \) is graceful; and all unicyclic caterpillar are graceful.

**Definition 1.3.1:** \( (P_2 + N_n)\circ \) is \( (P_2 + N_n) \) merging by an edge and it is a connected graph such that every vertex of \( (P_2 + N_n) \) is adjacent to every vertex of null graph \( N_n \) together with adjacent edges in \( (P_2 + N_n) \).
**Definition 1.3.2:** $P_k + N_n$ is a connected graph such that every vertex of $P_k$ is adjacent to every vertex of null graph $N_n$ together with adjacent edges in $P_k$.

**Definition 1.3.3:** $K_1 + P_n$ is a connected graph obtained from 1 copy of $P_n$ (whose vertices $u_1, u_2, \ldots, u_{n-1}, u_n; u_1, v_2, v_3, \ldots, v_{n-1}, u_n$ is first copy of $P_n$) and a null vertex $t(n_1)$ whose adjacency edges other than existing edges are $tu_i$ (for every $i = 1, 2, \ldots, n$); $u_iv_i, i = 2, 3, \ldots, (n-1))$.

**Definition 1.3.4:** $K_1 + 2P_n$ is a connected graph obtained from 2 copies of $P_n$ (whose vertices are $u_1, u_2, \ldots, u_{n-1}, u_n; u_1, v_2, v_3, \ldots, v_{n-1}, u_n$ as first copy of $P_n$; $u_1, w_2, \ldots, w_{n-1}, u_n$ is second copy of $P_n$) and a null vertex $t(n_1)$ whose adjacency edges other than existing edges are $tu_i$ (for every $i = 1, 2, \ldots, n$); $u_iv_i, i = 2, 3, \ldots, (n-1); v_jw_j, j = 2, 3, \ldots, (n-1))$.

**Definition 1.3.5:** $K_1 + 3P_n$ is a connected graph obtained from 3 copies of $P_n$ (whose vertices are $u_1, u_2, \ldots, u_{n-1}, u_n; u_1, v_2, v_3, \ldots, v_{n-1}, u_n$ as first copy of $P_n$; $u_1, w_2, \ldots, w_{n-1}, u_n$ is second copy of $P_n$; $u_1, s_2, \ldots, s_{n-1}, u_n$ is third copy of $P_n$) and a null vertex $t(n_1)$ whose adjacency edges other than existing edges are $tu_i$ (for every $i = 1, 2, \ldots, n$); $u_iv_i, i = 2, 3, \ldots, (n-1); v_jw_j, j = 2, 3, \ldots, (n-1); w_ks_k, k = 2, 3, \ldots, (n-1))$.

**Definition 1.3.6:** $K_1 + 4P_n$ is a connected graph obtained from 4 copies of $P_n$ (whose vertices are $u_1, u_2, \ldots, u_{n-1}, u_n; u_1, v_2, v_3, \ldots, v_{n-1}, u_n$ as first copy of $P_n$; $u_1, w_2, \ldots, w_{n-1}, u_n$ is second copy of $P_n$; $u_1, s_2, \ldots, s_{n-1}, u_n$ is third copy of $P_n$; $u_1, x_2, \ldots, x_{n-1}, u_n$ is fourth copy of $P_n$) and a null vertex $t(n_1)$ whose adjacency edges other than existing edges are $tu_i$.
(for every \( i = 1, 2, \ldots, n \); \( u_i v_i, i = 2, 3, \ldots, (n-1); v_j w_j, j = 2, 3, \ldots, (n-1) \), \( w_k s_k, k = 2, 3, \ldots, (n-1) \); \( s_l x_l, l = 2, 3, \ldots, (n-1) \)).

**Definition 1.3.7:** \( K_1 + 5P_n \) is a connected graph obtained from 4 copies of \( P_n \) (whose vertices are \( u_1, u_2, \ldots, u_{n-1}, u_n; u_1, v_2, v_3, \ldots, v_{n-1}, u_n \) as first copy of \( P_n \); \( u_1, w_2, \ldots, w_{n-1}, u_n \) is second copy of \( P_n \); \( u_1, s_2, \ldots, s_{n-1}, u_n \) is third copy of \( P_n \); \( u_1, x_2, \ldots, x_{n-1}, u_n \) is fourth copy of \( P_n \); \( u_1, y_2, \ldots, y_{n-1}, u_n \) is fifth copy of \( P_n \)) and a null vertex \( t(n_1) \) whose adjacency edges other than existing edges are \( t u_i \) (for every \( i = 1, 2, \ldots, n \); \( u_i v_i, i = 2, 3, \ldots, (n-1); v_j w_j, j = 2, 3, \ldots, (n-1) \); \( w_k s_k, k = 2, 3, \ldots, (n-1) \); \( s_l x_l, l = 2, 3, \ldots, (n-1) \)).

In this chapter the following graphs are proved as edge - odd graceful.

7. \((P_2 + N_n)^\circ e\) \hspace{1cm} 8. \(P_3 + N_n\) \hspace{1cm} 9. \(P_4 + N_n\)

10. \(P_5 + N_n\) \hspace{1cm} 11. \(K_1 + P_n\) \hspace{1cm} 12. \(K_1 + 2P_n\)

13. \(K_1 + 2P_n\) \hspace{1cm} 14. \(K_1 + 3P_n\) \hspace{1cm} 15. \(K_1 + 4P_n\) \hspace{1cm} 16. \(K_1 + 5P_n\)

**Example 1.3.8:** The following graphs are examples for edge - odd graceful due to various rules.

(i). \((P_2 + N_n)^\circ e\)

(ii). \(P_3 + N_n\), \( n \) is congruent to \((mod\ 6)\), when \( n \) is odd; and \( n \) is congruent to \((mod\ 8)\), when \( n \) is even

(iii). \(P_4 + N_n\), \( n \) is congruent to \((mod\ 6)\)

(iv). \(P_5 + N_n\), \( n \) is congruent to \((mod\ 6)\)

(v). \(K_1 + P_n\), \( n \) is congruent to \((mod\ 8)\)

(vi). \(K_1 + 2P_n\), \( n \) is congruent to \((mod\ 6)\)
(vii) $K_1 + 3P_n$, $n$ is even and $n$ is congruent to 2, 4, 6, 8 (mod 14); $n$ is congruent to 10, 12, 14 (mod 14); and $n$ is odd

(viii) $K_1 + 4P_n$, $n$ is even; and $n$ is odd congruent to (mod 4)

(ix) $K_1 + 5P_n$, $n$ is odd; and $n$ is even congruent to (mod 4)

**Section 1.4- Various types of connected graphs containing squares**

**Introduction:** Rosa [1967] proved that the $n$–cycle $C_n$ is graceful if and only if $n \equiv 0$ or 3 (mod 4). Delorme, Maheo, Thuillier, Koh, [1980], and Teo and Ma and Feng [1984] obtained that any cycle with a chord is graceful.

Koh and Yap [1985] generalized this by defining a cycle with a $P_k$-chord to be a cycle with the path $P_k$ joining two nonconsecutive vertices of the cycle. They found that these graphs are graceful when $k = 3$ and conjectured that all cycles with a $P_k$-chord are graceful.

Ropp and e.t.al [1990] got that for every $n$ and $t$ the class $C_n$ contains a graceful graph. They also conjectured that for all $n$ and $t$, all members of $C_n + t$ are graceful.

Abhyanker [2002] investigated various unicyclic (graphs with exactly one cycle) graphs. He analysed that the unicyclic graphs obtained by identifying one vertex of $C_4$ with the root of the olive tree with 2$n$ branches and identifying an adjacent vertex on $C_4$ with the end point of the path $P_{2n-2}$ are graceful. He discussed that if one attaches any number of pendant edges to these unicyclic graphs at the vertex of $C_4$ that is adjacent to the root of the olive tree but not adjacent to the end vertex of the attached path, the resulting graphs are graceful.

Sekar [2002] defined a chain of cycles $C_{2m,n}$ as the graph obtained by identifying $v_{i,m}$ and $v_{i+1,m}$ for $i = 1, 2, \ldots, n - 1$. He proved that $C_{6,2k}$ and $C_{8,n}$ are graceful for all $k$ and all $n$. He has shown that the graph
C_m ⊗ P_n obtained by attaching the path P_n to each vertex of C_m is graceful. He also determined that the graph obtained by identifying an endpoint of a star with a vertex of a cycle is graceful.

Sethuraman and Elmalai [2005] defined a cycle with parallel $P_k$-chords as a graph obtained from a cycle $C_n$ ($n \geq 6$) with consecutive vertices $v_0, v_1, ..., v_{n-1}$ by adding a disjoint path $P_k$ ($k \geq 3$), between each pair of nonadjacent vertices $v_i, v_{n-i}, v_2, v_{n-2}, ..., v_\alpha, v_\beta$ where $\alpha = \lfloor n/2 \rfloor - 1$ and $\beta = \lfloor n/2 \rfloor + 2$ if $n$ is odd or $\beta = \lfloor n/2 \rfloor + 1$ if $n$ is even. They proved that every cycle $C_n$ ($n \geq 6$) with parallel $P_k$-chords is graceful for $k = 3, 4, 6, 8$, and 10 and they conjectured that the cycle $C_n$ with parallel $P_k$-chords is graceful for all even $k$.

Barrientos [2007] obtained that all $C, C_{12,n}$ and $C_{6,2k}$ are graceful. He also proved that helms (graphs obtained from a wheel by attaching one pendant edge to each vertex) are graceful.

**Definition 1.4.1 Generalized n-squares (Type I) (1-nC_4):** $n$-C_4 is a collection of $n$ number of C_4. 1-nC_4 ($n \geq 1$) is a connected graph whose vertex set is $\{v_1, v_2, v_3, v_4, ..., v_{4n}\}$ whose edge set is $\{v_1v_2, v_2v_3, v_3v_4, ..., v_{4n-1}v_{4n}, v_{4n}v_{4n-3}, v_2v_6, v_6v_{10}, ..., v_{4n-6}v_{4n-2}\}$.

**Definition 1.4.2 Generalized n-squares (Type II) (2-nC_4):** $n$-C_4 is a collection of $n$ number of C_4. 2-nC_4 ($n \geq 1$) is a connected graph whose vertex set is $\{v_1, v_2, v_3, v_4, ..., v_{4n}\}$ whose edge set is $\{v_1v_2, v_2v_3, v_3v_4, ..., v_{4n-1}v_{4n}, v_{4n}v_{4n-3}, v_2v_6, V_6V_{10}, ..., v_{4n-6}v_{4n-2}, v_3v_7, v_7v_{11}, ..., v_{4n-5}v_{4n-1}\}$.

**Definition 1.4.3 Generalized n-squares (Type III) (3-nC_4):** $n$-C_4 is a collection of $n$ number of C_4. 3-nC_4 ($n \geq 1$) is a connected graph whose
Definition 1.4.4 Generalized n-squares (Type IV) (4-nC4): n-C4 is a collection of n number of C4. 4-nC4 (n ≥ 1) is a connected graph whose vertex set is \{v_1, v_2, v_3, ..., v_{4n}\} whose edge set is \{v_1v_2, v_2v_3, v_3v_4, ..., v_{4n-1}v_{4n}, v_{4n}v_{4n-3}, v_2v_6, v_6v_{10}, ..., v_{4n-6}v_{4n-2}, v_3v_7, v_7v_{11}, ..., v_{4n-5}v_{4n-1}, v_4v_8, v_8v_{12}, ..., v_{4n-4}v_{4n}\}.

In this chapter, the following graphs are obtained as graceful.

16. Generalized square 1-2KC4, where n is even
17. Generalized square 2-nC4
18. Generalized square 3-nC4
19. Generalized square 4-nC4

Example 1.4.5: The following graphs are as examples for edge - odd graceful for the above graphs.
(i). 1-2KC4, n is even
(ii). 2-nC4, n is even and odd
(iii). 3-nC4
(iv). 4-nC4

Section 1.5 - Edge - odd gracefulness of Cartesian product of path, circuit and wheel:

Introduction: Chen, L, and Yeh [1997] defined a firecracker as a graph obtained from the concatenation of stars by linking one leaf from each. They also defined a banana tree as a graph obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the
stars). They obtained that firecrackers are graceful and conjectured that banana trees are graceful. Lee [1989] has conjectured that all trees of odd order are edge-graceful.

Kanetkar and Sane [2007] found that trees formed by identifying one end vertex of each of six or fewer paths whose lengths determine an arithmetic progression are graceful.

Shiu, Ling and Low [2008] determined the edge-graceful spectra of all connected bicyclic graphs without pendant edges. Sethuraman and Jesintha [2008] got that all rooted trees in which every level contains pendant vertices and the degrees of the internal vertices in the same level are equal are graceful.

**Definition 1.5.1 Spanning tree:** A spanning tree of a connected, undirected graph is a tree that includes every vertex of that graph.

**Definition 1.5.2 Wheel:** A wheel is a graph obtained from a cycle by adding a new vertex and edges joining it to all the vertices of the cycle; the new edges are called the spokes of the wheel.

**In this chapter the following graphs are proved as edge-odd graceful.**

20. Cartesian Product of $C_3$ and $C_n$
21. Cartesian Product of $P_2$ and $W_n$
22. Spanning tree of $P_3 \square C_n$, $n$ is even
23. Spanning tree of $P_5 \square C_n$, $n$ is even
24. Spanning tree of $P_6 \square C_n$, $n$ is even

**Example 1.5.3:** The following graphs are given as examples.
(i). $C_3 \square C_n$, $n$ is congruent to $(mod 4)$ and $n \neq 8
(ii). $P_2 \boxplus W_n$, $n$ is congruent to (mod 8)
(iii). Spanning tree of $P_3$ and $C_n$, $n$ is even
(iv). Spanning tree of $P_5$ and $C_n$, $n$ is even
(v). Spanning tree of $P_6$ and $C_n$, $n$ is even.

**Section 1.6 - Edge - odd gracefulness of spanning trees:**

**Introduction:** Bermond and Sotteau [1976] proved that a rooted tree in which every level contains vertices of the same degree (symmetrical trees) are graceful. Rogers and e.tal [1978] gave methods for combining graceful trees to yield larger graceful trees. They also obtained recursive constructions to create graceful trees. Bermond [1979] conjectured that lobsters are graceful (a lobster is a tree with the property that the removal of the endpoints leaves a caterpillar).

Burzio and Ferrarese [1998] found that the graph obtained from any graceful tree by subdividing every edge is also graceful. Barrientos [2007] defined a $y$-tree as a graph obtained from a path by appending an edge to a vertex of a path adjacent to an end point. He got that graphs obtained from a $y$-tree $T$ by replacing every edge $e_i$ of $T$ by a copy of $K_{2,ni}$ in such a way that the ends of $e_i$ are merged with the two independent vertices of $K_{2,ni}$ after removing the edge $e_i$ from $T$ are graceful.

Krishnaa [2004] proved that all trees have graceful labelings. Kanetkar and Sane [2007] investigated that trees formed by identifying one end vertex of each of six or fewer paths whose lengths determine an arithmetic progression are graceful.

Sethuraman and Jesintha [2008] discussed that rooted trees obtained by identifying one of the end vertices adjacent to either of the
penultimate vertices of any number of caterpillars having equal diameter at least 3 with the property that all the degrees of internal vertices of all such caterpillars have the same parity are graceful. They also determined that rooted trees obtained by identifying either of the penultimate vertices of any number of caterpillars having equal diameter at least 3 with the property that all the degrees of internal vertices of all such caterpillars have the same parity are graceful. They also proved that all rooted trees in which every level contains pendant vertices and the degrees of the internal vertices in the same level are equal are graceful.

Sethuraman and Jesintha [2009] obtained that all banana trees and extended banana trees (graphs obtained by joining a vertex to one leaf of each of any number of stars by a path of length of at least two) are graceful.

In this chapter, the following graphs are proved as edge – odd graceful.

25. A spanning tree of $P_2 \Box C_n$, $n > 2$
26. A spanning tree of $P_4 \Box C_n$
27. A spanning tree of $P_m \Box C_n$, $n > 4$, $m$ is even and odd, $n$ is odd

Example 1.6.1: The following graphs are given as examples.
(i). A spanning tree of $P_2$ and $C_n$, $n > 2$
(ii). A spanning tree of $P_m$ and $C_n$, $n > 4$, $m$ is even and odd, $n$ is odd.

Also the following graph is proved as graceful:
28. A spanning tree of $P_m \Box C_n$

Section 1.7 - Strong product of Path and Circuit:

Introduction: Koh e.tal [1980] defined a web graph as one obtained by joining the pendant points of a helm to form a cycle and then adding a
single pendant edge to each vertex of this outer cycle. They asked whether such graphs are graceful. Lo [1985] found that all odd cycles are edge-graceful.

In [1996] this was proved by Kang, Liang, Gao, and Yang. Yang has extended the notion of a web by iterating the process of adding pendant points and joining them to form a cycle and then adding pendant points to the new cycle. In his notation, $W(2, n)$ is the web graph whereas $W(t, n)$ is the generalized web with $t_n$ cycles. Yang has shown that $W(3, n)$ and $W(4, n)$ are graceful. Wilson and Riskin [1998] got the Cartesian product of any number of odd cycles is edge-graceful.

Shiu, Lee and Schaffer [2001] discussed that the edge-gracefulness of multigraphs derived from paths, combs, and spiders obtained by replacing each edge by $k$ parallel edges. Shiu, Lam, and Cheng [2002] obtained that the composition of the path $P_3$ and any null graph of odd order is edge-graceful. Lee, Ma, Valdes and Tong [2002] investigated the edge-gracefulness of grids $P_m \times P_n$.

Gayathri et.al [2007] analysed the following: cycles are even edge-graceful if and only if the cycles are odd; even cycles with one pendant edge are even edge-graceful; wheels are even edge-graceful; gears are not even edge-graceful; fans $P_n + K_1$ are even edge-graceful; $C_4 \cup P_m$ for all $m$ are even edge-graceful; $C_{2n+1} \cup P_{2n+1}$ are even edge-graceful; crowns $C_n \circ K_1$ are even edge-graceful; $C_n$ are even edge-graceful; sunflowers are even edge-graceful; triangular snakes are even edge-graceful; closed helms with the centre vertex removed are even edge-graceful; graphs decomposable into two odd Hamiltonian cycles are even edge-graceful; and odd order graphs that are decomposable into three Hamiltonian cycles are even edge-graceful.
Definition 1.7.1 Strong Product Graph: The product is denoted by $G \Box H$ and defined by the vertex set $V(G) \times V(H)$, and $(g, h)$ is adjacent to $(g', h')$ if $(g = g' \text{ and } h \text{ adj } h')$ or $(g \text{ adj } g' \text{ and } h = h')$ or $(g \text{ adj } g' \text{ and } h \text{ adj } h')$.

Definition 1.7.2 Web graph: The web graph is a graph consisting of $n$ copies of cycles $C_n$ with corresponding vertices connected by spokes.

In this chapter, the following graphs are proved as edge - odd graceful.

29. Strong Product of $P_2$ and $C_n$
30. Strong Product of $P_3$ and $C_n$
31. Web graph $P_n$ and $C_7$

Example 1.7.3: The following graphs are given as examples.
(i). Strong Product of $P_2$ and $C_n$
(ii). Strong Product of $P_3$ and $C_n$
(iii). Web graph $P_n$ and $C_7$

Also the following graph is proved as even – edge graceful:

32. Strong Product of $P_2$ and $C_n$. 