Generalized Poisson Processes

8.1 Introduction

Poisson processes \( \{X(t) : t \geq 0\} \) are suitable models for a wide variety of counting processes. Ecological and genetical problems are mostly connected with counting processes and hence a Poisson process can give a suitable description under some basic assumptions. For details see Consul and Jain (1973) and Karlin and Taylor (1975). By modifying the Poisson process, Janardan (1980, 2002) Janardan et al (1981) derived a model for the number of eggs laid by a parasite on a host and this model was applied to study the variation of the distribution of chromosome aberrations in human and animal cells. He also developed a continuous time Markov chain \( \{X(t) : t \geq 0\} \) with \( X(0) = 0 \) and the probability mass function (p.m.f.) is given as an integral representation which is very helpful to derive the moments and other properties.

Some results in this chapter form part of Naik and Jose (2008b).
In this chapter we introduce two counting processes as generalizations of Poisson processes. First, we introduce a pure birth process which is a generalization of the Poisson process, suitable for modeling births where couples behave rationally during the periods when they have no child, one child, and more than one child. Then we introduce another generalized counting process called $q$-Weibull counting process with $q$-Weibull inter arrival times.

### 8.2 A Generalized Poisson Process

Let $P_n(t) = P\{X(t) = n\}, \ t \geq 0, n \geq 0$ be the probability mass function of a stochastic process $\{X(t) : t \geq 0\}$. Janardan et al (1981) developed a stochastic process with the following p.m.f.

1. $P_0(t) = e^{-\lambda t}$  
2. $P_n(t) = \frac{\lambda \mu^{n-1}}{(\mu - \lambda)^n} \left\{ e^{-\lambda t} - e^{-\mu t} \sum_{k=0}^{n-1} \frac{(\mu - \lambda)^k t^k}{k!} \right\}$ for $n \geq 1, \lambda > 0, \mu > 0$.  

The transition probabilities are assumed to be the following form:

$$
\lim_{h \to 0} \frac{P\{X(t+h) = j | X(t) = i\}}{h} = \begin{cases} 
\lambda_j & \text{if } j = i + 1 \\
0 & \text{if } j \neq i + 1
\end{cases}
$$

Using these basic assumptions, an integral representation corresponding to (8.2.2) is obtained by considering $\lambda_j = \lambda$ for $j = 0$ and $\lambda_j = \mu$ for all $j = 1, 2, \ldots$, which is given below:

$$
P_n(t) = \frac{\lambda \mu^{n-1}}{(n-1)!} e^{-\lambda t} \int_0^t u^{n-1} e^{-(\mu-\lambda)u} \, du.
$$

This representation is very useful to derive the moments and other properties of the p.m.f. Prabhakar (1971), Saxena et al (2004), Haubold et al (2002) discussed various generalizations of Mittag-Leffler functions along with applications. The new model developed in this chapter is connected to Mittag-Leffler function and we have obtained some interesting properties.
Now we introduce the new model and obtain the estimates for the parameters and
the results are illustrated for a data given by Norman and Sasaki (1966) for the number
of dicentric breaks in a cell exposed to 800 radiations. Also we connect these models to
some special functions which help us to use the properties of special functions in stochas-
tic processes. It is interesting to observe that one can easily connect this p.m.f. with the
Mittag-Leffler function and its generalizations. We establish a connection with the gener-
alized special functions such as G-function and H-function discussed in Mathai (1993), by
using the Mellin-Bernes integral representation.

8.3 A New Pure Birth Process

Now consider a continuous time Markov chain \( \{ X(t) : t \geq 0 \} \) with transition probabilities,

\[
\lim_{h \to 0} \frac{P\{ X(t+h) = j | X(t) = i \}}{h} = \begin{cases} 
\lambda_j & \text{if } j = i + 1 \\
0 & \text{if } j \neq i + 1.
\end{cases}
\]

Here we assume that \( \lambda_j = \lambda_0 \), for \( j = 0 \), \( \lambda_j = \lambda_1 \) for \( j = 1 \) and \( \lambda_j = \lambda_2 \) for \( j = 2, 3, \ldots \).

It can be seen that when \( \lambda_0 = \lambda_1 = \lambda_2 = \lambda \), it reduces to the Poisson process and when
\( \lambda_0 = \lambda \) and \( \lambda_n = \mu \) for \( n \geq 1 \), it reduces to the process discussed in Janardan (2002). Now we have,

\[
P\{ X(t) = 0 \} = e^{-\lambda_0 t} \quad \text{(8.3.1)}
\]

\[
P\{ X(t) = 1 \} = \lambda_1 t e^{-\lambda_1 t}. \quad \text{(8.3.2)}
\]

Let \( T_2 \) be the time to second occurrence of the process.

\[
P\{ T_2 \leq t_2 \} = 1 - e^{-\lambda_0 t_2} - \lambda_1 t_2 e^{-\lambda_1 t_2}. \quad \text{(8.3.3)}
\]

The p.d.f. \( f_2(t_2) \) of \( T_2 \) is given by
\[ f_2(t_2) = \frac{d}{dt_2} P[T_2 \leq t_2] \]
\[ = \lambda_0 e^{-\lambda_0 t_2} - \lambda_1 e^{-\lambda_1 t_2} + \lambda_1^2 t_2 e^{-\lambda_1 t_2}, \quad 0 \leq t_2 < \infty. \]

The conditional probability of \( n \) events \((n \geq 2)\) in \((0, t)\) is given by,
\[ P\{X(t) = n | T_2 = t_2\} = \frac{\lambda_2(t - t_2)^{n-2} e^{-\lambda_2(t-t_2)}}{(n-2)!} \quad n \geq 2. \quad (8.3.4) \]

Then the unconditional p.d.f. is given by
\[ P_n(t) = \int_{t_2=0}^{t} \frac{\lambda_2(t - t_2)^{n-2} e^{-\lambda_2(t-t_2)}}{(n-2)!} \left(\lambda_0 e^{-\lambda_0 t_2} - \lambda_1 e^{-\lambda_1 t_2} + \lambda_1^2 t_2 e^{-\lambda_1 t_2}\right) dt_2 \]
\[ = \frac{t e^{-\lambda_2 t}(\lambda_2 t)^{n-2}}{(n-2)!} \int_0^1 (1 - u)^{n-2} \]
\[ \times \left\{ \lambda_0 e^{-(\lambda_0 - \lambda_2)tu} - \lambda_1 e^{-(\lambda_1 - \lambda_2)tu} + \lambda_1^2 t e^{-(\lambda_1 - \lambda_2)tu} \right\} du. \quad (8.3.5) \]

**Result 8.3.1.** \( P_n(t) \) defined above can be expressed as a finite sum of the generalized Poisson density function as,
\[ P_n(t) = \lambda_2^{n-2} \left\{ \frac{\lambda_0 e^{-\lambda_0 t}}{(\lambda_2 - \lambda_0)^{n-1}} \left[ 1 - \sum_{k=0}^{n-2} \frac{[(\lambda_2 - \lambda_0) t]^k}{k!} e^{-(\lambda_2 - \lambda_0) t} \right] \right. \]
\[ + \frac{(\lambda_2^2 t - \lambda_1) e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)^{n-1}} \left[ 1 - \sum_{k=0}^{n-2} \frac{[(\lambda_2 - \lambda_1) t]^k}{k!} e^{-(\lambda_2 - \lambda_1) t} \right] \]
\[ - (n-1) \frac{\lambda_1^2 t e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)^n} \left[ 1 - \sum_{k=0}^{n-1} \frac{[(\lambda_2 - \lambda_1) t]^k}{k!} e^{-(\lambda_2 - \lambda_1) t} \right] \right\}. \quad (8.3.6) \]

The proof is obvious. The equation (8.3.6) can be given in terms of confluent hypergeometric function. The \( P_n(t) \) obtained is clearly a p.m.f. for \( \lambda_0 \neq \lambda_1 \neq \lambda_2 \). The probability generating function corresponding to this p.m.f. is
\[ G(s, t) = P_0(t) + sP_1(t) + \sum_{n=2}^{\infty} s^n P_n(t) \]

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\[ G(s, t) = e^{-\lambda_0 t} + \lambda_1 st e^{-\lambda_1 t} + s^2 e^{-\lambda_2 t(1-s)} \left\{ \lambda_0 \left( 1 - e^{-(\lambda_0 - \lambda_2)(1-s)t} \right) \over \lambda_0 - \lambda_2 + \lambda_2 s \right\} + \lambda_1 \lambda_2 (1-s) \left( 1 - e^{-(\lambda_1 - \lambda_2)(1-s)t} \right) \over (\lambda_1 - \lambda_2 + \lambda_2 s)^2 \]  

It can be verified that for \( \lambda_0 = \lambda_1 = \lambda_2 = \lambda \) the above equation reduces to the probability generating function of the Poisson distribution with parameter \( \lambda (> 0) \). The first moment is also derived to estimate \( \lambda_2 \) as given below.

\[ E[X(t)] = \sum_{n=0}^{\infty} n P_n(t) = \lambda_1 t e^{-\lambda_1 t} + \sum_{n=2}^{\infty} n P_n(t) 
= \lambda_1 t e^{-\lambda_1 t} + t \int_0^1 [\lambda_2 t(1-u) + 2](\lambda_0 e^{-\lambda_0 tu} - \lambda_1 e^{-\lambda_1 tu} + \lambda_2 t e^{-\lambda_2 tu}) du. \]

On simplifying this we get,

\[ E[X(t)] = 2(1 - e^{-\lambda_0 t}) - \lambda_1 t e^{-\lambda_1 t} + \lambda_2 t \left( 1 + e^{-\lambda_1 t} + {e^{-\lambda_0 t} - 1 \over \lambda_0 t} + {e^{-\lambda_1 t} - 1 \over \lambda_1 t} \right). \quad (8.3.7) \]

It is easy to verify that the first derivative of \( G(s, t) \) at \( s = 0 \) gives the expectation of \( X(t) \).

8.4 Estimation of Parameters \( \lambda_0, \lambda_1, \lambda_2 \)

In ecological and genetical problems usually the number of observations is small compared to other events. For the estimation of parameters of given distribution it is better to estimate it using relative frequencies so that it will give same expected frequencies as that of the observed frequencies, which will reduce the error sum of squares. If the number of estimated parameters is increased then it is likely that the distribution will give a better fit. Let \( \hat{P}_0, \hat{P}_1, \ldots \) be the relative frequencies from the data, then \( \hat{\lambda}_0 \) and \( \hat{\lambda}_1 \) are estimated from

\[ \hat{P}_0(t) = e^{-\hat{\lambda}_0 t} \]

and

\[ \hat{P}_1(t) = \hat{\lambda}_1 t e^{-\hat{\lambda}_1 t}. \quad (8.4.1) \]
\( \hat{\lambda}_0 \) can be estimated for any values of \( \hat{P}_0 \). But \( \hat{\lambda}_1 \) can be estimated only for \( \hat{P}_1 \leq 0.3678 \). If \( \hat{P}_1 = 0.3678 \) then there exist a unique estimate for \( \hat{\lambda}_1 \). If \( \hat{P}_1 < 0.3678 \) there are two estimates for \( \hat{\lambda}_1 \). Among the two the larger one is taken as an estimate for \( \hat{\lambda}_1 \), which gives a better fit for the model. The estimation of \( \hat{\lambda}_2 \) is done by using the estimate of \( \hat{\lambda}_0 \) and \( \hat{\lambda}_1 \) and \( E[X(t)] \). It is obvious that if \( \hat{\lambda}_0 \) is estimated using (8.3.1), then depending upon \( \hat{P}_1 \) we may have either no solution or unique solution or two solutions. For \( t = 1 \), \( \hat{\lambda}_1 \) can be estimated from (8.3.2). Figure 8.1 shows that for any value of \( \hat{P}_0 \), \( \hat{\lambda}_0 \) can be uniquely estimated and for \( \hat{P}_1 \) there exist two estimates for \( \hat{\lambda}_1 \) with \( t = 1 \) if \( \hat{P}_1 < 0.3678 \).

If the relative frequency \( \hat{P}_1 > 0.3678 \) then any version of Poisson distribution will not give a model for the given data because the maximum of \( \hat{P}_1 \) is at \( \hat{\lambda}_1 = 1 \) and at \( \hat{\lambda}_1 = 1, \hat{\lambda}_1 e^{-\hat{\lambda}_1 t} = 0.3678 \). Now the number of dicentric breaks in a cell exposed to 800 radiations is given in Table 8.1 [Norman and Sasaki (1966)]. Let us take \( \hat{\lambda}_0 \) such that it satisfies (8.3.1) and estimate \( \hat{\lambda}_1 \) by using (8.3.2) and \( \hat{\lambda}_2 \) by using the expected value in (8.3.7) respectively. Thus if Pearson’s \( \chi^2 \) goodness-of-fit statistic is used, then we obtain the result as shown in Table 8.1.

Here Model (1) refers to Janardan’s model given by (8.2.2) and (8.2.4) and Model (2) refers to New model given by (8.3.5) and (8.3.6). Lagrangian Poisson model (LPD) has
Table 8.1: Distribution of Dicentrics [Norman and Sasaki (1966)]

<table>
<thead>
<tr>
<th>Abberations</th>
<th>No. of cells</th>
<th>LPD</th>
<th>Model (1)</th>
<th>Model (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
<td>8.5</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>24</td>
<td>23.3</td>
<td>25.8</td>
<td>24</td>
</tr>
<tr>
<td>2</td>
<td>34</td>
<td>31.1</td>
<td>33.1</td>
<td>33.99</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
<td>27</td>
<td>27</td>
<td>27.89</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>17.8</td>
<td>16</td>
<td>15.99</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>8.4</td>
<td>7.6</td>
<td>7.089</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>5</td>
<td>4.4</td>
<td>5.2714</td>
</tr>
</tbody>
</table>

\[ \hat{\lambda}_0 = 3.0 \]
\[ \bar{x} = 2.53 \]
\[ s^2 = 2.09 \]
\[ \hat{\lambda}_1 = 2.783 \]
\[ \hat{\lambda}_2 = -0.10 \]
\[ \hat{\mu} = 2.313 \]
\[ \hat{\lambda}_2 = 2.0793 \]
\[ \chi^2 = 2.3581 \]
\[ \chi^2 = 2.045 \]
\[ \chi^2 = 1.9946 \]
\[ \lambda_0 = 3.0 \]
\[ \chi^2 = 2.3581 \]
\[ \chi^2 = 2.045 \]
\[ \chi^2 = 1.9946 \]
\[ d.f = 3 \]
\[ p-value = 0.5105 \]
\[ p-value = 0.5631 \]
\[ p-value = 0.5735 \]

been studied by Consul and Jain (1973). When using Pearson’s goodness-of-fit statistic we reject the hypothesis that the selected model is the true model for large values of \( \chi^2 \) because \( \chi^2 \) is a generalized measure of distance between the vector of relative frequencies and the vector of true probabilities under the selected model. Thus for small values of \( \chi^2 \) we do not reject the hypothesis and we may conclude that for smaller values of \( \chi^2 \) we get better models. But since \( \chi^2 \) is a “distance” statistic we may take the view that smaller the values of \( \chi^2 \) we have a better model. In this respect our model in (8.3.5) is better than Janardan’s and Lagrangian models for the data in Table 8.1. Pearson’s \( \chi^2 \) is approximately a chi-square with \( k - r - 1 \) degrees of freedom for \( np_i \geq 5 \) for all \( i = 1, 2, \ldots, k, \ k \geq 5 \) where \( k \) is the number of cells, \( p_i \) is the true probability in the \( i^{th} \)-cell, \( r \) is the number of parameters estimated and \( n \) is the total frequency. In Table 1 since the expected frequency corresponding to the cell “6” is less than 5 it is pooled with the cell “5” thereby the number of effective cells is 6 and not 7. In LPD and in Model (1) we are estimating 2 parameters thereby \( r = 2 \). Thus the effective number of degrees of freedom in LPD and Model (1) is \( 6 - 1 - 2 = 3 \). For comparing the new model with LPD and Model (1) we need to keep the
d.f. = 3. In order to do this we are going to fit a subfamily from the family in equations (8.3.5) and (8.3.6) by preselecting the value of $\lambda_0 = 3$ and estimating $\lambda_1$ and $\lambda_2$ so that the final d.f. = 3. This fitted model is compared with LPD and Model (1). Since the value of $\chi^2$ is less than the model given by Janardan et al (1981), the model is a fit which is better than Lagrangian model and the model introduced by Janardan et al (1981).

8.5 Representation of Stochastic Process in Terms of Mittag-Leffler Function

Prabhakar (1971), Saxena et al (2004), and Haubold et al (2002) discussed various generalizations of Mittag-Leffler functions along with applications. An interesting result observed from (8.2.1) and (8.2.2) is that a Poisson process $\{X(t), t \geq 0\}$ with parameters $\mu$ and $\lambda$ can be easily connected with the famous Mittag-Leffler function and its generalizations and can be connected with the generalized special functions such as $G$-function and $H$-function by taking the Mellin-Bernes integral representation. These connections will be stated as theorems.

**Theorem 8.5.1.** For the Poisson process $\{X(t), t \geq 0\}$ in (8.2.1) and (8.2.4) and for an integer $n \geq 1$, $P_n(t)$ defined in (8.2.2) can be expressed in terms of Mittag-Leffler function as

$$P_n(t) = n! \left( \frac{\lambda}{\mu} \right) f_n(\mu t) E_{1,n+1}(xt) \quad \text{for } n \geq 1$$

where $x = \mu - \lambda$ and the Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \\alpha, \beta \in C, \quad \Re(\alpha) > 0, \Re(\beta) > 0$$

and

$$f_n(y) = \frac{y^n e^{-y}}{n!} \quad \text{for } n = 1, 2, \ldots, \text{and for } y > 0$$

where $\Re(\cdot)$ denotes the real part of $(\cdot)$ and $C$ is the complex domain.
Proof: Consider $P_n(t)$ as in (8.2.2)

$$P_n(t) = \frac{\lambda \mu^{n-1}}{(\mu - \lambda)^n} \left\{ e^{-\lambda t} - e^{-\mu t} \sum_{k=0}^{n-1} \frac{(\mu - \lambda)^k t^k}{k!} \right\} \quad \text{for } n \geq 1$$

$$= \frac{\lambda \mu^{n-1}}{x^n} e^{-xt} \left\{ 1 - e^{-xt} \sum_{k=0}^{n-1} \frac{(xt)^k}{k!} \right\}$$

where $x = \mu - \lambda$

$$= \frac{\lambda}{\mu} e^{-xt} (\mu t)^n \sum_{j=0}^{\infty} \frac{(xt)^j}{(j+n)!}$$

$$= n! \frac{\lambda}{\mu} f_n(\mu t) E_{1,n+1}(xt) \quad \text{for } n \geq 1$$

which is the required result.

**Property 1.** For an integer $n \geq 1$ the $P_n(t)$ defined in (8.2.2) is closely related to the Mellin-Ross function as

$$P_n(t) = \lambda \mu^{n-1} e^{-\mu t} E_t(n, a) \quad \text{for } n \geq 1 \quad (8.5.1)$$

where the Mellin-Ross function is defined as

$$E_t(\gamma, a) = t^{\gamma} \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\gamma + k + 1)} \quad \Re(\gamma + 1) > 0.$$

**Property 2.** For the Poisson process $\{X(t), t \geq 0\}$ in (8.2.1) and (8.2.4) and for an integer $n \geq 1$, $P_n(t)$ in (8.2.2) is related to the confluent hypergeometric function as

$$P_n(t) = \left( \frac{\mu}{\lambda} \right)^{n-1} f_n(\lambda t) {}_1F_1(n; n+1; -xt) \quad n \geq 1 \quad (8.5.2)$$

where $x = \mu - \lambda$ and $f_n(\cdot)$ is defined as in Theorem 8.5.1.

**Theorem 8.5.2.** Let $\{X(t), t \geq 0\}$ be a continuous time Markov chain with $X(0)=0$ and the transition probabilities as in (8.2.3). Then (8.2.4) is connected with a gener-
alized Mittag-Leffler function as

\[ P_n(t) = \left( \frac{\mu}{\lambda} \right)^{n-1} e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(-xt)^k}{k! \Gamma(n+k+1)} \]  \[ n \geq 1 \]  \hspace{1cm} (8.5.3)

where the generalized Mittag-Leffler function is defined as

\[ E_{\alpha,\beta}^{\gamma}(z) = \sum_{j=0}^{\infty} \frac{(\gamma)^j z^j}{j! \Gamma(\gamma j + \beta)} \quad \Re(\alpha) > 0, \quad \Re(\beta) > 0. \]

Proof: Consider the p.m.f. in (8.2.4)

\[ P_n(t) = \frac{\lambda \mu^{n-1} t^n}{(n-1)!} e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(-xt)^k}{k! \Gamma(n+k+1)} \]

where \((a)_j\) is the Pochhammer symbol defined as \((a)_j = a(a+1) \cdots (a+j-1), a \neq 0, (a)_0 = 1.\)

**Theorem 8.5.3.** For the Poisson process \( \{X(t), t \geq 0\} \) and for an integer \( n \geq 1 \) the \( P_n(t) \) of the p.m.f. in (8.2.4) is transformed to generalized hypergeometric functions such as \( G \)-function and \( H \)-function by taking the Mellin-Bernes integral representation as follows:

\[ P_n(t) = n \left( \frac{\mu}{\lambda} \right)^{n-1} f_n(\lambda t) H_{1,2}^{1,1} \left[ xt \left| \begin{array}{c} (1-n,1) \\ (0,1),(-n,1) \end{array} \right. \right] \]  \hspace{1cm} (8.5.4)

Proof: Consider the Poisson process \( \{X(t), t \geq 0\} \) with \( P_n(t) = P\{X(t) = n\} \) for \( n \geq 1 \) as in (8.2.4) and by taking the Mellin-Bernes integral representation, we get

\[ P_n(t) = \frac{\lambda \mu^{n-1}}{(n-1)!} e^{-\lambda t} \frac{1}{2\pi i} \int_{L} \int_{0}^{t} u^{n-1} \Gamma(s)(xu)^{-s} du ds, \quad i = \sqrt{-1} \]
where $du \wedge ds$ is the usual wedge product of differentials

$$P_n(t) = \left( \frac{\mu}{\lambda} \right)^{n-1} \frac{(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!} \frac{1}{2\pi i} \int_L \frac{\Gamma(s) \Gamma(n-s)}{\Gamma(n-s+1)} (xt)^{-s} ds$$

$$= n \left( \frac{\mu}{\lambda} \right)^{n-1} f_n(\lambda t) \, H^1_{1,2} \left[ xt^{(1-n,1),(-n,1)} \right].$$

Remark 8.5.1. The above result can also be obtained by applying the following relation

$$1 F_1(a; b : -x) = \frac{\Gamma(b)}{\Gamma(a)} H^1_{1,2} \left[ x^{(1-a,1),(0,1),(-b,1)} \right].$$

8.6 Some Interesting Results on Confluent Hypergeometric Functions

From the Poisson process $\{X(t), t \geq 0\}$ with parameters $\mu$ and $\lambda$ the p.m.f. as given in (8.2.2) and (8.2.4) we obtain a finite sum representation for the confluent hypergeometric function for an integer $n \geq 1$.

Result 8.6.1. In a continuous time Markov chain $\{X(t), t \geq 0\}$ with $X(0) = 0$ and the transition probabilities given as in (8.2.3) with parameters $\mu$ and $\lambda$ and for an integer $n \geq 1$ the confluent hypergeometric function in Property 2 can be expressed as a finite sum which is given below

$$1 F_1(n; n + 1; -xt) = \frac{n!}{(xt)^n} \left[ 1 - e^{-xt} \sum_{k=0}^{n-1} \frac{(xt)^k}{k!} \right] \quad n \geq 1. \quad (8.6.1)$$

Proof: On comparing $P_n(t)$ of the p.m.f. given in (8.2.2) and (8.2.4) we get,

$$\frac{1}{(x)^n} \left\{ 1 - e^{-xt} \sum_{k=0}^{n-1} \frac{(xt)^k}{k!} \right\} = \frac{1}{(n-1)!} \int_0^t u^{n-1} e^{-xu} du$$

$$= \frac{1}{(n-1)!} \frac{t^n}{n} 1 F_1(n; n + 1; -xt)$$

$$= \frac{t^n}{n!} 1 F_1(n; n + 1; -xt).$$
Therefore
\[ _1F_1(n; n + 1; -xt) = \frac{n!}{(xt)^n} \left\{ 1 - e^{-xt} \sum_{k=0}^{n-1} \frac{(xt)^k}{k!} \right\} \]

**Remark 8.6.1.** From (8.6.1) it follows that for all points for which \( xt = 1 \) and for integer \( n \geq 1 \)
\[ _1F_1(n; n + 1; -1) = (n!)e^{-1} \left\{ e - \sum_{k=0}^{n-1} \frac{1}{k!} \right\} \]
which gives a cut-off tail of the exponential function. Also from (8.6.1) it follows that for all points for which \( xt = -1 \) and for an integer \( n \geq 1 \)
\[ _1F_1(n; n + 1; -1) = (n!)(-1)^n e^{-1} \left\{ e - \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right\} \]
which gives a cut-off tail of the inverse exponential function.

**Result 8.6.2.** For a Poisson process \( \{X(t), t \geq 0\} \) and for an integer \( n \geq 1 \) if the probability of exactly \( n \) events occurring in a time interval \( (0, t) \) is known, then it is possible to predict more than \( n \) events occurring in the same time interval \( (0, t) \) by using the recurrence relation
\[ _1F_1(n + 1; n + 2; -xt) = \frac{n + 1}{xt} \left\{ _1F_1(n; n + 1; -xt) - e^{-xt} \right\} \quad n \geq 1 \]
provided \( n \) is a positive integer.

**Proof:** Using Result 8.3.1
\[
_1F_1(n + 1; n + 2; -xt) = \frac{(n + 1)!}{(xt)^{n+1}} \left\{ 1 - e^{-xt} \sum_{k=0}^{n} \frac{(xt)^k}{k!} \right\} \\
= \frac{n + 1}{xt} \left\{ \frac{n!}{(xt)^n} \left\{ 1 - e^{-xt} \sum_{k=0}^{n-1} \frac{(xt)^k}{k!} \right\} - e^{-xt} \right\} \\
= \frac{n + 1}{xt} \left\{ _1F_1(n; n + 1; -xt) - e^{-xt} \right\}
\]
which is the required result.

**Remark 8.6.2.** From Result 8.6.2 it follows that for all points for \( xt = 1 \) and for an integer \( n \geq 1 \),

\[ \text{}_{1}F_{1}(n + 1; n + 2; -1) = (n + 1)\{n\}_{n; n + 1; -1} - e^{-1} \]

and for all points for which \( xt = -1 \) and for an integer \( n \geq 1 \),

\[ \text{}_{1}F_{1}(n + 1; n + 2; 1) = (n + 1)\{e - 1\}_{n; n + 1; -1} \].

**Result 8.6.3.** In the Poisson process \( \{X(t), t \geq 0\} \) in (8.2.1) and (8.2.4) the successive values of the \( G \)-function and \( H \)-function are connected as follows:

\[
\begin{align*}
G_{1,2}^{1,1}\left[ xt \big|^{(1-n,1)}_{(0,1),(-n,1)} \right] &= xtG_{1,2}^{1,1}\left[ xt \big|^{(-n,1)}_{(0,1),(-n-1,1)} \right] + e^{-xt} \quad (8.6.5) \\
\text{and corresponding } H \text{-function is} \\
H_{1,2}^{1,1}\left[ xt \big|^{(1-n,1)}_{(0,1),(-n,1)} \right] &= xtH_{1,2}^{1,1}\left[ xt \big|^{(-n,1)}_{(0,1),(-n-1,1)} \right] + e^{-xt} \quad (8.6.6)
\end{align*}
\]

provided \( n \geq 1 \), an integer.

**Remark 8.6.3.** The above result shows that the difference between two \( G \)-functions or two \( H \)-functions with the suitable choices of parameters and for positive integers \( n \geq 1 \) gives a negative exponential function.

**Remark 8.6.4.** From (8.6.6) it follows that for all points for which \( xt = 1 \)

\[ n G_{1,2}^{1,1}\left[ 1 \big|^{(1-n,1)}_{(0,1),(-n,1)} \right] - G_{1,2}^{1,1}\left[ 1 \big|^{(-n,1)}_{(0,1),(-n-1,1)} \right] = e^{-1} \text{ for } n \geq 1. \]

and for all points for which \( xt = -1 \)

\[ G_{1,2}^{1,1}\left[ -1 \big|^{(1-n,1)}_{(0,1),(-n,1)} \right] - G_{1,2}^{1,1}\left[ -1 \big|^{(-n,1)}_{(0,1),(-n-1,1)} \right] = e \text{ for } n \geq 1. \]
Corresponding relations can be obtained for $H$-functions also. All these results in this paper are derived from the new generalized Poisson process. This is the novelty in these results.

8.7 The $q$-Weibull Counting Process

In this section we generalize the Poisson process in another direction by introducing a counting process using $q$-Weibull inter arrival times. For the Poisson process mean and variance are equal, which is not realistic. Now following \( q \), we introduce a model which give more flexibility to the mean-variance relationship.

Let $Y_n$ be the time from the measurement origin at which the $n$th event occurs. Let $X(t)$ be the number of events that have occurred until the time $t$. Then $Y_n \leq t \iff X(t) \geq n$. In other words the amount of time at which the $n$th event occurs from the time origin is less than or equal to $t$ iff the number of events that have occurred by time $t$ is greater than or equal to $n$. Now the count model, $P_n(t)$ is given by

$$P_n(t) = P[X(t) = n] = P[X(t) \geq n] - P[X(t) \geq n + 1] = P[Y_n \leq t] - P[Y_{n+1} \leq t] = F_n(t) - F_{n+1}(t)$$

where $F_n(t)$ be the CDF of $Y_n$. When the counting process coincide with the occurrence of an event, then $F_n(t)$ is the $n$-fold convolution of the common interarrival time distribution.

Now we consider the count model for the $q$-Weibull distribution by assuming that the inter arrival times are i.i.d $q$-Weibull distribution, with cumulative distribution function (CDF), $F(x) = 1 - [1 - (1 - q)(\lambda x)^\alpha]^{\frac{2-q}{1-q}}$ for $0 \leq x \leq \frac{1}{\lambda(1-q)^{\frac{1}{\alpha}}}$, $1$ for $x \geq \frac{1}{\lambda(1-q)^{\frac{1}{\alpha}}}$ and $0$ otherwise.

Here we consider the case for $q < 1$. The series expansion for the CDF is given by

$$F(t) = 1 - \sum_{j=0}^{\infty} \frac{(-1)^j(2-q)_{j+1}(1-q)(\lambda t)^\alpha}{\Gamma(j+1)}$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^j(2-q)_{j+1}(1-q)(\lambda t)^\alpha}{\Gamma(j+1)}$$
where \((\alpha)_j\) is the Pochammer symbol. Then corresponding \(f(t)\) is given by

\[
f(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} \alpha_j (2-q) \Gamma(j+1)}{(1-q)^{j+1}}.
\]

Also we have the recursive relationship

\[
P_n(t) = \int_0^t [F_{n-1}(t) - F_n(t)] f(s) ds = \int_0^t P_{n-1}(t-s) f(s) ds.
\]

Now \(F_0(t) = 1, \forall t\) and \(F_1(t) = F(t).\) Also

\[
P_0(t) = F_0(t) - F_1(t) = [1 - (1-q)(\lambda t)\alpha]^{\frac{2-q}{1-q}} = \sum_{j=0}^{\infty} \frac{(-1)^{j+1} (2-q)^j [(1-q)(\lambda t)\alpha]^j}{\Gamma(j+1)}.
\]

Using the recursive formula

\[
P_1(t) = \int_0^t P_0(t-s) f(s) ds
\]

\[
= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{j+i+1} (2-q)^j (1-q)^{i+j} (\lambda t)^{\alpha(i+j)} \Gamma(\alpha i + 1) \Gamma(\alpha j + 1)}{\Gamma(j+1) \Gamma(i+1) \Gamma(\alpha(i+j)+1)}
\]

\[
= \sum_{j=1}^{\infty} \frac{(-1)^{j+1} [(1-q)(\lambda t)\alpha]^j}{\Gamma(\alpha j + 1)} \left[ \sum_{i=0}^{j-1} \frac{(2-q)^j \Gamma(\alpha i + 1) \Gamma(\alpha (j-i) + 1)}{\Gamma(j+1) \Gamma(i+1) \Gamma(j-i+1)} \right]
\]

\[
= \sum_{j=1}^{\infty} \frac{(-1)^{j+1} [(1-q)(\lambda t)\alpha]^j a_j^2}{\Gamma(\alpha j + 1)}
\]

where \(a_j^2 = \sum_{i=0}^{j-1} \frac{(2-q)^j \Gamma(\alpha i + 1) \Gamma(\alpha (j-i) + 1)}{\Gamma(j+1) \Gamma(i+1) \Gamma(j-i+1)}\) which suggest a general form

\[
P_{n+1}(t) = \int_0^t P_n(t-s) f(s) ds
\]

\[
= \sum_{j=n}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^{i+j+n+1} (2-q)^i (1-q)^{i+j} (\lambda t)^{\alpha(i+j)} \Gamma(\alpha i + 1) \Gamma(\alpha j + 1)}{\Gamma(\alpha j + 1) \Gamma(i+1) \Gamma(\alpha(i+j)+1)}
\]
On simplification, we get

\[ P_{n+1}(t) = \sum_{l=n+1}^{\infty} \frac{(-1)^{l+n+1}[(1-q)(\lambda t)]^\alpha}{l!} \sum_{k=n}^{l-1} \frac{a_k^{n}(2-q)}{\Gamma(l-k)} \frac{\Gamma(\alpha(l-k)+1)}{\Gamma(\alpha-k+1)} \]

Thus we have the model,

\[ P[N(t) = n] = \sum_{l=n}^{\infty} \frac{(-1)^{l+n}[(1-q)(\lambda t)^\alpha]a_l^n}{\Gamma(\alpha l+1)}, n = 0, 1, \ldots, \]

where \( a_j^0 = \frac{(2-q)}{\Gamma(j+1)} \) for \( j = 0, 1, \ldots \), and \( a_j^{n+1} = \sum_{i=0}^{\infty} \frac{(-1)^{l+n}[(1-q)(\lambda t)^\alpha]a_i^n}{\Gamma(\alpha(i+j)+1)} \) for \( n = 0, 1, \ldots, j = n+1, n+2, \ldots \), and the mean count is given by

\[ E(N) = \sum_{n=1}^{\infty} \sum_{l=n}^{\infty} \frac{n(-1)^{l+n}[(1-q)(\lambda t)^\alpha]a_l^n}{\Gamma(\alpha l+1)}, \]

with variance given by

\[ \text{Var}(N) = \sum_{n=2}^{\infty} \sum_{l=n}^{\infty} \frac{n^2(-1)^{l+n}[(1-q)(\lambda t)^\alpha]a_l^n}{\Gamma(\alpha l+1)} - \left( \sum_{n=1}^{\infty} \sum_{l=n}^{\infty} \frac{n(-1)^{l+n}[(1-q)(\lambda t)^\alpha]a_l^n}{\Gamma(\alpha l+1)} \right)^2. \]

Then the moment generating function can be readily obtained as

\[ M_t(\theta) = E[e^{\theta N}] = \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \frac{e^{im(-1)^{j+i}[(1-q)(\lambda t)^\alpha]a_j^i}}{\Gamma(\alpha j+1)}. \]

Now this model generalizes the commonly used Poisson and Weibull count models and we call the new model as \( q \)-Weibull count model. In a similar manner we can construct a heterogeneous \( q \)-Weibull count model with \( \lambda \) replaced by \( \lambda_i \) where \( \lambda_i \) follows gamma
distribution with parameters $m$ and $p$. The p.m.f. of the process is given by

$$P[N(t) = n] = \int_0^\infty \sum_{l=n}^{\infty} \frac{(-1)^{l+n}[(1-q)(\lambda t)^\alpha]a^n_l}{\Gamma(\alpha l + 1)} f(\lambda_l|m,p)d\lambda_l$$

$$= \sum_{l=n}^{\infty} \frac{(-1)^{l+n}[(1-q)t^\alpha]a^n_l}{\Gamma(\alpha l + 1)}$$

$$\times \int_0^\infty \frac{m^p}{\Gamma(p)}e^{-m\lambda_l}\lambda_l^{p+\alpha l-1}d\lambda_l$$

$$= \sum_{l=n}^{\infty} \frac{(-1)^{l+n}[(1-q)t^\alpha]a^n_l}{\Gamma(\alpha l + 1)} \frac{\Gamma(p + \alpha l)}{m^{\alpha l}\Gamma(p)}.$$

This probability model provides an entirely new class of counting processes derived using $q$-Weibull inter arrival times and is an improvement over the traditional Poisson process.

### 8.8 Conclusion

In this chapter we have introduced two generalizations of the Poisson process namely, a new pure birth process and a new $q$-Weibull counting process. The pure birth process introduced here assumes different rate parameters. A new discrete distribution is also developed, parameters are estimated and applied to a real data set. As a biproduct, we have obtained many results useful in special function theory. Finally we developed a $q$-Weibull counting process which provides more flexibility in the relationship between mean and variance compared to Poisson processes.

### References


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