CHAPTER VII

STRONG PRODUCT OF PATH AND CIRCUIT

Introduction: Koh. etal [1980] defined a web graph as one obtained by joining the pendant points of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. They asked whether such graphs are graceful. In [1996] this was proved by Kang, Liang, Gao, and Yang.

Yang has extended the notion of a web by iterating the process of adding pendant points and joining them to form a cycle and then adding pendant points to the new cycle. In his notation, $W(2, n)$ is the web graph whereas $W(t, n)$ is the generalized web with $t$ $n$ cycles. Yang has shown that $W(3, n)$ and $W(4, n)$ are graceful.

Lo [1985] proved that all odd cycles are edge - graceful and Wilson and Riskin [1998] proved the Cartesian product of any number of odd cycles is edge - graceful.

Kuang, Lee, Mitchem, and Wang [1988] have conjectured that unicyclic graphs of odd order are edge - graceful. They have verified this conjecture in the following cases: graphs obtained by identifying an end point of a path $P_m$ with a vertex of $C_n$ when $m + n$ is even; crowns with one pendant edge deleted; graphs obtained from crowns by identifying an endpoint of $P_m$, $m$ odd, with a vertex of degree 1; amalgamations of a cycle and a star obtained by identifying the centre of the star with a cycle vertex where the resulting graph has odd order; graphs obtained from $C_n$ by joining a pendant edge to $n-1$ of the cycle vertices and two pendant edges to the remaining cycle vertex.
Section 7.1 - Preliminaries and Previous works:

Lee [1989] has conjectured that all trees of odd order are edge - graceful. Lee and Seah [1989] have also investigated edge - gracefulfulness of various multigraphs. Small [1990] proved that spiders for which every vertex has odd degree with the property that the distance from the vertex of degree greater than 2 to each end vertex is the same are edge - graceful.

Lee and Seah [1990] showed that $K_{n,n,...,n}$ is edge - graceful if and only if $n$ is odd and the number of partite sets is either odd or a multiple of 4 and they have also proved that $C_n^k$ (the $k$th power of $C_n$) is edge - graceful for $k < n/2$ if and only if $n$ is odd and $C_n$ is edge - graceful for $k \geq n/2$ if and only if $n \equiv 2 \pmod{4}$. They have also proved that for $k \leq n/2$, $C_n^k$ is edge - graceful if and only if $n$ is odd, and for $k \geq n/2$, $C_n^k$ is edge - graceful if and only if $n \equiv 2 \pmod{4}$.

Ho, Lee, and Seah [1991] use $S(n; a_1, a_2, \ldots, a_k)$ where $n$ is odd and $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k < n/2$ to denote the $(n, nk)$ multigraph with vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ and edge set $\{v_i v_j / i = j, i-j \equiv a_t \pmod{n}\}$ for $t = 1, 2, \ldots, k$. They proved that all such multigraphs are edge graceful.

Lee, Seah, and Lo [1991] have proved that for $n$ odd, $C_{2n} \cup C_{2n+1}$, $C_n \cup C_{2n+2}$ and $C_n \cup C_{4n}$ are edge - graceful. They also shown that for odd $k$ and odd $n$, $K_{C_n}$ is edge - graceful.

Kendrick and Lee [1991] proved that there are only finitely many $n$ for which $K_{m,n}$ is edge - graceful and they completely solve the problem for $m = 2$ and $m = 3$. Lee and Pritikin [1991] proved that the Mobius ladders of order $4n$ are edge - graceful.
Lee and Seah [1991] found that Petersen graph P(n, k) is edge - graceful if and only if n is even and k < n/2. In particular, P (n, 1) = C_n × P_2 is edge-graceful if and only if m and n are odd. Cabaniss, Low, and Mitchem [1992] have shown that regular spiders of odd order are edge - graceful. Keene and Simoson [1996] got that all spiders of odd order with exactly three end vertices are edge-graceful.

Lee, Tong, and Seah [1999] have conjectured that the total graph of a (p, p)-graph is edge - graceful if and only if p is even. They have proved this conjecture for cycles. Shiu, Lee, and Schaffer [2001] investigated the edge - gracefulness of multigraphs derived from paths, combs, and spiders obtained by replacing each edge by k parallel edges.

Shiu, Lam, and Cheng [2002] obtained that the composition of the path P_3 and any null graph of odd order is edge - graceful. Lee, Ma, Valdes, and Tong [2002] investigated the edge - gracefulness of grids P_m × P_n.

Lee et al. discussed the following: P_2 × P_n is not edge - graceful for all n > 1; P_3 × P_n is edge - graceful if and only if n = 1 or n = 4; P_4 × P_n is edge - graceful if and only if n = 3 or n = 4; P_5 × P_n is edge - graceful if and only if n = 1; P_2m × P_2n is edge-graceful if and only if m = n = 2. They conjectured that for all m, n ≥ 10 of the form m = (2k + 1)(4k + 1), n = (2k + 1)(4k + 3), the grids P_m × P_n are edge - graceful.

Duan and Qi [2002] used G_t(m_1, n_1; m_2, n_2; . . . ; m_s, n_s) to denote the graph composed of the complete bipartite graphs K_{m_1,n_1}, K_{m_2,n_2}, . . . , K_{m_s,n_s} that have only t (1 ≤ t ≤ min{m_1, m_2, . . . , m_s}) common vertices but no common edge and G(m_1, n_1; m_2, n_2) to one common edge. They proved that these graphs are k - graceful graphs for
all k. Youssef [2003] analysed that if G is Skolem - graceful, then G + K_n is graceful.

Hegde [2003] found that the following: if a graph is (k, d) - graceful for odd k and even d, then the graph is bipartite; if a graph is (k, d)-graceful and contains C_{2j+1} as a subgraph, then k \leq jd(q - j - 1); K_n is (k, d) - graceful if and only if n \leq 4; C_{4t} is (k, d)-graceful for all k and d; C_{4t+1} is (2t, 1) - graceful; C_{4t+2} is (2t - 1, 2) - graceful; and C_{4t+3} is (2t + 1, 1) - graceful.

Lu, Pan, and Li [2004] obtained that K_{1,m} \cup K_{p,q} is k - graceful when k > 1, and p and q are at least 2. Jirimutu, Bao, and Kong [2004] have shown that the graphs obtained from K_{2,n} (n \geq 2) and K_{3,n} (n \geq 3) by attaching \ r \geq 2 edges at each vertex is k-graceful for all k \geq 2.

Shiu [2006] investigated that the following: C_n \times P_2 is super – edge graceful for all n \geq 2 ; P (n; \theta) is super – edge graceful for all n \geq 2 ; certain other families of connected cubic multigraphs are super – edge graceful and conjectures that every connected cubic of multigraph except K_4 and the graph with 2 vertices and 3 edges is super – edge graceful. Youssef [2006] has shown that for all \theta \in \mathbb{R}, P_n \cup S_m is Skolem - graceful if and only if n \geq 3 or n = 2 and m is even.

Gayathri et al [2007] discussed the following: cycles are even edge - graceful if and only if the cycles are odd; even cycles with one pendant edge are even edge-graceful; wheels are even edge - graceful; gears are not even edge - graceful; fans P_n + K_1 are even edge - graceful; C_4 \cup P_m for all m are even edge - graceful; C_{2n+1} \cup P_{2n+1} are even edge - graceful; crowns C_n \circ K_1 are even edge - graceful; C_n are even edge - graceful; sunflowers are even edge - graceful.
triangular snakes

are even edge -graceful; closed helms with the centre vertex removed are even edge - graceful; graphs decomposable into two odd Hamiltonian cycles are even edge - graceful; and odd order graphs that are decomposable into three Hamiltonian cycles are even edge-graceful.

Riskin Weidman [2008] analysed: if G is an edge - graceful 2r-regular graph with p vertices and q edges and (r, kp) = 1, then kG is edge-graceful when k is odd; when n and k are odd, kC_n is edge - graceful; and if G is the Cartesian product of an odd number of odd cycles and k is odd, then kG is edge - graceful. They have conjectured that the disjoint union of an odd number of copies of a 2r-regular edge-graceful graph is edge - graceful. Seoud and Elsakhawi [2008] proved: paths and ladders are arbitrarily graceful; and for n ≥ 3, K_n is k - graceful if and only if k = 1 and n = 3 or 4.

In this chapter, the following graphs, Strong Product of P_2 and C_n, Strong Product of P_3 and C_n, and Web graph P_n and C_7 are proved as edge - odd graceful and Strong Product of P_2 and C_n is got as even - edge graceful.

Section 7.2 - Edge - odd gracefulness of strong product of P_2 and C_n:

Definition 7.2.1: Strong Product Graph: The product is denoted G △ H and defined by the vertex set V(G) X V(H), and (g, h) is
adjacent to \((g', h')\) if \((g = g' \text{ and } h \text{ adj } h')\) or \((g \text{ adj } g' \text{ and } h = h')\) or \((g \text{ adj } g' \text{ and } h \text{ adj } h')\).

**Theorem 7.2.2:** The strong product of \(P_2 \boxtimes C_n\) is edge-odd graceful.

**Proof:** The strong product of a path \(P_2\) and a circuit \(C_n\) is given and the arbitrary labelings for vertices and edges for \(P_2 \boxtimes C_n\) are mentioned below:

To find edge-odd graceful, define \(f : E(P_2 \boxtimes C_n) \rightarrow \{1, 3, \ldots, 2q-1\}\) by

**Case i.** \(n \equiv 0\text{(mod 5)}\)

\[
\begin{align*}
  f(e_1) &= 5, \\  f(e_2) &= 1, \\  f(e_3) &= 7, \\  f(e_4) &= 3 \\  f(e_i) &= 2i-1, \quad i = 5, 6, 7, \ldots, 5n
\end{align*}
\]

Rule (1)

**Case ii.** \(n \equiv 2\text{(mod 5)}\)

\[
\begin{align*}
  f(e_1) &= 5, \\  f(e_3) &= 1 \\  f(e_i) &= 2i-1, \quad i = 2, 4, 5, 6, 7, \ldots, (2n+2), (3n+3), (3n+4), \ldots, 5n \\  f(e_{3n+3}) &= f(e_{2n+2}) + 2i, \quad i = 1, 2, \ldots, n
\end{align*}
\]

Rule (2)

**Case iii.** \(n \equiv 4\text{(mod 5)}\)

Figure 7.01: Edge-odd graceful labeling of \(P_2 \boxtimes C_n\)
\[ f(e_1) = 5, \ f(e_2) = 7, \ f(e_3) = 1, \ f(e_4) = 3 \]

\[ f(e_i) = 2i - 1, \quad i = 5, 6, 7, \ldots, (2n+2), (3n+3), \]
\[ (3n+4), \ldots, 5n \quad \text{Rule (3)} \]

\[ f(e_{3n+3-i}) = f(e_{2n+2}) + 2i, \quad i = 1, 2, \ldots, n \]

**Case iv. \( n \equiv 3(\text{mod } 5) \)**

\[
\begin{align*}
  f(e_1) &= 7, \ f(e_2) = 1, \ f(e_3) = 3, \ f(e_4) = 5 \\
  f(e_i) &= 2i - 1, \quad i = 5, 6, 7, \ldots, (2n+2), (3n+3), \\
  (3n+4), \ldots, 5n \quad \text{Rule (4)} \]
\[
  f(e_{3n+3-i}) = f(e_{2n+2}) + 2i, \quad i = 1, 2, \ldots, n
\]

For \( n \equiv 1(\text{mod } 5) \), the arbitrary labelings for vertices and edges for \( P_2 \Box C_n \) are mentioned below:

**Figure 7.02** : Edge - odd graceful labeling of \( P_2 \Box C_n \)

**Case iv. \( n \equiv 1(\text{mod } 5) \)**

\[ f(e_i) = 2i - 1, \quad i = 1, 2, 3, \ldots, 5n \quad \text{Rule (5)} \]
Define $f_+: V(G) \rightarrow \{0, 1, 2, \ldots, (2k - 1)\}$ by $f_+(v) \equiv \sum f(uv) \mod[2k]$, where this sum run over all edges through $v$. .......... Rule (6)

Hence the induced map $f_+$ provides the distinct labels for vertices and also the edge labeling is distinct. So the strong product graph $P_2$ and $C_n$ is edge - odd graceful.

**Example 7.2.3:** The strong product graph $P_2 \boxtimes C_5$ is edge - odd graceful.

**Proof:** The strong product graph $P_2 \boxtimes C_5$ is a connected graph with 10 vertices and 25 edges, where $n \equiv 0 (\text{mod } 5)$. Due to the rules (1) & (6) in theorem (7.2.2), edge - odd graceful labeling of the required graph is obtained as follows:

![Figure 7.03: Edge - odd graceful labeling of $P_2 \boxtimes C_5$](image)

**Example 7.2.4:** The strong product graph $P_2 \boxtimes C_7$ is edge - odd graceful.

**Proof:** The strong product graph $P_2 \boxtimes C_7$ is a connected graph with 14 vertices and 35 edges, where $n \equiv 2 (\text{mod } 5)$. Due to the rules (2) & (6) in theorem (7.2.2), edge - odd graceful labeling of the required graph is obtained as follows:

![Figure 7.03: Edge - odd graceful labeling of $P_2 \boxtimes C_7$](image)
Example 7.2.5: The strong product graph $P_2 \boxtimes C_9$ is edge - odd graceful.  

**Proof:** The strong product graph $P_2 \boxtimes C_9$ is a connected graph with 18 vertices and 45 edges, where $n \equiv 4 \pmod{5}$. Due to the rules (3) & (6) in theorem (7.2.2), edge - odd graceful labeling of the required graph is obtained as follows:

![Figure 7.04: Edge - odd graceful labeling of $P_2 \boxtimes C_7$](image)

Example 7.2.6: The strong product graph $P_2 \boxtimes C_8$ is edge - odd graceful.  

**Proof:** The strong product graph $P_2 \boxtimes C_8$ is a connected graph with 16 vertices and 40 edges, where $n \equiv 3 \pmod{5}$. Due to the rules (4) & (6) in theorem (7.2.2), edge - odd graceful labeling of the required graph is obtained as follows:

![Figure 7.05: Edge - odd graceful labeling of $P_2 \boxtimes C_9$](image)
Figure 7.06: Edge - odd graceful labeling of $P_2 \boxtimes C_8$

Example 7.2.7: The strong product graph $P_2 \boxtimes C_6$ is edge - odd graceful.

Proof: The strong product graph $P_2 \boxtimes C_6$ is a connected graph with 12 vertices and 30 edges, where $n \equiv 1 \pmod{5}$. Due to the rules (5) & (6) in theorem (7.2.2), edge - odd graceful labeling of the required graph is obtained as follows:

Figure 7.07: Edge - odd graceful labeling of $P_2 \boxtimes C_6$

Section 7.3 - Edge-odd graceful labeling for the strong product of $P_3$ and $C_n$:

Lemma 7.3.1: A strong product of $P_3$ and $C_n$ is edge – odd graceful where $n = 2, 3$.  

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The strong product of $P_3$ and $C_2$ is a connected graph with 6 vertices and 11 edges. The arbitrary labeling of edge - odd graceful of the required graph is obtained as follows:

![Figure 7.08: Edge – odd graceful labeling of $P_3 \boxtimes C_2$](image)

The strong product of $P_3$ and $C_3$ is a connected graph with 9 vertices and 27 edges. The arbitrary labeling of edge - odd graceful of the required graph is obtained as follows:

![Figure 7.09: Edge – odd graceful labeling of $P_3 \boxtimes C_3$](image)

**Theorem 7.3.2:** A strong product of $P_3$ and $C_n$ is edge – odd graceful. The following graph is strong product of $P_3$ and $C_n$ with $5n$ vertices and $9n$ edges with some arbitrarily labeling in vertices and edges.

![Graph of $P_3 \boxtimes C_n$](image)
To find edge - odd graceful, define $f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ by

**n is even**

**Case i.** $n \equiv 0(\text{mod } 6)$

\[
\begin{align*}
    f(e_1) &= 5, f(e_3) = 1 \\
    f(e_i) &= 2i-1, \quad i = 2, 4, 5, 6, \ldots, (2n+5), (2n+8), (2n+9)\ldots.9n \\
    f(e_{2n+6}) &= 4n+13, f(e_{2n+7}) = 4n+11
\end{align*}
\]

Rule (1)

**Case ii.** $n \equiv 2(\text{mod } 6)$

\[
\begin{align*}
    f(e_1) &= 3, f(e_2) = 5, f(e_3) = 1 \\
    f(e_i) &= 2i-1, \quad i = 4, 5, 6, \ldots, (3n+6), (4n+6), (4n+7), \ldots, 9n \\
    f(e_{4n+6+i}) &= f(e_{3n+6}) + 2i, \quad i = 1, 2, 3, \ldots, (n-1)
\end{align*}
\]

Rule (2)

**Case iii.** $n \equiv 4(\text{mod } 6)$

\[
\begin{align*}
    f(e_1) &= 5, f(e_3) = 1 \\
    f(e_i) &= 2i-1, \quad i = 2, 4, 5, 6, \ldots, (n+7), (n+9), (n+10), \ldots, (2n+6), (2n+8), \ldots, 9n
\end{align*}
\]
f(e_{n+8}) = 4n+13, f(e_{2n+7}) = 2n+15 \quad \text{Rule (3)}

**n is odd**

**Case iv.** \( n \equiv 1 \pmod{6} \)

\[
\begin{align*}
 f(e_1) = 5, & \quad f(e_3) = 1 \\
 f(e_i) = 2i-1, & \quad i = 2, 4, 5, \ldots, 9n \\
\end{align*}
\]

\quad \text{Rule (4)}

**Case v.** \( n \equiv 3 \pmod{6} \)

\[
\begin{align*}
 f(e_1) = 5, & \quad f(e_3) = 1 \\
 f(e_i) = 2i-1, & \quad i = 2, 4, 5, \ldots, 9n \\
\end{align*}
\]

\quad \text{Rule (5)}

**Case vi.** \( n \equiv 5 \pmod{6} \)

\[
\begin{align*}
 f(e_1) = 3, & \quad f(e_2) = 5, f(e_3) = 1 \\
 f(e_i) = 2i-1, & \quad i = 4, 5, 6, \ldots, (2n+3), (2n+8), \ldots, (4n+5), (5n+5), (5n+6), \ldots, 9n \\
 f(e_{5n+5-i}) = f(e_{4n+5}) + 2i, & \quad i = 1, 2, 3, \ldots, (n-1) \\
 f(e_{2n+5}) = f(e_{2n+6}) - 2, f(e_{2n+4}) = f(e_{2n+5}) + 4, \\
 f(e_{2n+7}) = f(e_{2n+8}) + 2, f(e_{2n+6}) = f(e_{2n+7}) + 4 \\
\end{align*}
\]

\quad \text{Rule (6)}

Define \( f_+: V(G) \rightarrow \{0, 1, 2, \ldots, (2k-1)\} \) by \( f_+(v) \equiv \sum f(uv) \pmod{(2k)}, \) where this sum run over all edges through \( v \) \ldots \text{Rule (7)}

Hence the map \( f \) and the induced map \( f_+ \) provide labels as odd numbers for edges with all distinct and also the labeling for vertex set has distinct values in \( \{0, 1, 2, \ldots, (2k-1)\} \). So \( G \) is edge - odd graceful.

**Example 7.3.3:** The strong product graph \( P_3 \Box C_6 \) is edge - odd graceful.

**Proof:** The strong product graph \( P_3 \Box C_6 \) is a connected graph with 18 vertices and 54 edges, where \( n \equiv 0 \pmod{6} \). Due to the rules (1) & (7) in
theorem (7.3.2), edge - odd graceful labeling of the required graph is obtained as follows:

![Figure 7.11: Edge – odd graceful labeling of \( P_3 \square C_6 \)]

**Example 7.3.4:** The strong product graph \( P_3 \square C_8 \) is edge - odd graceful.

**Proof:** The strong product graph \( P_3 \square C_8 \) is a connected graph with 24 vertices and 72 edges, where \( n \equiv 2 \pmod{6} \). Due to the rules (2) & (7) in theorem (7.3.2), edge - odd graceful labeling of the required graph is obtained as follows:
Example 7.3.5: The strong product graph $P_3 \boxtimes C_4$ is edge - odd graceful.

Proof: The strong product graph $P_3 \boxtimes C_4$ is a connected graph with 12 vertices and 36 edges, where $n \equiv 4 \pmod{6}$. Due to the rules (3) & (7) in theorem (7.3.2), edge - odd graceful labeling of $P_3 \boxtimes C_4$ is obtained.

Figure 7.13: Edge – odd graceful labeling of $P_3 \boxtimes C_4$
**Example 7.3.6:** The strong product graph $P_3 \Box C_7$ is edge - odd graceful.

**Proof:** The strong product graph $P_3 \Box C_7$ is a connected graph with 21 vertices and 63 edges, where $n \equiv 1 \pmod{6}$. Due to the rules (4) & (7) in theorem (7.3.2), edge - odd graceful labeling of the required graph is obtained as follows:

![Figure 7.14: Edge – odd graceful labeling of $P_3 \Box C_7$](image)

**Example 7.3.7:** The strong product graph $P_3 \Box C_9$ is edge - odd graceful.

**Proof:** The strong product graph $P_3 \Box C_9$ is a connected graph with 27 vertices and 81 edges, where $n \equiv 3 \pmod{6}$. Due to the rules (5) & (7) in theorem (7.3.2), edge - odd graceful labeling of the required graph is obtained as follows:
**Example 7.3.8:** The strong product graph $P_3 \Box C_5$ is edge - odd graceful.

**Proof:** The strong product graph $P_3 \Box C_5$ is a connected graph with 15 vertices and 45 edges, where $n \equiv 5 \pmod{6}$. Due to the rules (6) & (7) in theorem (7.3.2), edge - odd graceful labeling of the required graph is obtained as follows:
Section 7.4 - Edge - odd gracefulness of web graph $P_n$ and $C_7$:

Definition 7.4.1: Web graph: The web graph is a graph consisting of $n$ copies of cycles $C_n$ with corresponding vertices connected by spokes.

Theorem 7.4.2: The web graph $P_n$ and $C_7$ is edge - odd graceful.

Proof: The connected graph $P_n$ and the circuit $C_7$ is a connected graph with $7n$ vertices and $(14n-7)$ edges and the arbitrary labelings for vertices and edges for $P_n$ and $C_7$ are mentioned below:

For $n \equiv 0, 1 \pmod{4}$

![Figure 7.17: Edge - odd graceful labeling of $P_n$ and $C_7$](image)
For $n \equiv 2 \pmod{4}$

\[ 8n-1 \equiv 7n \equiv 14n-7 \equiv 2n-1 \equiv 5n+2 \equiv 8n+1 \equiv 3n \equiv 13n-6 \equiv 12n-3 \equiv 9n-1 \quad 5n-1 \equiv 4n+3 \equiv 4n+2 \equiv 4n+1 \quad 2n+1 \equiv 10n-3 \equiv 3n+2 \equiv 4n-1 \equiv 12n-5 \equiv 11n-4 \quad 4n \]

Figure 7.18: Edge - odd graceful labeling of $P_n$ and $C_7$
For \( n \equiv 3 \pmod{4} \)

\[
\begin{align*}
\text{Figure 7.19: Edge - odd graceful labeling of } & P_n \text{ and } C_7 \\
\text{To find edge - odd graceful, define } f: E(P_n \text{ and } C_7) & \to \{1, 3, \ldots, 2q-1\} \text{ by} \\
\text{ } & n \equiv 0, 1 \pmod{4} \\
f(e_i) &= 2i-1, i = 1, 2, 3, \ldots, (14n-7) \quad \text{Rule (1)} \\
n \equiv 2 \pmod{4} \\
f(e_i) &= 2i-1, i = 1, 2, 3, \ldots, (14n-7) \quad \text{Rule (2)} \\
n \equiv 3 \pmod{4} \\
f(e_i) &= 2i-1, i = 1, 2, 3, \ldots, (14n-7), \quad i \neq 5n-1, i \neq 6n-2 \quad \text{Rule (3)} \\
f(e_{5n-1}) &= 12n-5; \ f(e_{6n-2}) = 10n-3 
\end{align*}
\]
Define \( f_+ : V(G) \rightarrow \{0, 1, 2, \ldots, (2k - 1)\} \) by \( f_+(v) = \sum f(uv) \mod (2k) \), where this sum run over all edges through \( v \) \( \ldots \) Rule (4)

Both of \( f \) and \( f_+ \) finds the distinct labels for vertices and also the edge labeling is distinct. Hence the web graph \( P_n \) and \( C_7 \) is edge - odd graceful.

**Example 7.4.3:** A web graph \( P_4 \) and \( C_7 \) is edge - odd graceful.

**Proof:** A web graph \( P_4 \) and \( C_7 \) is a connected graph with 28 vertices and 49 edges, where \( n \equiv 0 \pmod{4} \). Due to the rules (1) & (4) in theorem (7.4.2), edge - odd graceful labeling of the required graph is obtained as follows:

![Graph Image](image-url)

*Figure 7.20: Edge - odd graceful graph of \( P_4 \) and \( C_7 \)*
Example 7.4.4: A web graph $P_5$ and $C_7$ is edge - odd graceful.

**Proof:** A web graph $P_5$ and $C_7$ is a connected graph with 35 vertices and 63 edges, where $n \equiv 1 \pmod{4}$. Due to the rules (1) & (4) in theorem (7.4.2), edge - odd graceful labeling of the required graph is obtained as follows:

![Figure 7.21: Edge - odd graceful labeling of $P_5$ and $C_7$](image)

Example 7.4.5: A web graph $P_6$ and $C_7$ is edge - odd graceful.

**Proof:** A web graph $P_6$ and $C_7$ is a connected graph with 42 vertices and 77 edges, where $n \equiv 2 \pmod{4}$. Due to the rules (2) & (4) in theorem (7.4.2), edge - odd graceful labeling of the required graph is obtained as follows:
Example 7.4.6: A web graph $P_3$ and $C_7$ is edge - odd graceful.

Proof: A web graph $P_3$ and $C_7$ is a connected graph with 21 vertices and 35 edges, where $n \equiv 3 \pmod{4}$. Due to the rules (3) & (4) in theorem (7.4.2), edge-odd graceful labeling of the required graph is obtained as follows:
Section 7.5 - Even - edge graceful labeling of strong product of $P_2$ and $C_n$:

**Lemma 7.5.1:** The strong product of $P_2$ and $C_n$ is even-edge graceful, where $n = 2, 3, 4, 5, 6$ and 7.

**Proof:** The strong product of the path $P_2$ and the circuit $C_2$ is a connected graph with 4 vertices and 6 edges. The arbitrary labeling of even - edge graceful of the required graph is obtained as follows:

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Figure 7.24: Even - edge graceful labeling of $P_2 \boxtimes C_2$
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The strong product of the path $P_2$ and the circuit $C_3$ is a connected graph with 6 vertices and 15 edges. The arbitrary labeling of even-edge graceful of the required graph is obtained as follows:

![Figure 7.25: Even-edge graceful labeling of $P_2 \boxtimes C_3$]

The strong product of the path $P_2$ and the circuit $C_4$ is a connected graph with 8 vertices and 20 edges. The arbitrary labeling of even-edge graceful of the required graph is obtained as follows:

![Figure 7.26: Even-edge graceful labeling of $P_2 \boxtimes C_4$]

The strong product of the path $P_2$ and the circuit $C_5$ is a connected graph with 10 vertices and 25 edges. The arbitrary labeling of even-edge graceful of the required graph is obtained as follows:
The strong product of the path $P_2$ and the circuit $C_6$ is a connected graph with 12 vertices and 30 edges. The arbitrary labeling of even-edge graceful of the required graph is obtained as follows:

The strong product of the path $P_2$ and the circuit $C_7$ is a connected graph with 14 vertices and 35 edges. The arbitrary labeling of even-edge graceful of the required graph is obtained as follows:
**Theorem 7.5.2:** The strong product of path $P_2$ and circuit $C_n$ ($n \geq 8$) is even-edge graceful.

**Proof:** The strong product of a path $P_2$ and a circuit $C_n$ is given below. The arbitrary labelings for vertices and edges are as follows:

![Figure 7.30: Even-edge graceful labeling of $P_2 \boxtimes C_n$](image)

To find even-edge graceful, define $f: E(G) \rightarrow \{1, 2, \ldots, 2q\}$ by

$$
\begin{align*}
f(e_1) &= 2q = 10n \\
f(e_i) &= 2i+2, \quad i = 2, 3, \ldots, n-1 \\
f(s_n) &= 4, \quad f(t_n) = 2, \quad f(s_1) = 28, \quad f(t_1) = 26 \\
f(s_i) &= 18 + 10i, \quad i = 2, 3, \ldots, (n-4) \\
f(t_i) &= 16 + 10i, \quad i = 2, 3, \ldots, (n-4) \\
f(s_{n-3}) &= f(n-4) + 6 \\
f(t_{n-3}) &= f(n-4)+6 \\
f(s_{n-2}) &= f(s_{n-3}) + 10 \\
f(t_{n-2}) &= f(t_{n-3}) + 10 \\
f(s_{n-1}) &= f(s_{n-2}) + 4 \\
f(t_{n-1}) &= f(t_{n-2}) + 4 \\
f(\ell_1) &= 5, \quad f(\ell_2) = 11, \quad f(u_1) = 7, \quad f(v_1) = 9 \\
f(u_2) &= 13, \quad f(v_2) = 15 \\
f(u_i) &= 4i + 5, \quad i = 3, 4, \ldots, (n-1) \\
f(v_i) &= 4i + 7, \quad i = 3, 4, \ldots, (n-1) \\
f(e_n) &= 6n-14
\end{align*}
$$

Rule (1)
Define \( f_+ : V(G) \rightarrow \{0, 1, 2, \ldots, (2k-2)\} \) by \( f_+(v) \equiv \sum f(uv) \mod (2k) \), where this sum run over all edges through \( v \). Rule (2)

So the induced map \( f_+ \) provides the even number labels for vertices with all distinct and also the edge labeling is distinct. \( f \) and \( f_+ \) satisfy all conditions for even-edge graceful. Hence the strong product graph \( P_2 \) and \( C_n \) is even-edge graceful.

**Example 7.5.3:** A strong product of \( P_2 \) and \( C_{10} \) is even-edge graceful.

**Proof:** The following is a connected graph with 20 vertices and 50 edges. Due to the rules (1) & (2) in theorem (7.5.2), the even-edge graceful labeling of the \( P_2 \square C_{10} \) is obtained as follows:

![Figure 7.31: Even-edge graceful labeling of \( P_2 \square C_{10} \)](image)