CHAPTER 4

EQUITABLE COLORING OF WHEEL GRAPH FAMILIES

In this chapter, the equitable chromatic number $\chi_e$ for the corona product of $G \circ W_n$, the sun let graph $S_n$, line graph of sun let graph $L(S_n)$, middle graph of sun let graph $M(S_n)$, total graph of sun let graph $T(S_n)$, the helm graph $H_n$, the line graph of helm graph $L(H_n)$, middle graph of helm graph $M(H_n)$, total graph of helm graph $T(H_n)$, the gear graph $G_n$, the line graph of gear graph $L(G_n)$, the middle graph of gear graph $M(G_n)$, the total graph of gear graph $T(G_n)$.

4.1 PRELIMINARIES

For any integer $n \geq 4$, the wheel graph $W_n$ is the $n$–vertex graph obtained by joining a vertex $v_1$ to each of the $n - 1$ vertices $\{w_1, w_2, \ldots, w_{n-1}\}$ of the cycle graph $C_{n-1}$.

For any integer $l \geq 2$, we define the graph $G \circ^l H$ recursively from $G \circ H$ as $G \circ^l H = (G \circ^{l-1} H) \circ H$. Graph $G \circ^l H$ is also named $l$-corona product of $G$ and $H$. Such type of graph product was introduced by Frucht and Harary in 1970 [17].

The $n$–sun let graph on $2n$ vertices is obtained by attaching $n$ pendant edges to the cycle $C_n$ and is denoted by $S_n$.

The Helm graph $H_n$ is the graph obtained from an $n$-wheel graph by adjoining a pendent edge at each node of the cycle.

The Gear graph $G_n$, also known as a bipartite wheel graph, is a wheel graph
with a graph vertex added between each pair of adjacent graph vertices of the outer cycle.

The line graph \([4, 16]\) of \(G\), denoted by \(L(G)\) is the graph with vertices are the edges of \(G\) with two vertices of \(L(G)\) adjacent whenever the corresponding edges of \(G\) are adjacent.

The middle graph \([14]\) of \(G\), is defined with the vertex set \(V(G) \cup E(G)\) where two vertices are adjacent if they are either adjacent edges of \(G\) or one is the vertex and the other is an edge incident with it and it is denoted by \(M(G)\).

Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). The total graph \([4, 14, 16]\) of \(G\), denoted by \(T(G)\) is defined in the following way. The vertex set of \(T(G)\) is \(V(G) \cup E(G)\). Two vertices \(x, y\) in the vertex set of \(T(G)\) are adjacent in \(T(G)\) in case one of the following holds: (i) \(x, y\) are in \(V(G)\) and \(x\) is adjacent to \(y\) in \(G\). (ii) \(x, y\) are in \(E(G)\) and \(x, y\) are adjacent in \(G\). (iii) \(x\) is in \(V(G)\), \(y\) is in \(E(G)\), and \(x, y\) are incident in \(G\).

## 4.2 EQUITABLE COLORING OF CORONA PRODUCT OF WHEEL GRAPH

Let us start with giving results for coronas of single vertex and wheels.

**Theorem 4.2.1.** Let \(n\) be a positive integer, \(n > 4\). Then

\[
\chi_e(K_1 \circ W_n) = \left\lceil \frac{n-1}{2} \right\rceil + 2.
\]

*Proof.* The color used for coloring the vertex of \(K_1\) and the color used for coloring vertex \(v_1\) cannot be used more times, so one can use any other color at most twice. Hence the value of equitable chromatic number is equal to \(\left\lceil \frac{n-1}{2} \right\rceil + 2\). \(\square\)

Let us notice that \(\Delta(K_1 \circ W_n) = n \geq \lceil (n-1)/2 \rceil + 2\) for \(n > 4\). This means that ECC holds for \(K_1 \circ W_n, n > 4\).
Next, let us consider coronas, where set of vertices of graph $G$ includes more than one element.

**Theorem 4.2.2.** Let $G$ be an equitably 4-colorable graph on $m$, $m \geq 2$, vertices and let $m$ is even, $n$ is odd, $n \geq 4$, then $\chi(G \circ W_n) = 4$.

**Proof.** Let $n_i(k)$ be the number of color $k$ appearance, $1 \leq k \leq 4$, in the $i$th copy of $W_n$ corresponding to vertex $u_i$ of $G$ in $G \circ W_n$, $i = 1, 2, \ldots, m$.

Let $f(u_i) = j$ be the color assigned to vertex $u_i$ ($1 \leq i \leq m$) of $G$. Since $G$ is 4-colorable $j$ takes the values in the range $1 \leq j \leq 4$.

Let us color the graph $G$ equitably with four colors. Let us order the vertices of $G$: $u_1, u_2, \ldots, u_m$ in such a way that vertex $v_i$ is colored with color $i \mod 4$ - let us use the color 4 instead of color 0 (in some cases recoloring is needed). Extending this coloring into the whole graph $G \circ W_n$ due to the following conditions. Let us consider the following two cases:

1. $m \mod 4 = 0$

   If $f(u_i) = j$, $u_i \in V(G)$, $1 \leq j \leq 4$, then
   
   
   
   \begin{align*}
   & \bullet \ n_i((j + 1) \mod 4 + 4 \cdot [(j + 1) \mod 4 = 0]) = 1, \\
   & \bullet \ n_i((j + 2) \mod 4 + 4 \cdot [(j + 2) \mod 4 = 0]) = \frac{n-1}{2} \text{ and} \\
   & \bullet \ n_i((j + 3) \mod 4 + 4 \cdot [(j + 3) \mod 4 = 0]) = \frac{n-1}{2}.
   \end{align*}

   In the above coloring, let us use each color exactly $(n + 1)m/4$ times. Graph $G \circ W_n$ is colored equitably.

2. $m \mod 4 = 2$

   Let us color the first $m - 2$ copies of $W_n$ in the same way as one have colored the corresponding vertices in Case (1). Let us color the last two copies in the following way. For each vertex $u_i, i = m - 1, m$, if $f(u_i) = j$, $1 \leq j \leq 2$, then the extended coloring must fulfill the following conditions.
• $n_i((j+2)) = 1$,
• $n_i((j+3) \mod 4 + 4 \cdot [(j+3) \mod 4 = 0]) = \frac{n-1}{2}$,
• $n_i(j+1) = \frac{n-1}{2}$.

Let us use each of the four colors exactly $(n+1)[m/4] + (n+1)/2$ times. Graph $G \circ W_n$ is colored equitably.

Hence $\chi_e(G \circ W_n) \leq 4$. By the definition of corona graph, graph $G \circ W_n$ contains $K_4$. Hence $\chi_e(G \circ W_n) = 4$.

**Theorem 4.2.3.** Let $G$ be an equitably 4-colorable graph on 5 vertices, then $\chi_e(G \circ W_5) = 4$.

**Proof.** Since $W_5$ has the cycle $C_4$, $\chi_e(W_4) \geq 3$. By the definition of corona, each vertex $u_i$ of $G$ is adjacent to every vertex of its copy of $W_n$. Hence $\chi_e(G \circ W_5) \geq 4$.

By assigning the colors 1, 2, 3 and 4 as given below, it is concluded that the 1 appears 7 times, 2 appears 8 times, 3 appears 8 times and 4 appears 7 times. (i.e) The difference between number of appearance of each pair of colors does not exceed one. Hence $\chi_e(G \circ W_5) \leq 4$. Hence $\chi_e(G \circ W_5) = 4$.

![Figure 4.1: An equitable 4-coloring of $K_{1,1,1,2} \circ W_5$ with $n(1) = n(4) = 7$ and $n(2) = n(3) = 8$.](image)
Now, let us consider the remaining cases of $m$ and $n$. It turns out that in these cases five colors are desirable for proper equitable coloring.

**Theorem 4.2.4.** Let $G$ be an equitably 5-colorable graph on $m$ vertices. If $m \mod 2 = 1$, $n \geq 7$ or $m, n$ even with $n \geq 4$ then $\chi_e(G \circ W_n) = 5$.

**Proof.** Let $n_i(k)$ be the number of color $k$ appearance, $1 \leq k \leq 5$, in the $i$th copy of $W_n$ corresponding to vertex $u_i$ of $G$ in $G \circ W_n$, $i = 1, 2, \ldots, m$.

Let $f(u_i) = j$ be the color assigned to vertex $u_i$ ($1 \leq i \leq m$) of $G$. Since $G$ is 5-colorable $j$ takes the values in the range $1 \leq j \leq 5$.

Let us color the graph $G$ equitably with five colors. Let us order the vertices of $G$: $u_1, u_2, \ldots, u_m$ in such a way that vertex $v_i$ is colored with color $i \mod 5$-let us use color 5 instead of color 0 (in some cases recoloring is needed). By extending this coloring into the whole graph $G \circ W_n$ due to the following conditions. Let us consider five cases dependently on the value of $m$.

1. $m \mod 5 = 0$

For each vertex $u_i \in V(G)$ if $f(u_i) = j$, $1 \leq j \leq 5$, then

- $n_i((j + 1) \mod 5 + 5 \cdot [(j + 1) \mod 5 = 0]) = 1$,
- $n_i((j + 2) \mod 5 + 5 \cdot [(j + 2) \mod 5 = 0]) = 1$,
- $n_i((j + 3) \mod 5 + 5 \cdot [(j + 3) \mod 5 = 0]) = \left\lceil \frac{n - 2}{2} \right\rceil$,
- $n_i((j + 4) \mod 5 + 5 \cdot [(j + 4) \mod 5 = 0]) = \left\lfloor \frac{n - 2}{2} \right\rfloor$.

Let us use each of the five colors exactly $(n + 1)m/5$ times. Graph $G \circ W_n$ is colored equitably.

2. $m \mod 5 = 1$

First, let us color $m - 6$ copies of $W_n$ in the same way as one have colored the corresponding vertices in Case (1). Let us color the last six copies in the
following way. For each vertex $u_i \ (m - 5 \leq i \leq m)$ by extending the coloring due to the following conditions, dependently on $n$.

(a) $n \text{ mod } 5 = 0$

- For vertex $u_{m-5} \ (f (u_{m-5}) = 1)$ one has $n_{m-5}(2) = 1$, $n_{m-5}(3) = n_{m-5}(4) = \frac{2n - 5}{5}$, $n_{m-5}(5) = \frac{n + 5}{5}$.
- For vertex $u_{m-4} \ (f (u_{m-4}) = 2)$ one has $n_{m-4}(3) = 1$, $n_{m-4}(1) = n_{m-4}(5) = \frac{2n - 5}{5}$, $n_{m-4}(4) = \frac{n + 5}{5}$.
- For vertex $u_{m-3} \ (f (u_{m-3}) = 3)$ one has $n_{m-3}(4) = 1$, $n_{m-3}(2) = \frac{2n}{5}$, $n_{m-3}(5) = \frac{2n - 10}{5}$, $n_{m-3}(1) = \frac{n + 5}{5}$.
- For vertex $u_{m-2} \ (f (u_{m-2}) = 4)$ one has $n_{m-2}(5) = 1$, $n_{m-2}(3) = \frac{2n}{5}$, $n_{m-2}(2) = \frac{2n - 5}{5}$, $n_{m-2}(1) = \frac{n}{5}$.
- For vertex $u_{m-1} \ (f (u_{m-1}) = 5)$ one has $n_{m-1}(3) = 1$, $n_{m-1}(1) = n_{m-1}(4) = \frac{2n - 5}{5}$, $n_{m-1}(2) = \frac{n + 5}{5}$.
- For vertex $u_m \ (f (u_m) = 1)$ one has $n_{m}(2) = 1$, $n_{m}(3) = n_{m}(4) = \frac{2n - 5}{5}$, $n_{m}(5) = \frac{n + 5}{5}$.

Each of colors 1, 2, 3 and 5 is used $(6n + 5)/5$ times and color 4 is used $(6n + 5)/5 + 1$ times.

(b) $n \text{ mod } 5 = 1$ or $n \text{ mod } 5 = 4$

- For vertex $u_{m-5} \ (f (u_{m-5}) = 1)$ one has $n_{m-5}(2) = 1$, $n_{m-5}(3) = n_{m-5}(4) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_{m-5}(5) = \left\lceil \frac{n - 1}{5} \right\rceil$.
- For vertex $u_{m-4} \ (f (u_{m-4}) = 2)$ one has $n_{m-4}(3) = 1$, $n_{m-4}(1) = n_{m-4}(5) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_{m-4}(4) = \left\lceil \frac{n - 1}{5} \right\rceil$.
- For vertex $u_{m-3} \ (f (u_{m-3}) = 3)$ one has $n_{m-3}(4) = 1$, $n_{m-3}(2) = n_{m-3}(5) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_{m-3}(1) = \left\lceil \frac{n - 1}{5} \right\rceil$.
- For vertex $u_{m-2} \ (f (u_{m-2}) = 4)$ one has $n_{m-2}(5) = 1$, $n_{m-2}(2) = n_{m-2}(3) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_{m-2}(1) = \left\lceil \frac{n - 1}{5} \right\rceil$. 
• For vertex $u_{m-1}$ ($f(u_{m-1}) = 5$) one has $n_{m-1}(3) = 1, n_{m-1}(1) =$ 
$n_{m-1}(2) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_{m-1}(4) = \left\lfloor \frac{n-1}{5} \right\rfloor$.

• For vertex $u_m$ ($f(u_m) = 1$) one has $n_m(2) = 1, n_m(3) = n_m(4) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_m(5) = \left\lfloor \frac{n-1}{5} \right\rfloor$.

Each of colors 1, 4 and 5 is used $2 + 2\left\lfloor \frac{2n}{5} \right\rfloor + 2\left\lfloor (n - 1)/5 \right\rfloor$ times and

colors 2 and 3 are used, each one, $3 + 3\left\lfloor 2n/5 \right\rfloor$ times. For $n$ mod $5 = 1$
or $n$ mod $5 = 4$, the difference does not exceed one.

(c) $n$ mod $5 = 2$

• For vertex $u_{m-5}$ ($f(u_{m-5}) = 1$) one has $n_{m-5}(2) = 1, n_{m-5}(3) =$
$n_{m-5}(4) = \frac{2n + 1}{5}, n_{m-5}(5) = \frac{n - 2}{5}$.

• For vertex $u_{m-4}$ ($f(u_{m-4}) = 2$) one has $n_{m-4}(3) = 1, n_{m-4}(1) =$
$n_{m-4}(5) = \frac{2n - 4}{5}, n_{m-4}(4) = \frac{n - 2}{5}$.

• For vertex $u_{m-3}$ ($f(u_{m-3}) = 3$) one has $n_{m-3}(4) = 1, n_{m-3}(2) =$
$n_{m-3}(5) = \frac{2n + 1}{5}, n_{m-3}(1) = \frac{n - 2}{5}$.

• For vertex $u_{m-2}$ ($f(u_{m-2}) = 4$) one has $n_{m-2}(5) = 1, n_{m-2}(2) =$
$n_{m-2}(3) = \frac{2n - 9}{5}, n_{m-2}(1) = \frac{n + 3}{5}$.

• For vertex $u_{m-1}$ ($f(u_{m-1}) = 5$) one has $n_{m-1}(3) = 1, n_{m-1}(1) =$
$n_{m-1}(2) = \frac{2n - 9}{5}, n_{m-1}(4) = \frac{n + 3}{5}$.

• For vertex $u_m$ ($f(u_m) = 1$) one has $n_m(2) = 1, n_m(3) = \frac{2n + 1}{5}$,
$n_m(4) = \frac{2n - 4}{5}, n_m(5) = \frac{n - 2}{5}$.

Each of colors 1, 2 and 3 is used $(6n + 3)/5 + 1$ times and colors 4 and 5

are used, each one, $(6n + 3)/5$ times.

(d) $n$ mod $5 = 3$

• For vertex $u_{m-5}$ ($f(u_{m-5}) = 1$) one has $n_{m-5}(2) = 1, n_{m-5}(3) =$
$n_{m-5}(4) = \frac{2n - 1}{5}, n_{m-5}(5) = \frac{n - 3}{5}$.

• For vertex $u_{m-4}$ ($f(u_{m-4}) = 2$) one has $n_{m-4}(3) = 1, n_{m-4}(1) =$
$n_{m-4}(5) = \frac{2n - 1}{5}, n_{m-4}(4) = \frac{n - 3}{5}$.
• For vertex $u_{m-3}$ ($f(u_{m-3}) = 3$) one has $n_{m-3}(4) = 1$, $n_{m-3}(2) = n_{m-3}(5) = \frac{2n-1}{5}$, $n_{m-3}(1) = \frac{n-3}{5}$.

• For vertex $u_{m-2}$ ($f(u_{m-2}) = 4$) one has $n_{m-2}(5) = 1$, $n_{m-2}(2) = \frac{2n-1}{5}$, $n_{m-2}(3) = \frac{2n-6}{5}$, $n_{m-2}(1) = \frac{n+2}{5}$.

• For vertex $u_{m-1}$ ($f(u_{m-1}) = 5$) one has $n_{m-1}(3) = 1$, $n_{m-1}(1) = \frac{2n-1}{5}$, $n_{m-1}(2) = \frac{2n-6}{5}$, $n_{m-1}(4) = \frac{n+2}{5}$.

• For vertex $u_{m}$ ($f(u_{m}) = 1$) one has $n_{m}(2) = 1$, $n_{m}(3) = n_{m}(4) = \frac{2n-1}{5}$, $n_{m}(5) = \frac{n-3}{5}$.

Each of colors 1, 2, 3 and 4 is used $(6n+2)/5 + 1$ times and color 5 is used $(6n+2)/5$ times.

In all the above cases the difference between the cardinalities of color classes does not exceed one, so our coloring is equitable.

3. $m \mod 5 = 2$

Let us color the first $m - 2$ copies of $W_n$ in the same way as one have colored the corresponding vertices in Case (1). Let us color the last two copies (for $u_{m-1}$ and $u_{m}$) in the following way. Let us consider the five cases dependently on $n$.

(a) $n \mod 5 = 0$

• If $f(u_i) = 1$, $n_i(3) = 1$, $n_i(2) = \frac{2n-5}{5}$, $n_i(4) = \frac{2n-5}{5}$, $n_i(5) = \frac{n+5}{5}$.

• If $f(u_i) = 2$, $n_i(4) = 1$, $n_i(1) = n_i(3) = \frac{2n}{5}$, $n_i(5) = \frac{n-5}{5}$.

(b) $n \mod 5 = 1$ or $n \mod 5 = 4$

• If $f(u_i) = 1$, $n_i(3) = 1$, $n_i(2) = n_i(4) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_i(5) = \left\lceil \frac{n-1}{5} \right\rceil$.

• If $f(u_i) = 2$, $n_i(4) = 1$, $n_i(1) = n_i(3) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_i(5) = \left\lceil \frac{n-1}{5} \right\rceil$.

(c) $n \mod 5 = 2$
• If $f(u_i) = 1$, $n_i(3) = 1$, $n_i(2) = n_i(4) = \frac{2n - 4}{5}$, $n_i(5) = \frac{n + 3}{5}$.

• If $f(u_i) = 2$, $n_i(4) = 1$, $n_i(1) = n_i(3) = \frac{2n - 4}{5}$, $n_i(5) = \frac{n + 3}{5}$.

(d) $n \mod 5 = 3$

• If $f(u_i) = 1$, $n_i(3) = 1$, $n_i(2) = \frac{2n - 6}{5}$, $n_i(4) = \frac{2n - 1}{5}$, $n_i(5) = \frac{n + 2}{5}$.

• If $f(u_i) = 2$, $n_i(4) = 1$, $n_i(1) = n_i(3) = \frac{2n - 1}{5}$, $n_i(5) = \frac{n - 3}{5}$.

In all the above cases the difference between the cardinalities of color classes does not exceed one, so our coloring is equitable.

4. $m \mod 5 = 3$

Let us color the first $m - 8$ copies of $W_n$ in the same way as one have colored the corresponding vertices in Case (1). For each vertex $u_i$ ($m - 7 \leq i \leq m$) by extending the coloring due to following conditions, dependently on $n$.

(a) $n \mod 5 = 0$ or $n \mod 5 = 3$

• For vertex $u_{m-7}$ ($f(u_{m-7}) = 1$) one has $n_{m-7}(2) = 1$, $n_{m-7}(3) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_{m-7}(4) = \left\lceil \frac{2n}{5} \right\rceil - 1$, $n_{m-7}(5) = \left\lceil \frac{n}{5} \right\rceil$.

• For vertex $u_{m-6}$ ($f(u_{m-6}) = 2$) one has $n_{m-6}(1) = 1$, $n_{m-6}(3) = \left\lceil \frac{2n}{5} \right\rceil - 1$, $n_{m-6}(4) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_{m-6}(5) = \left\lceil \frac{n}{5} \right\rceil$.

• For vertex $u_{m-5}$ ($f(u_{m-5}) = 3$) one has $n_{m-5}(4) = 1$, $n_{m-5}(1) = \left\lceil \frac{2n}{5} \right\rceil$, $n_{m-5}(2) = \left\lfloor \frac{2n}{5} \right\rfloor - 1$, $n_{m-5}(5) = \left\lceil \frac{n}{5} \right\rceil$.

• For vertex $u_{m-4}$ ($f(u_{m-4}) = 4$) one has $n_{m-4}(3) = 1$, $n_{m-4}(1) = \left\lceil \frac{2n}{5} \right\rceil - 1$, $n_{m-4}(2) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_{m-4}(5) = \left\lceil \frac{n}{5} \right\rceil$.

• For vertex $u_{m-3}$ ($f(u_{m-3}) = 5$) one has $n_{m-3}(1) = 1$, $n_{m-3}(2) = \left\lceil \frac{2n}{5} \right\rceil$, $n_{m-3}(3) = \left\lfloor \frac{2n}{5} \right\rfloor - 1$, $n_{m-3}(4) = \left\lceil \frac{n}{5} \right\rceil$.

• For vertex $u_{m-2}$ ($f(u_{m-2}) = 1$) one has $n_{m-2}(2) = 1$, $n_{m-2}(3) = \left\lceil \frac{2n}{5} \right\rceil$, $n_{m-2}(5) = \left\lfloor \frac{2n}{5} \right\rfloor - 1$, $n_{m-2}(4) = \left\lceil \frac{n}{5} \right\rceil$.
• For vertex $u_{m-1}$ ($f(u_{m-1}) = 2$) one has $n_{m-1}(3) = 1$, $n_{m-1}(1) = \left\lfloor \frac{2n}{5} \right\rfloor - 1$, $n_{m-1}(5) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_{m-1}(4) = \left\lceil \frac{n}{5} \right\rceil$.

• For vertex $u_m$ ($f(u_m) = 3$) one has $n_m(5) = 1$, $n_m(1) = \left\lfloor \frac{2n}{5} \right\rfloor$, $n_m(2) = \left\lceil \frac{2n}{5} \right\rceil - 1$, $n_m(4) = \left\lceil \frac{n}{5} \right\rceil$.

Each of colors 1, 2 and 3 is used $2 + 4 \lfloor 2n/5 \rfloor$ times and colors 4 and 5 are used, each one, $2 \lfloor 2n/5 \rfloor + 4 \lceil n/5 \rceil + 1$ times. For $n \mod 5 = 0$ or $n \mod 5 = 3$, the difference does not exceed one.

(b) $n \mod 5 = 1$

• For vertex $u_{m-7}$ ($f(u_{m-7}) = 1$) one has $n_{m-7}(2) = 1$, $n_{m-7}(3) = \frac{2n - 2}{5}$, $n_{m-7}(5) = \frac{n - 1}{5}$.

• For vertex $u_{m-6}$ ($f(u_{m-6}) = 2$) one has $n_{m-6}(1) = 1$, $n_{m-6}(3) = \frac{2n - 2}{5}$, $n_{m-6}(4) = \frac{2n - 7}{5}$, $n_{m-6}(5) = \frac{n + 4}{5}$.

• For vertex $u_{m-5}$ ($f(u_{m-5}) = 3$) one has $n_{m-5}(4) = 1$, $n_{m-5}(1) = \frac{2n - 2}{5}$, $n_{m-5}(5) = \frac{n - 1}{5}$.

• For vertex $u_{m-4}$ ($f(u_{m-4}) = 4$) one has $n_{m-4}(3) = 1$, $n_{m-4}(1) = \frac{2n - 2}{5}$, $n_{m-4}(2) = \frac{2n - 7}{5}$, $n_{m-4}(5) = \frac{n + 4}{5}$.

• For vertex $u_{m-3}$ ($f(u_{m-3}) = 5$) one has $n_{m-3}(1) = 1$, $n_{m-3}(2) = \frac{2n - 2}{5}$, $n_{m-3}(4) = \frac{n - 1}{5}$.

• For vertex $u_{m-2}$ ($f(u_{m-2}) = 1$) one has $n_{m-2}(2) = 1$, $n_{m-2}(3) = \frac{2n - 2}{5}$, $n_{m-2}(5) = \frac{2n - 7}{5}$, $n_{m-2}(4) = \frac{n + 4}{5}$.

• For vertex $u_{m-1}$ ($f(u_{m-1}) = 2$) one has $n_{m-1}(3) = 1$, $n_{m-1}(1) = \frac{2n - 2}{5}$, $n_{m-1}(4) = \frac{n - 1}{5}$.

• For vertex $u_m$ ($f(u_m) = 3$) one has $n_m(5) = 1$, $n_m(1) = \frac{2n - 7}{5}$, $n_m(2) = \frac{2n - 2}{5}$, $n_m(4) = \frac{n + 4}{5}$.

Each of colors 1, 2, 4 and 5 is used $(8n + 7)/5$ times and color 3 is used $(8n + 7)/5 + 1$ times.
(c) \( n \mod 5 = 2 \)

- For vertex \( u_{m-7} \) (\( f(u_{m-7}) = 1 \)) one has \( n_{m-7}(2) = 1, n_{m-7}(3) = \frac{2n+1}{5}, n_{m-7}(4) = \frac{2n-4}{5}, n_{m-7}(5) = \frac{n-2}{5} \).

- For vertex \( u_{m-6} \) (\( f(u_{m-6}) = 2 \)) one has \( n_{m-6}(1) = 1, n_{m-6}(3) = n_{m-6}(4) = \frac{2n-4}{5}, n_{m-6}(5) = \frac{n+3}{5} \).

- For vertex \( u_{m-5} \) (\( f(u_{m-5}) = 3 \)) one has \( n_{m-5}(4) = 1, n_{m-5}(1) = \frac{2n+1}{5}, n_{m-5}(2) = \frac{2n-4}{5}, n_{m-5}(5) = \frac{n-2}{5} \).

- For vertex \( u_{m-4} \) (\( f(u_{m-4}) = 4 \)) one has \( n_{m-4}(3) = 1, n_{m-4}(1) = n_{m-4}(2) = \frac{2n-4}{5}, n_{m-4}(5) = \frac{n+3}{5} \).

- For vertex \( u_{m-3} \) (\( f(u_{m-3}) = 5 \)) one has \( n_{m-3}(1) = 1, n_{m-3}(2) = \frac{2n+1}{5}, n_{m-3}(3) = \frac{2n-4}{5}, n_{m-3}(4) = \frac{n-2}{5} \).

- For vertex \( u_{m-2} \) (\( f(u_{m-2}) = 1 \)) one has \( n_{m-2}(2) = 1, n_{m-2}(3) = n_{m-2}(5) = \frac{2n-4}{5}, n_{m-2}(4) = \frac{n+3}{5} \).

- For vertex \( u_{m-1} \) (\( f(u_{m-1}) = 2 \)) one has \( n_{m-1}(3) = 1, n_{m-1}(1) = \frac{2n-4}{5}, n_{m-1}(5) = \frac{2n+1}{5}, n_{m-1}(4) = \frac{n-2}{5} \).

- For vertex \( u_{m} \) (\( f(u_{m}) = 3 \)) one has \( n_{m}(5) = 1, n_{m}(1) = n_{m}(2) = \frac{2n-4}{5}, n_{m}(4) = \frac{n+3}{5} \).

Each of colors 1,2,3 and 5 is used \((8n+4)/5 + 1\) times and color 4 is used \((8n+4)/5\) times.

(d) \( n \mod 5 = 4 \)

- For vertex \( u_{m-7} \) (\( f(u_{m-7}) = 1 \)) one has \( n_{m-7}(2) = 1, n_{m-7}(3) = n_{m-7}(4) = \frac{2n-3}{5}, n_{m-7}(5) = \frac{n+1}{5} \).

- For vertex \( u_{m-6} \) (\( f(u_{m-6}) = 2 \)) one has \( n_{m-6}(1) = 1, n_{m-6}(3) = n_{m-6}(4) = \frac{2n-3}{5}, n_{m-6}(5) = \frac{n+1}{5} \).

- For vertex \( u_{m-5} \) (\( f(u_{m-5}) = 3 \)) one has \( n_{m-5}(4) = 1, n_{m-5}(1) = n_{m-5}(2) = \frac{2n-3}{5}, n_{m-5}(5) = \frac{n+1}{5} \).
• For vertex \( u_{m-4} \) \( f(u_{m-4}) = 4 \) one has \( n_{m-4}(3) = 1, n_{m-4}(1) = \frac{2n-3}{5}, n_{m-4}(5) = \frac{n+1}{5} \).
• For vertex \( u_{m-3} \) \( f(u_{m-3}) = 5 \) one has \( n_{m-3}(1) = 1, n_{m-3}(2) = \frac{2n-3}{5}, n_{i}(4) = \frac{n+1}{5} \).
• For vertex \( u_{m-2} \) \( f(u_{m-2}) = 1 \) one has \( n_{m-2}(2) = 1, n_{m-2}(3) = \frac{2n-3}{5}, n_{m-2}(4) = \frac{n+1}{5} \).
• For vertex \( u_{m-1} \) \( f(u_{m-1}) = 2 \) one has \( n_{m-1}(3) = 1, n_{m-1}(1) = \frac{2n-3}{5}, n_{m-1}(4) = \frac{n+1}{5} \).
• For vertex \( u_{m} \) \( f(u_{m}) = 3 \) one has \( n_{m}(5) = 1, n_{m}(1) = n_{m}(2) = \frac{2n-3}{5}, n_{m}(4) = \frac{n+1}{5} \).

Each of colors is used \((8n+8)/5\) times.

In all the above cases the difference between the cardinalities of color classes does not exceed one, so our coloring is equitable.

5. \( m \mod 5 = 4 \)

Let us color the first \( m-4 \) copies of \( W_n \) in the same way as one have colored the corresponding vertices in Case (1). Then, let us color the last four copies in the following way. For each vertex \( u_i \), \( (m-3 \leq i \leq m) \), let us color the corresponding copy of \( W_n \) due the following conditions, dependently on \( n \).

(a) \( n \mod 5 = 0 \)

If \( f(u_i) = j, 1 \leq j \leq 4 \), then

• \( n_i((j+1) \mod 4 + 4 \cdot [(j+1) \mod 4 = 0]) = 1 \),
• \( n_i((j+2) \mod 4 + 4 \cdot [(j+2) \mod 4 = 0]) = \frac{2n}{5}, \)
• \( n_i((j+3) \mod 4 + 4 \cdot [(j+3) \mod 4 = 0]) = \frac{2n-5}{5}, \)
• \( n_i(5) = \frac{n}{5} \).
(b) \( n \mod 5 = 1 \)

For vertex \( u_{m-3} (f(u_{m-3}) = 1) \) one has \( n_{m-3}(2) = 1, n_{m-3}(3) = \frac{2n-2}{5}, n_{m-3}(4) = \frac{2n-7}{5}, n_{m-3}(5) = \frac{n+4}{5}. \)

For vertices \( u_i, m - 2 \leq i \leq m, \) if \( f(u_i) = j, 1 \leq j \leq 4, \) then

- \( n_i((j + 1) \mod 4 + 4 \cdot [j + 1 \mod 4 = 0]) = 1, \)
- \( n_i((j + 2) \mod 4 + 4 \cdot [j + 2 \mod 4 = 0]) = n_i((j + 3) \mod 4 + 4 \cdot [j + 3 \mod 4 = 0]) = \frac{2n - 2}{5}, \)
- \( n_i(5) = \frac{n - 1}{5}. \)

(c) \( n \mod 5 = 2 \)

For vertices \( u_i, m - 3 \leq i \leq m - 2, \) if \( f(u_i) = j, 1 \leq j \leq 2, \) then

- \( n_i((j + 1) \mod 4 + 4 \cdot [j + 1 \mod 4 = 0]) = 1, \)
- \( n_i((j + 2) \mod 4 + 4 \cdot [j + 2 \mod 4 = 0]) = n_i((j + 3) \mod 4 + 4 \cdot [j + 3 \mod 4 = 0]) = \frac{2n - 4}{5}, \)
- \( n_i(5) = \frac{n + 3}{5}. \)

For vertices \( u_i, m - 1 \leq i \leq m, \) if \( f(u_i) = j, 3 \leq j \leq 4, \) then

- \( n_i((j + 1) \mod 4 + 4 \cdot [j + 1 \mod 4 = 0]) = 1, \)
- \( n_i((j + 2) \mod 4 + 4 \cdot [j + 2 \mod 4 = 0]) = \frac{2n + 1}{5}, \)
- \( n_i((j + 3) \mod 4 + 4 \cdot [j + 3 \mod 4 = 0]) = \frac{2n - 4}{5}, \)
- \( n_i(5) = \frac{n - 2}{5}. \)

(d) \( n \mod 5 = 3 \)

If \( f(u_i) = j, 1 \leq j \leq 4, \) then

- \( n_i((j + 1) \mod 4 + 4 \cdot [j + 1 \mod 4 = 0]) = 1, \)
- \( n_i((j + 2) \mod 4 + 4 \cdot [j + 2 \mod 4 = 0]) = \frac{2n - 1}{5}, \)
- \( n_i((j + 3) \mod 4 + 4 \cdot [j + 3 \mod 4 = 0]) = \frac{2n - 6}{5}, \)
- \( n_i(5) = \frac{n + 2}{5}. \)
\((e)\) \(n \mod 5 = 4\)

If \(f(u_i) = j, 1 \leq j \leq 4\), then

- \(n_i((j + 1) \mod 4 + 4 \cdot [j + 1 \mod 4 = 0]) = 1\),
- \(n_i((j + 2) \mod 4 + 4 \cdot [j + 2 \mod 4 = 0]) = n_i((j + 3) \mod 4 + 4 \cdot [j + 3 \mod 4 = 0]) = \frac{2n - 3}{5}\),
- \(n_i(5) = \frac{n + 1}{5}\).

In all the above cases the difference between the cardinalities of color classes does not exceed one, so our coloring is equitable. Hence \(\chi = (G \circ W_n) \leq 5\). By the definition of corona graph for each vertex \(u_i\) of \(G\), there exists a copy of \(W_n\) whose vertices are adjacent to the vertex \(u_i\).

**Case 1:** If \(m \mod 2 = 1, n \geq 7\)

In this case either both \(m\) and \(n\) are odd (or) \(m\) is odd and \(n\) is even.

(a) If \(m\) and \(n\) are odd.

Since \(\chi(W_n) = 3\) for odd \(n\), one need at least 4 colors for coloring each copy of \(W_n\) and the corresponding vertex of \(G\). In this coloring, since \(m\) is odd there exists atleast one color which reappears in \(\langle \{u_i : 1 \leq i \leq m\} \rangle\). Let the color \(j (1 \leq j \leq 4)\) reappears at the vertex \(u_i (5 \leq i \leq m)\). Then the center vertex of the copy \(W_n\) corresponding to the vertex \(u_i\), receives a color \(k (1 \leq k \leq 4)\), where \(k \neq j\). Other vertices of \(W_n\) receive the colors other than \(j\) and \(k\). (i.e) The number of possible colors to color these vertices is two. Hence it is clear that for the case of \(n \geq 5\), it is not possible to color the vertices of the cycle \(C_{n-1}\) of \(W_n\) equitably with two colors. Therefore \(\chi = (G \circ W_n) \geq 5\). Hence \(\chi = (G \circ W_n) = 5\) for \(m\) and \(n\) are odd.

(b) If \(m\) is odd and \(n\) is even.

Since \(\chi(W_n) = 4\) for even \(n\), the graph \(G \circ W_n\) requires at least 5 colors. Hence \(\chi = (G \circ W_n) = 5\) for \(m\) is odd and \(n\) is even.
Case 2: If $m$ and $n$ are even, $n \geq 4$

Since $\chi(W_n) = 4$ for even $n$, graph $G \circ W_n$ requires at least 5 colors.
Therefore $\chi=(G \circ W_n) \geq 5$. Hence $\chi=(G \circ W_n) = 5$ for even $n$.

\[\square\]

**Theorem 4.2.5.** Let $G$ be an equitably 3-colorable graph with $m = 3$ vertices. Then

1. $\chi=(G \circ W_3) = 4$.

2. $\chi=(G \circ W_n) = 5 \quad n = 7, 9, 11, 13, 15, 17$.

3. $\chi=(G \circ W_n) = 5 \quad n \geq 19, \text{ if } n \text{ is odd}$.

4. $\chi=(G \circ W_n) = 5 \quad n = 4, 6, 8, 10$.

5. $\chi=(G \circ W_n) = 6 \quad n \geq 12, \text{ if } n \text{ is even}$.

**Proof.** Let $\{u_i : 1 \leq i \leq 3\}$ be the set of vertices of $G$.

1. Let us color $G \circ W_3$ as for the following procedure.

   - For vertex $u_1$ ($f(u_1) = 1$) one has $n_1(2) = 1$, $n_1(3) = n_1(4) = 2$.
   - For vertex $u_2$ ($f(u_2) = 2$) one has $n_2(3) = 1$, $n_2(4) = n_2(1) = 2$.
   - For vertex $u_3$ ($f(u_3) = 3$) one has $n_3(4) = 1$, $n_3(1) = n_3(2) = 2$.

   In the above cases the difference between the cardinalities of color classes does not exceed one, so our coloring is equitable. Hence $\chi=(G \circ W_3) \leq 4$. Since $W_3$ is 3-colorable, at each copy of $W_3$ of $G \circ W_3$, there exists one more color. Therefore $\chi=(G \circ W_3) \geq 4$. hence $\chi=(G \circ W_3) = 4$.

2. Assign the color $i$ to the vertex $u_i$ ($1 \leq i \leq 3$), color 4 to the vertex $u_{1n}$, color 5 to the vertex $u_{2n}$ and color 1 to the vertex $u_{3n}$. Since $C_{n-1}$ is of even order, one require only two colors for proper coloring of $C_{n-1}$. Let us use three colors in each $C_{n-1}$ of $W_n$ in $G \circ W_n$. One can use the colors 2,3,5 to the vertices
of $C_{n-1}$ of $W_n$ at $u_1$. Similarly one can use the colors 1,3,4 and 4,5,2 to the
vertices of $C_{n-1}$ of $W_n$ at $u_2$ and $u_3$ respectively. The number of appearance
of the colors are given in the following cases.

(a) $n = 7, 17$
- For vertex $u_1 (f (u_1 = 1))$ one has $n_1 (4) = 1$, $n_1 (2) = \frac{2n + 1}{5}$,
  $n_1 (3) = \frac{2n - 4}{5}$, $n_1 (5) = \frac{n - 2}{5}$.
- For vertex $u_2 (f (u_2 = 2))$ one has $n_2 (5) = 1$, $n_2 (1) = \frac{n - 1}{2}$, $n_2 (3) = \left\lceil \frac{n - 1}{4} \right\rceil$, $n_2 (4) = \left\lfloor \frac{n - 1}{4} \right\rfloor$.
- For vertex $u_3 (f (u_3 = 3))$ one has $n_3 (1) = 1$, $n_3 (4) = \frac{2n - 4}{5}$,
  $n_3 (5) = \frac{2n + 1}{5}$, $n_3 (2) = \frac{n - 2}{5}$.

(b) $n = 9$
- For vertex $u_1 (f (u_1 = 1))$ one has $n_1 (4) = 1$, $n_1 (2) = n_1 (3) = 3$,
  $n_1 (5) = 2$.
- For vertex $u_2 (f (u_2 = 2))$ one has $n_2 (5) = 1$, $n_2 (1) = 4$, $n_2 (3) = n_2 (4) = 2$.
- For vertex $u_3 (f (u_3 = 3))$ one has $n_3 (1) = 1$, $n_3 (4) = n_3 (5) = 3$,
  $n_3 (2) = 2$.

(c) $n = 11$
- For vertex $u_1 (f (u_1 = 1))$ one has $n_1 (4) = 1$, $n_1 (2) = n_1 (3) = 4$,
  $n_1 (5) = 2$.
- For vertex $u_2 (f (u_2 = 2))$ one has $n_2 (5) = 1$, $n_2 (1) = 5$, $n_2 (3) = 3$,
  $n_2 (4) = 2$.
- For vertex $u_3 (f (u_3 = 3))$ one has $n_3 (1) = 1$, $n_3 (4) = n_3 (5) = 4$,
  $n_3 (2) = 2$.

(d) $n = 13$
• For vertex $u_1 (f (u_1) = 1)$ one has $n_1 (4) = 1$, $n_1 (2) = n_1 (3) = 5$, $n_1 (5) = 2$.
• For vertex $u_2 (f (u_2) = 2)$ one has $n_2 (5) = 1$, $n_2 (1) = 6$, $n_2 (3) = 3$, $n_2 (4) = 3$.
• For vertex $u_3 (f (u_3) = 3)$ one has $n_3 (1) = 1$, $n_3 (4) = n_3 (5) = 5$, $n_3 (2) = 2$.

(e) $n = 15$

• For vertex $u_1 (f (u_1) = 1)$ one has $n_1 (4) = 1$, $n_1 (2) = n_1 (3) = 6$, $n_1 (5) = 2$.
• For vertex $u_2 (f (u_2) = 2)$ one has $n_2 (5) = 1$, $n_2 (1) = 7$, $n_2 (3) = 3$, $n_2 (4) = 4$.
• For vertex $u_3 (f (u_3) = 3)$ one has $n_3 (1) = 1$, $n_3 (4) = 4$, $n_3 (5) = 7$, $n_3 (2) = 2$.

In the above cases the difference between the cardinalities of color classes does not exceed one, so our coloring is equitable. Hence $\chi_c (G \circ W_n) \leq 5$.

Since $G$ is 3-colorable, let $i$ be the color assigned to the vertex $u_i \ (1 \leq i \leq 3)$ of $G \circ W_n$. Let $j \ (1 \leq j \leq 4), \ (i \neq j)$ be the color assigned to the center vertices of each copy $W_n$ of $G \circ W_n$. The other vertices of these copies receive the colors other than $i$ and $j$. (i.e) The number of possible colors to color these vertices is two. Hence it is clear that for the case of $n = 7, 9, 11, 13, 15, 17$, it is not possible to color the vertices of the cycle $C_{n-1}$ of $W_n$ equitably with two colors. Therefore $\chi_c (G \circ W_n) \geq 5$. Hence $\chi_c (G \circ W_n) = 5$ for $n = 7, 9, 11, 13, 15, 17$.

3. Suppose that $G \circ W_n$ is 4—equitably colorable. Since $G$ is 3—colorable, let it be colored by the color 1,2 and 3. Let $u_i$ receives the color $i \ (1 \leq i \leq 3)$. Then $u_{1n}, u_{2n}$ and $u_{3n}$ should receive any two of the three color 1,2,3 and the color 4.
Let $u_{1n}$ receive 4, $u_{2n}$ receive 1 and $u_{2n}$ receive 2. Then $u_{1i} (1 \leq i \leq n - 1)$ receives the color 2, $\frac{n - 1}{2}$ times and 3, $\frac{n - 1}{2}$ times. $u_{2i} (1 \leq i \leq n - 1)$ receives the color 3, $\frac{n - 1}{2}$ times, the color 4, $\frac{n - 1}{2}$ times. Similarly $u_{3i}$ receives the color 1, $\frac{n - 1}{2}$ times and the color 1, $\frac{n - 1}{2}$ times.

Number of appearance of color 1 = $\frac{n + 3}{2}$.
Number of appearance of color 2 = $\frac{n + 3}{2}$.
Number of appearance of color 3 = $n$.
Number of appearance of color 4 = $n$.

As the above mentioned partition does not imply the equitable partition, it is concluded that $G \circ W_n$ should not be equitable 4-colorable.

Hence $\chi = (G \circ W_n) \geq 5$

Suppose that $G \circ W_n$ is 5-equitable colorable. Let $G$ be colored by the colors 1, 2 and 3. Let $u_i$ receives the color $i$ ($1 \leq i \leq 3$). Since $G \circ W_n$ is 5-equitable colorable, any two of the vertices $u_{1n}, u_{2n}$ and $u_{3n}$ receives the color 4 and 5 (Say $u_{1n}, u_{2n}$) and remaining vertex $u_{3n}$ should receive the color 1.

For the case of $n \geq 19$, if we use the above coloring with 5 colors, then the maximum of appearance of color 1 = $\frac{n - 1}{2} + 2 = \frac{n + 3}{2}$.

Remaining number of vertices to be colored = $3n + 3 - \frac{n + 3}{2} = \frac{5n + 3}{2}$.

Number of vertices which receive each colors of 2, 3, 4 and 5 = $\frac{5n + 3}{4} = \frac{5n + 3}{8}$.

For $n \geq 19$, $\left[ \frac{5n + 3}{2} \right] - \left[ \frac{n + 3}{2} \right] \geq 2$.

(i.e) it may not be possible to equitably color $G \circ W_n$ with 5 colors.

$\chi = (G \circ W_n) \geq 6$.

- For vertex $u_1 (f(u_1) = 1)$ one has $n_1 (4) = 1, n_1 (2) = n_1 (3) = \frac{n - 1}{2}$.
- For vertex $u_2 (f(u_2) = 2)$ one has $n_2 (5) = 1, n_2 (1) = n_2 (6) = \frac{n - 1}{2}$.
- For vertex $u_3 (f(u_3) = 3)$ one has $n_3 (6) = 1, n_3 (5) = n_3 (4) = \frac{n - 1}{2}$.
In the above cases the difference between the cardinalities of color classes does not exceed one, so our coloring is equitable. Hence $\chi = (G \circ W_n) = 6$, $n \geq 19$, if $n$ is odd.

4. Since $n$ is even $W_n$ has odd cycle $C_{n−1}$. Minimum number of colors assigned to color any cycle is 3. Hence $u_{in}$ $(1 \leq i \leq n)$ should have a fourth color and hence $u_i$ $(1 \leq i \leq n)$ must receive a fifth color. Hence $\chi = (G \circ W_n) \geq 5$.

Now let us partition the vertex set $V(G \circ W_n)$ as follows,

$$V_1 = \{u_1, u_{21}, u_{25}, u_{28}, u_{3n}\}$$
$$V_2 = \{u_2, u_{11}, u_{14}, u_{33}, u_{36}, u_{39}\}$$
$$V_3 = \{u_3, u_{12}, u_{15}, u_{24}, u_{27}\}$$
$$V_4 = \{u_{1n}, u_{22}, u_{26}, u_{29}, u_{32}, u_{35}, u_{37}\}$$
$$V_5 = \{u_{2n}, u_{13}, u_{16}, u_{19}, u_{31}, u_{34}, u_{38}\}$$

Clearly $V_1, V_2, V_3, V_4$ and $V_5$ are independent set of $G \circ W_n$. Hence $||V_i|−|V_j|| \leq 1$ for every $i \neq j$. Hence $\chi = (G \circ W_n) = 5$, $4 \leq n \leq 10$, if $n$ is even.

5. Let $n_i(k)$ be the number of appearance of the color $k$ in the copy of $W_n$ corresponding to the vertex $u_i$ of $G$ in $G \circ W_n$.

Let $f(u_i) = j$ be the color assigned to each vertices $u_i (1 \leq i \leq m)$ of $G$. Since $G$ is 6-colorable $j$ takes the values in the range $1 \leq j \leq 6$.

- For vertex $u_1$ ($f(u_1) = 1$) one has $n_1(2) = n_1(5) = 1$, $n_1(3) = n_1(4) = \frac{n - 2}{2}$.
- For vertex $u_2$ ($f(u_2) = 2$) one has $n_2(3) = n_2(1) = 1$, $n_2(5) = n_2(6) = \frac{n - 2}{2}$.
- For vertex $u_3$ ($f(u_3) = 3$) one has $n_3(6) = n_3(4) = 1$, $n_3(1) = n_3(2) = \frac{n - 2}{2}$. 
In the above cases the difference between the cardinalities of color classes does not exceed one, so our coloring is equitable. Hence $\chi(G \circ W_n) \leq 6$

Since $n$ is even, one requires at least 3 colors to color each $C_{n-1}$ of $W_n$, one color for the centre vertex of $W_n$ and one color corresponding to the vertex of $G$. Hence one may assume that $\chi(G \circ W_n) = 5$. It is clear that one of these five colors appears twice in $\langle \{u_i : 1 \leq i \leq 3\} \cup \{u_{in} : 1 \leq i \leq 3\} \rangle$, let it be color $j (1 \leq j \leq 5)$. This color $j$ can be assigned only $\frac{n-2}{2}$ times in any of the $C_{n-1}$ copy of $W_{n-1}$. This violate the equitable conclusion. Therefore $\chi(G \circ W_n) \geq 6$. Hence $\chi(G \circ W_n) = 6$. \hfill $\Box$

Let us notice that our results can be extended into further products of graphs.

**Corollary 4.2.6.** Let $G$ be an equitably 4-colorable graph on $m$, $m \geq 2$, vertices, let $m$ is even, $n$ is odd, $n \geq 4$, and $l \geq 1$. Then

$$\chi(G \circ^l W_n) = 4.$$  

**Proof.** Let us use the principle of mathematical induction due to number $l$.

1. $l=1$
   
The truth follows immediately from Theorem 4.2.2.

2. Induction hypothesis for $l$. It means that $\chi(G \circ^l W_n) = 4$ for $n$ odd and $m = |V(G)|$ even.

3. One must show that $\chi(G \circ^{l+1} W_n) = 4$ for graphs under consideration.
   
   Let us notice that graph from induction hypothesis $G \circ^l W_n$ is an equitably 4-colorable graph, it means a graph fulfilling the assumption of Theorem 4.2.2. Its number of vertices, equals to $m(n+1)^l$ is an even number for $m$ even. So, $\chi(G \circ^{l+1} W_n) = 4$. \hfill $\Box$
Corollary 4.2.7. Let $G$ be an equitably 5-colorable graph on $m$ vertices and let $m \geq 2$, $n \geq 4$, $l \geq 1$. Then

$$\chi_e(G \circ^l W_n) = \begin{cases} 
5 & \text{for } n \text{ even}, \\
\leq 5 & \text{for } m \text{ and } n \text{ odd}.
\end{cases}$$

Proof. Follows immediately from Theorem 4.2.4.

4.3 EQUITABLE COLORING OF SUN LET GRAPH AND IT’S LINE, MIDDLE AND TOTAL GRAPH

Theorem 4.3.1. If $n \geq 2$ the equitable chromatic number of sun let graph $S_n$,

$$\chi_e(S_n) = \begin{cases} 
2 & \text{if } n \text{ is even} \\
3 & \text{if } n \text{ is odd}.
\end{cases}$$

Proof. Let $S_n$ be the sun let graph on $2n$ vertices. Let $V(S_n) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ where $v_i$’s are the vertices of cycles taken in cyclic order and $u_i$’s are pendant vertices such that each $v_iu_i$ is a pendant edge.

Figure 4.2: Sunlet Graph $S_n$. 
Case 1: If $n$ is even.

Now, partition the vertex set $V(S_n)$ as $V_1 = \{v_1, v_3, \ldots, v_{n-1}\} \cup \{u_2, u_4, \ldots, u_n\}$; $V_2 = \{v_2, v_4, \ldots, v_n\} \cup \{u_1, u_3, \ldots, u_{n-1}\}$. Clearly $V_1$ and $V_2$ are independent sets of $V(S_n)$. Also $|V_1| = |V_2| = n$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair $(i, j)$. $\chi_=(S_n) \leq 2$. Since $\chi(S_n) \geq 2$, $\chi_=(S_n) \geq \chi(S_n) \geq 2$, $\chi_=(S_n) \geq 2$. Therefore $\chi_=(S_n) = 2$.

Case 2: If $n$ is odd.

1. If $n = 6k - 3$ for some positive integer $k$, then set the partition of $V$ as below. $V_1 = \{v_{3i-2} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-1} : 1 \leq i \leq 2k - 1\}$; $V_2 = \{v_{3i-1} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i} : 1 \leq i \leq 2k - 1\}$; $V_3 = \{v_{3i} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-2} : 1 \leq i \leq 2k - 1\}$. Clearly $V_1, V_2, V_3$ are independent sets of $V(S_n)$. Also $|V_1| = |V_2| = |V_3| = 4k - 2$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair $(i, j)$.

2. If $n = 6k - 1$ for some positive integer $k$, then set the partition of $V$ as below. $V_1 = \{v_{3i-2} : 1 \leq i \leq 2k\} \cup \{u_{3i-1} : 1 \leq i \leq 2k\}$; $V_2 = \{v_{3i-1} : 1 \leq i \leq 2k\} \cup \{u_{3i} : 1 \leq i \leq 2k - 1\}$; $V_3 = \{v_{3i} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-2} : 1 \leq i \leq 2k\}$. Clearly $V_1, V_2, V_3$ are independent sets of $V(S_n)$. Also $|V_1| = 4k$ and $|V_2| = |V_3| = 4k - 1$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair $(i, j)$.

3. If $n = 6k + 1$ for some positive integer $k$, then set the partition of $V$ as below. $V_1 = \{v_{3i-2} : 1 \leq i \leq 2k\} \cup \{u_{3i-1} : 1 \leq i \leq 2k\}$; $V_2 = \{v_{3i-1} : 1 \leq i \leq 2k\} \cup \{v_{6k+1}\} \cup \{u_{3i} : 1 \leq i \leq 2k\}$; $V_3 = \{v_{3i} : 1 \leq i \leq 2k\} \cup \{u_{3i-2} : 1 \leq i \leq 2k + 1\}$. Clearly $V_1, V_2, V_3$ are independent sets of $V(S_n)$. $|V_1| = 4k$ and $|V_2| = |V_3| = 4k + 1$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair $(i, j)$.

From Case 2, $\chi_=(S_n) \leq 3$. Since $\chi(S_n) \geq 3$, $\chi_=(S_n) \geq \chi(S_n) \geq 3$, $\chi_=(S_n) \geq 3$. Therefore $\chi_=(S_n) = 3$.  

□
Theorem 4.3.2. If $n \geq 3$ the equitable chromatic number on line graph of sun let graph $L(S_n)$, $\chi=(L(S_n)) = 3$.

Proof. Let $V(S_n) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ and $E(S_n)= \{e_i: 1 \leq i \leq n\} \cup \{e_i: 1 \leq i \leq n-1\} \cup \{e_n\}$ where $e_i$ is the edge $v_i v_{i+1}$ ($1 \leq i \leq n-1$), $e_n$ is the edge $v_n v_1$ and $e'_i$ is the edge $v_i u_i$ ($1 \leq i \leq n$).

By the definition of line graph $V(L(S_n)) = E(S_n) = \{u'_i: 1 \leq i \leq n\} \cup \{v'_i: 1 \leq i \leq n-1\} \cup \{v'_n\}$ where $v'_i$ and $u'_i$ represents the edge $e_i$ and $e'_i$ ($1 \leq i \leq n$) respectively.

Case 1: If $n$ is odd.

1. If $n = 6k - 3$ for some positive integer $k$, then set the partition of $V$ as below. $V_1 = \{v'_{3i-2}: 1 \leq i \leq 2k - 1\} \cup \{u'_{3i}: 1 \leq i \leq 2k - 1\};
\quad V_2 = \{v'_{3i-1}: 1 \leq i \leq 2k - 1\} \cup \{u'_{3i-2}: 1 \leq i \leq 2k - 1\};
\quad V_3 = \{v'_{3i}: 1 \leq i \leq 2k - 1\} \cup \{u'_{3i-1}: 1 \leq i \leq 2k - 1\}$. Clearly $V_1, V_2, V_3$ are independent.
Case 2: If \( n \) is even.

1. If \( n = 6k - 2 \) for some positive integer \( k \), then set the partition of \( V \) as below. \( V_1 = \{v'_{3i-1} : 1 \leq i \leq 2k - 1\} \cup \{u'_{3i-2} : 1 \leq i \leq 2k\} \cup \{u'_{6k-2}\} \); \( V_2 = \{v'_{3i-2} : 1 \leq i \leq 2k\} \cup \{u'_{6k}\} \cup \{u'_{6k-3}\} \cup \{u'_{3i} : 1 \leq i \leq 2k - 2\} \); \( V_3 = \{v'_{3i} : 1 \leq i \leq 2k - 2\} \cup \{v'_{6k-2}\} \cup \{u'_{6k-3}\} \). Clearly \( V_1, V_2, V_3 \) are independent sets of \( V(L(S_n)) \). Also \( |V_1| = 4k - 2 \) and \( |V_1| = |V_3| = 4k - 1 \), it holds the inequality \( ||V_i| - |V_j|| \leq 1 \) for every pair \((i, j)\).

2. If \( n = 6k \) for some positive integer \( k \), then set the partition of \( V \) as below. \( V_1 = \{v'_{3i-2} : 1 \leq i \leq 2k\} \cup \{u'_{3i} : 1 \leq i \leq 2k\} \); \( V_2 = \{v'_{3i-1} : 1 \leq i \leq 2k\} \cup \{u'_{3i-2} : 1 \leq i \leq 2k\} \cup \{u'_{6k-2}\} \); \( V_3 = \{v'_{3i} : 1 \leq i \leq 2k\} \cup \{u'_{3i-1} : 1 \leq i \leq 2k\} \). Clearly \( V_1, V_2, V_3 \) are independent sets of \( V(L(S_n)) \). Also \( |V_1| = |V_2| = |V_3| = 4k \), it holds the inequality \( ||V_i| - |V_j|| \leq 1 \) for every pair \((i, j)\).
3. If $n = 6k + 2$ for some positive integer $k$, then set the partition of $V$ as below. $V_1 = \{v'_{3i-1} : 1 \leq i \leq 2k\} \cup \{u'_{3i-2} : 1 \leq i \leq 2k + 1\} \cup \{u'_{6k+2}\};$ $V_2 = \{v'_{3i-2} : 1 \leq i \leq 2k + 1\} \cup \{v'_{6k+1}\} \cup \{u'_{3i-1} : 1 \leq i \leq 2k\};$ $V_3 = \{v'_{3i} : 1 \leq i \leq 2k\} \cup \{v'_{6k+2}\} \cup \{u'_{3i-1} : 1 \leq i \leq 2k\}$. Clearly $V_1$, $V_2$, $V_3$ are independent sets of $V(L(S_n))$. Also $|V_1| = 4k + 2$ and $|V_2| = |V_3| = 4k + 1$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair $(i, j)$.

From Case 1 and Case 2, $\chi_e(L(S_n)) \leq 3$. Since $L(S_n)$ contains a clique of order 3, $\chi(L(S_n)) \geq 3$, $\chi_e(L(S_n)) \geq \chi(L(S_n)) \geq 3$, $\chi_e(L(S_n)) \geq 3$. Therefore $\chi_e(L(S_n)) = 3$.

**Theorem 4.3.3.** If $n \geq 3$ the equitable chromatic number on middle graph of sun let graph $M(S_n)$, $\chi_e(M(S_n)) = 4$.

**Proof.** Let $V(S_n) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ and $E(S_n) = \{e_i : 1 \leq i \leq n\}$ $\cup \{e_i : 1 \leq i \leq n - 1\} \cup \{e_n\}$ where $e_i$ is the edge $v_i v_{i+1}$ ($1 \leq i \leq n - 1$), $e_n$ is the edge $v_n v_1$ and $e'_i$ is the edge $v_i u_i$ ($1 \leq i \leq n$). By the definition of middle graph $V(M(S_n)) = V(S_n) \cup E(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\} \cup \{u'_i : 1 \leq i \leq n\}$ where $v'_i$ and $u'_i$ represents the edge $e_i$ and $e'_i$ ($1 \leq i \leq n$) respectively.

Case 1: If $n$ is even.

Now, partition the vertex set $V(M(S_n))$ as $V_1 = \{v_i : 1 \leq i \leq n\}; V_2 = \{v'_{2i-1} : 1 \leq i \leq \frac{n}{2}\} \cup \{u_{2i-1} : 1 \leq i \leq \frac{n}{2}\}$; $V_3 = \{v'_{2i} : 1 \leq i \leq \frac{n}{2}\} \cup \{u_{2i} : 1 \leq i \leq \frac{n}{2}\}$; $V_4 = \{u'_i : 1 \leq i \leq n\}$. Clearly $V_1$, $V_2$, $V_3$ and $V_4$ are independent sets of $M(S_n)$. Also $|V_1| = |V_2| = |V_3| = |V_4| = n$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair $(i, j)$. $\chi_e(M(S_n)) \leq 4$. Since $M(S_n)$ contains a clique of order 4, $\chi_e(M(S_n)) \geq 4$, $\chi_e(M(S_n)) \geq \chi(M(S_n)) \geq 4$, $\chi_e(M(S_n)) \geq 4$. Therefore $\chi_e(M(S_n)) = 4$.

Case 2: If $n$ is odd.
Figure 4.4: Middle graph of Sunlet Graph $M(S_n)$.

1. If $n = 6k - 3$ for some positive integer $k$, then set the partition of $V$ as below. $V_1 = \{v_{3i-2} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-2} : 1 \leq i \leq 2k - 1\} \cup \{v'_{3i-1} : 1 \leq i \leq 2k - 1\}; V_2 = \{v_{3i-1} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-1} : 1 \leq i \leq 2k - 1\} \cup \{v'_{3i} : 1 \leq i \leq 2k - 1\}; V_3 = \{v_{3i} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i} : 1 \leq i \leq 2k - 1\} \cup \{v'_{3i-2} : 1 \leq i \leq 2k - 1\}; V_4 = \{u'_i : 1 \leq i \leq 6k - 3\}$. Clearly $V_1, V_2, V_3$ and $V_4$ are independent sets of $M(S_n)$. Also $|V_1| = |V_2| = |V_3| = |V_4| = 6k - 3$.

2. If $n = 6k - 1$ for some positive integer $k$, then set the partition of $V$ as below. $V_1 = \{v_{3i} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-2} : 1 \leq i \leq 2k\} \cup \{v'_{3i-2} : 1 \leq i \leq 2k\}; V_2 = \{v_{3i-1} : 1 \leq i \leq 2k-1\} \cup \{u_{3i-1} : 1 \leq i \leq 2k\} \cup \{v'_{3i-1} : 1 \leq i \leq 2k\}; V_3 = \{v_{3i} : 1 \leq i \leq 2k\} \cup \{v_1\} \cup \{u_{3i} : 1 \leq i \leq 2k - 1\} \cup \{v'_{3i} : 1 \leq i \leq 2k - 1\}; V_4 = \{u'_i : 1 \leq i \leq 6k - 1\}$. Clearly $V_1, V_2, V_3$ and $V_4$ are independent sets of $M(S_n)$. Also $|V_1| = |V_2| = |V_3| = |V_4| = 6k - 1$.

3. If $n = 6k + 1$ for some positive integer $k$, then set the partition of $V$ as below. $V_1 = \{v_{3i} : 1 \leq i \leq 2k\} \cup \{u_{3i} : 1 \leq i \leq 2k\} \cup \{v_{6k+1}\} \cup \{u_{3i} : 1 \leq i \leq 2k\}; V_2 = \{v_{3i+1} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-2} : 1 \leq i \leq 2k + 1\} \cup \{v'_{3i-1} : 1 \leq i \leq 2k\}$; $V_3 = \{v_{3i} : 1 \leq i \leq 2k\} \cup \{v_{3i+1} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i} : 1 \leq i \leq 2k\} \cup \{v'_{3i-2} : 1 \leq i \leq 2k + 1\} \cup \{v'_{3i-1} : 1 \leq i \leq 2k\}$; $V_4 = \{v'_{3i} : 1 \leq i \leq 2k\} \cup \{v_{3i+1} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i} : 1 \leq i \leq 2k\} \cup \{v'_{3i-2} : 1 \leq i \leq 2k + 1\} \cup \{v'_{3i-1} : 1 \leq i \leq 2k\}$. Clearly $V_1, V_2, V_3$ and $V_4$ are independent sets of $M(S_n)$. Also $|V_1| = |V_2| = |V_3| = |V_4| = 6k + 1$. 


1 \leq i \leq 2k \cup \{v_{6k+1}'\}; V_3 = \{v_{3i-1} : 1 \leq i \leq 2k\} \cup \{u_{3i-1} : 1 \leq i \leq 2k\} \cup \{v_1\} \cup \{v_{3i}' : 1 \leq i \leq 2k\}; V_4 = \{u_i' : 1 \leq i \leq 6k+1\}. Clearly V_1, V_2, V_3 and V_4 are independent sets of \(M(S_n)\). Also \(\mid V_1 \mid = \mid V_2 \mid = \mid V_3 \mid = \mid V_4 \mid = 6k + 1\).

From Case 2, \(V\) can be partitioned into four independent sets satisfying the relation \(\mid \mid V_i \mid - \mid V_j \mid \leq 1\) for every pair \((i, j)\). \(\chi=(M(S_n)) \leq 4\). Since \(M(S_n)\) contains a clique of order 4, \(\chi(M(S_n)) \geq 4\), \(\chi=(M(S_n)) \geq \chi(M(S_n)) \geq 4\). Therefore \(\chi=(M(S_n)) = 4\).

**Theorem 4.3.4.** If \(n \geq 3\) the equitable chromatic number on total graph of sun let graph \(T(S_n)\), \(\chi=(T(S_n)) = 4\).

**Proof.** Let \(V(S_n) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}\) and \(E(S_n) = \{e_i' : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n-1\} \cup \{e_n\}\) where \(e_i\) is the edge \(v_iv_{i+1}\) \((1 \leq i \leq n-1)\), \(e_n\) is the edge \(v_nv_1\) and \(e_i'\) is the edge \(v_iu_i\) \((1 \leq i \leq n)\). By the definition of total graph \(V(T(S_n)) = V(S_n) \cup E(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{v_i' : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{u_i' : 1 \leq i \leq n\}\) where \(v_i'\) and \(u_i'\) represents the edge \(e_i\) and \(e_i'\) \((1 \leq i \leq n)\) respectively.

![Figure 4.5: Total graph of Sunlet Graph \(T(S_n)\).](image)
Case 1: If $n$ is even.

Now, partition the vertex set $V(T(S_n))$ as $V_1 = \{v_{2i-1} : 1 \leq i \leq \frac{n}{2}\} \cup \{u_{2i} : 1 \leq i \leq \frac{n}{2}\}$; $V_2 = \{v_{2i} : 1 \leq i \leq \frac{n}{2}\} \cup \{u'_{2i-1} : 1 \leq i \leq \frac{n}{2}\}$; $V_3 = \{u_{2i-1} : 1 \leq i \leq \frac{n}{2}\} \cup \{v'_{2i-1} : 1 \leq i \leq \frac{n}{2}\}$; $V_4 = \{u_{2i} : 1 \leq i \leq \frac{n}{2}\} \cup \{v'_{2i} : 1 \leq i \leq \frac{n}{2}\}$. Clearly $V_1, V_2, V_3$ and $V_4$ are the independent sets of $T(S_n)$. Also $|V_1| = |V_2| = |V_3| = |V_4| = n$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair $(i, j)$. Since $T(S_n)$ contains a clique of order 4, $\chi(T(S_n)) \geq 4$, $\chi(V(T(S_n))) \leq 4$. Therefore $\chi(V(T(S_n))) = 4$.

Case 2: If $n$ is odd.

1. If $n = 6k - 3$ for some positive integer $k$, then set the partition of $V$ as below. $V_1 = \{v_{3i-2} : 1 \leq i \leq 2k - 1\} \cup \{v'_{3i-2} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-1} : 1 \leq i \leq 2k - 1\}$; $V_2 = \{v_{3i} : 1 \leq i \leq 2k - 1\} \cup \{v'_{3i-2} : 1 \leq i \leq 2k - 1\}$; $V_3 = \{v_{3i} : 1 \leq i \leq 2k - 1\} \cup \{v'_{3i} : 1 \leq i \leq 2k - 1\}$; $V_4 = \{u'_i : 1 \leq i \leq 6k - 3\}$. Clearly $V_1, V_2, V_3$ and $V_4$ are independent sets of $T(S_n)$. Also $|V_1| = |V_2| = |V_3| = |V_4| = 6k - 3$.

2. If $n = 6k - 1$ for some positive integer $k$, then set the partition of $V$ as below. $V_1 = \{v_{3i-1} : 1 \leq i \leq 2k - 1\} \cup \{v'_{3i} : 1 \leq i \leq 2k - 1\} \cup \{v'_{6k-1}\} \cup \{u_{3i-2} : 1 \leq i \leq 2k\}$; $V_2 = \{v_{3i} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-1} : 1 \leq i \leq 2k\}$; $V_3 = \{v_{3i} : 1 \leq i \leq 2k - 1\} \cup \{v'_{3i-1} : 1 \leq i \leq 2k - 1\}$; $V_4 = \{u'_i : 1 \leq i \leq 6k - 2\} \cup \{v_{6k-1}\}$. Clearly $V_1, V_2, V_3$ and $V_4$ are independent sets of $T(S_n)$. Also $|V_1| = |V_2| = |V_3| = |V_4| = 6k - 1$.

3. If $n = 6k + 1$ for some positive integer $k$, then set the partition of $V$ as below. $V_1 = \{v_{4i-3} : 1 \leq i \leq 2k\} \cup \{u_{4i} : 1 \leq i \leq 2k - 1\} \cup \{u'_{4i-1} : 1 \leq i \leq 2k\}$; $V_2 = \{v_{4i-3} : 1 \leq i \leq 2k\} \cup \{u_{4i-2} : 1 \leq i \leq 2k\}$; $V_3 = \{v_{4i-2} : 1 \leq i \leq 2k\}$; $V_4 = \{v_{4i-1} : 1 \leq i \leq 2k\} \cup \{v'_{4i-1} : 1 \leq i \leq 2k\} \cup \{u'_{4i+1} : 1 \leq i \leq 2k - 1\}$;
\[ V_1 = \{v_{4i-1} : 1 \leq i \leq 2k\} \cup \{u_{4i} : 1 \leq i \leq 2k - 1\} \cup \{v'_{4i} : 1 \leq i \leq 2k - 1\} \cup \{u'_{4i} : 1 \leq i \leq 2k\} \cup \{u'_4\}. \]

Clearly \( V_1, V_2, V_3, \) and \( V_4 \) are independent sets of \( T(S_n) \). Also \( |V_1| = |V_2| = |V_3| = |V_4| = 8k - 1 \).

From Case 2, \( V \) can be partitioned into four independent set satisfying the relation \( ||V_i| - |V_j|| \leq 1 \) for every pair \((i, j)\). \( \chi(T(S_n)) \leq 4 \). Since \( T(S_n) \) contains a clique of order 4, \( \chi(T(S_n)) \geq 4 \), \( \chi(T(S_n)) \geq \chi(T(S_n)) \geq 4 \). Therefore \( \chi(T(S_n)) = 4 \).

4.4 EQUIitable COLORING OF HELM GRAPH AND IT’S LINE, MIDDLE AND TOTAL GRAPH

Theorem 4.4.1. If \( n \geq 4 \) the equitable chromatic number of Helm graph \( H_n \),

\[
\chi_e(H_n) = \begin{cases} 
3 & \text{if } n \text{ is even} \\
4 & \text{if } n \text{ is odd}
\end{cases}
\]

Proof. Let \( H_n \) be the Helm graph obtained by attaching a pendant edge at each vertex of the cycle. Let \( V(H_n) = \{v\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\} \) where \( v_i \)'s are the vertices of cycles taken in cyclic order and \( u_i \)'s are pendant vertices such that each \( v_i u_i \) is a pendant edge and \( v \) is a hub of the cycle.

Case 1: If \( n \) is even.

1. If \( n = 6k - 2 \) for some positive integer \( k \), then set the partition of \( V \) as below.

\[ V_1 = \{v\} \cup \{u_i : 2k+1 \leq i \leq 6k-2\}; V_2 = \{v_{2i-1} : 1 \leq i \leq 3k-1\} \cup \{u_{2i} : 1 \leq i \leq k\}; V_3 = \{v_{2i} : 1 \leq i \leq 3k-1\} \cup \{u_{2i-1} : 1 \leq i \leq k\}. \]

Clearly \( V_1, V_2, V_3 \) are independent sets of \( V(H_n) \). Also \( |V_1| = |V_2| = |V_3| = 4k - 1 \); it holds the inequality \( ||V_i| - |V_j|| \leq 1 \) for every pair \((i, j)\).
Figure 4.6: Helm Graph $H_n$

2. If $n = 6k$ for some positive integer $k$, then set the partition of $V$ as below.
   \[ V_1 = \{ v \} \cup \{ u_i : 2k + 2 \leq i \leq 6k \}; \]
   \[ V_2 = \{ v_{2i-1} : 1 \leq i \leq 3k \} \cup \{ u_{2i} : 1 \leq i \leq k \}; \]
   \[ V_3 = \{ v_{2i} : 1 \leq i \leq 3k \} \cup \{ u_{2i-1} : 1 \leq i \leq k + 1 \}. \]
   Clearly $V_1$, $V_2$, $V_3$ are independent sets of $V(H_n)$. Also $|V_1| = |V_2| = 4k$ and $|V_3| = 4k + 1$, it holds the inequality $|\|V_i| - |V_j|| \leq 1$ for every pair $(i, j)$.

3. If $n = 6k + 2$ for some positive integer $k$, then set the partition of $V$ as below.
   \[ V_1 = \{ v \} \cup \{ u_i : 2k+3 \leq i \leq 6k+2 \}; \]
   \[ V_2 = \{ v_{2i-1} : 1 \leq i \leq 3k+1 \} \cup \{ u_{2i} : 1 \leq i \leq k+1 \}; \]
   \[ V_3 = \{ v_{2i} : 1 \leq i \leq 3k+1 \} \cup \{ u_{2i-1} : 1 \leq i \leq k+1 \}. \]
   Clearly $V_1$, $V_2$, $V_3$ are independent sets of $V(H_n)$. $|V_1| = 4k + 1$ and $|V_2| = |V_3| = 4k + 2$, it holds the inequality $|\|V_i| - |V_j|| \leq 1$ for every pair $(i, j)$.

From case 1, $\chi_e(H_n) \leq 3$. Since $\chi(H_n) \geq 3$, $\chi_e(H_n) \geq \chi(H_n) \geq 3$, $\chi_e(H_n) \geq 3$. Therefore $\chi_e(H_n) = 3$. 
Case 2: If $n$ is odd.

1. If $n = 6k - 3$ for some positive integer $k$, then set the partition of $V$ as below.

$$V_1 = \{v\} \cup \{u_i : 3k + 1 \leq i \leq 5k - 1\}; V_2 = \{v_{3i-2} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i} : 1 \leq i \leq k\}; V_3 = \{v_{3i-1} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i} : 1 \leq i \leq k\}; V_4 = \{v_{3i} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-1} : 1 \leq i \leq k\}.$$ 

Clearly $V_1$, $V_2$, $V_3$ are independent sets of $V(H_n)$. Also $|V_1| = 3k - 2$ and $|V_2| = |V_3| = |V_4| = 3k - 1$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair $(i, j)$.

2. If $n = 6k - 1$ for some positive integer $k$, then set the partition of $V$ as below.

$$V_1 = \{v\} \cup \{u_i : 3k + 2 \leq i \leq 6k - 1\}; V_2 = \{v_{3i-2} : 1 \leq i \leq 2k\} \cup \{u_{3i-1} : 1 \leq i \leq 2k - 1\}; V_3 = \{v_{3i-1} : 1 \leq i \leq 2k\} \cup \{u_{3i} : 1 \leq i \leq 2k - 1\}; V_4 = \{v_{3i} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-2} : 1 \leq i \leq 2k\}.$$ 

Clearly $V_1$, $V_2$, $V_3$ are independent sets of $V(H_n)$. Also $|V_1| = 4k - 2$ and $|V_2| = |V_3| = |V_4| = 4k - 1$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair $(i, j)$.

3. If $n = 6k + 1$ for some positive integer $k$, then set the partition of $V$ as below.

$$V_1 = \{v\} \cup \{u_i : 3k + 2 \leq i \leq 6k + 1\}; V_2 = \{v_{3i-2} : 1 \leq i \leq 2k\} \cup \{u_{3i-1} : 1 \leq i \leq 2k - 1\}; V_3 = \{v_{3i-1} : 1 \leq i \leq 2k\} \cup \{v_n\} \cup \{u_{3i} : 1 \leq i \leq 2k - 1\}; V_4 = \{v_{3i} : 1 \leq i \leq 2k\} \cup \{u_1\} \cup \{u_{3i+1} : 1 \leq i \leq 2k - 1\}.$$ 

Clearly $V_1$, $V_2$, $V_3$ are independent sets of $V(H_n)$. $|V_1| = |V_3| = |V_4| = 4k$ and $|V_2| = 4k - 1$, it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair $(i, j)$.

From case 2, $\chi_e(H_n) \leq 4$. Since $\chi(H_n) \geq 4$, $\chi_e(H_n) \geq \chi(H_n) \geq 4$, $\chi_e(H_n) \geq 4$. Therefore $\chi_e(H_n) = 4$. 

**Theorem 4.4.2.** If $n \geq 4$ the equitable chromatic number on line graph of Helm graph $L(H_n)$, $\chi_e(L(H_n)) = n$.

**Proof.** Let $V(H_n) = \{v\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ and $E(H_n) = \{e_i :
1 ≤ i ≤ n} ∪ \{e'_i : 1 ≤ i ≤ n-1\} ∪ \{e'_n\} ∪ \{s_i : 1 ≤ i ≤ n\} where \(e_i\) is the edge \(vv_i\) (1 ≤ i ≤ n), \(e'_i\) is the edge \(v_i v_{i+1}\) (1 ≤ i ≤ n − 1), \(e'_n\) is the edge \(v_nv_1\) and \(s_i\) is the edge \(v_i u_i\) (1 ≤ i ≤ n). By the definition of line graph \(V(L(H_n)) = E(H_n) = \{e_i : 1 ≤ i ≤ n\} ∪ \{e'_i : 1 ≤ i ≤ n\} ∪ \{s_i : 1 ≤ i ≤ n\}\).

Figure 4.7: Line graph of Helm Graph \(L(H_n)\).

Now, set the partition of the vertex set of \(V(L(H_n))\) as below.

\[V_1 = \{e_1, e'_2, s_n\}; V_i = \{e_i, e'_{i+1}, s_{i-1} : 2 ≤ i ≤ n-1\}; V_n = \{e_n, e'_1, s_{n-1}\}.\]

Clearly \(V_1, V_i, V_n\) (2 ≤ i ≤ n − 1) are independent sets of \(L(H_n)\). Also \(|V_1| = |V_i| = |V_n| = 3\) (2 ≤ i ≤ n − 1), it holds the inequality \(||V_i| - |V_j|| ≤ 1\) for every pair \((i, j)\).

\(\chi = \chi(L(H_n)) \leq n\). Since \(e_i(1 ≤ i ≤ n)\) forms a clique of order \(n\), \(\chi(L(H_n)) ≥ n\), \(\chi(L(H_n)) ≥ \chi(L(H_n)) \geq n\), \(\chi(L(H_n)) ≥ n\). Therefore \(\chi = \chi(L(H_n)) = n\).

**Theorem 4.4.3.** If \(n ≥ 5\) the equitable chromatic number on middle graph of Helm graph \(M(H_n)\), \(\chi = \chi(M(H_n)) = n + 1\).

**Proof.** Let \(V(H_n) = \{v\} ∪ \{v_1, v_2, \ldots, v_n\} ∪ \{u_1, u_2, \ldots, u_n\}\) and \(E(H_n) = \{e_i : 1 ≤ i ≤ n\} ∪ \{e'_i : 1 ≤ i ≤ n-1\} ∪ \{e'_n\} ∪ \{s_i : 1 ≤ i ≤ n\}\) where \(e_i\) is the edge \(vv_i\) (1 ≤ i ≤ n), \(e'_i\) is the edge \(v_i v_{i+1}\) (1 ≤ i ≤ n − 1), \(e'_n\) is the edge \(v_nv_1\) and \(s_i\) is the edge \(v_i u_i\) (1 ≤ i ≤ n). By the definition of line graph \(V(L(H_n)) = E(H_n) = \{e_i : 1 ≤ i ≤ n\} ∪ \{e'_i : 1 ≤ i ≤ n\} ∪ \{s_i : 1 ≤ i ≤ n\}\).
and $s_i$ is the edge $v_iu_i$ ($1 \leq i \leq n$). By the definition of middle graph $V(M(H_n)) = V(H_n) \cup E(H_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

Figure 4.8: Middle graph of Helm Graph $M(H_n)$.

Now, partition the vertex set of $V(M(H_n))$ as below.

$V_1 = \{e_1, e'_2, u_1, s_n\}$; $V_i = \{v_{i-1}, u_i, e_i, e'_{i+1} : 2 \leq i \leq n-1\} \cup \{s_{i-2} : 3 \leq i \leq n+1\}$; $V_n = \{v_{n-1}, s_{n-2}, e_n, e'_1\}$; $V_{n+1} = \{v, v_n, s_{n-1}, u_n\}$. Clearly $V_1, V_2, \ldots, V_n, V_{n+1}$ are independent sets of $M(H_n)$. Also $|V_1| = |V_2| = |V_n| = |V_{n+1}| = 4$ and $|V_i| = 5$ ($3 \leq i \leq n-1$), it holds the inequality $||V_i| - |V_j|| \leq 1$ for every pair $(i, j)$. $\chi_e(M(H_n)) \leq n + 1$. Since $ve_i(1 \leq i \leq n)$ forms a clique of order $n + 1$, $\chi(M(H_n)) \geq n + 1$, $\chi_e(M(H_n)) \geq \chi(M(H_n)) \geq n + 1$, $\chi_e(M(H_n)) \geq n + 1$. Therefore $\chi_e(M(H_n)) = n + 1$.

Theorem 4.4.4. If $n \geq 5$ the equitable chromatic number on total graph of Helm graph $T(H_n)$, $\chi_e(T(H_n)) = n + 1$.

Proof. Let $V(H_n) = \{v\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ and $E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n-1\} \cup \{e''_n\} \cup \{s_i : 1 \leq i \leq n\}$ where $e_i$ is the
edge $vv_i (1 \leq i \leq n)$, $e_i'$ is the edge $v_iv_{i+1} (1 \leq i \leq n-1)$, $e_n'$ is the edge $v_nv_1$ and $s_i$ is the edge $v_iu_i (1 \leq i \leq n)$. By the definition of total graph $V(T(H_n)) = V(H_n) \cup E(H_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e_i' : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$.

![Figure 4.9: Total graph of Helm Graph $T(H_n)$](image)

Now, partition the vertex set of $V(T(H_n))$ as below.

$V_1 = \{e_1, e_2', u_3, v_n\}; V_2 = \{e_2, v_2, e_3', s_n, u_4\}; V_i = \{e_i, v_{i-1}, e_{i+1}', s_{i-2}, u_{i+2} : 3 \leq i \leq n-2\}; V_{n-1} = \{e_{n-1}, v_{n-2}, e_n', s_{n-3}\}; V_n = \{e_n, v_{n-1}, e_1', s_{n-2}\}; V_{n+1} = \{v, s_{n-1}, u_1, u_2\}.$

Clearly $V_1, V_2, V_i, V_{n-1}, V_n, V_{n+1} (3 \leq i \leq n-2)$ are independent sets of $T(H_n)$. Also $|V_1| = |V_n| = |V_{n+1}| = 4$ and $|V_2| = |V_i| = 5 (3 \leq i \leq n-2)$, it holds the inequality $|V_i| - |V_j| \leq 1$ for every pair $(i, j)$, $\chi_-(T(H_n)) \leq n + 1$. Since $v_{e_i}(1 \leq i \leq n)$ forms a clique of order $n + 1$, $\chi(T(H_n)) \geq n + 1, \chi_-(T(H_n)) \geq \chi(T(H_n)) \geq n + 1, \chi_-(T(H_n)) \geq n + 1$. Therefore $\chi_-(T(H_n)) = n + 1$. \qed
4.5 EQUITABLE COLORING OF GEAR GRAPH AND IT’S LINE, MIDDLE AND TOTAL GRAPH

Theorem 4.5.1. If \( n \geq 3 \) the equitable chromatic number of gear graph \( G_n \),
\[
\chi_e(G_n) = 2.
\]

Proof. Let \( V(G_n) = \{v\} \cup \{v_1, v_2, \ldots, v_{2n}\} \) where \( v_i \)'s are the vertices of cycles taken in cyclic order and \( v \) is adjacent with \( v_{2i-1} \) (\( 1 \leq i \leq n \)).

Now, partition the vertex set of \( V(G_n) \) as below.
\[
V_1 = \{v\} \cup \{v_{2i} : 1 \leq i \leq n - 1\}; \quad V_2 = \{v_{2i-1} : 1 \leq i \leq n\}.
\]
Clearly \( V_1, V_2 \) are independent sets of \( (G_n) \). Also \( |V_1| = n + 1 \) and \( |V_2| = n \), it holds the inequality
\[
||V_i|| - ||V_j|| \leq 1 \text{ for every pair } (i, j).
\]
\[
\chi_e(G_n) \leq 2. \quad \chi_e(G_n) \geq 2, \quad \chi_e(G_n) \geq \chi(G_n) \geq 2, \quad \chi_e(G_n) \geq 2. \quad \text{Therefore } \chi_e(G_n) = 2. \]

Theorem 4.5.2. If \( n \geq 3 \) the equitable chromatic number on line graph of Gear graph \( L(G_n) \), \( \chi_e(L(G_n)) = n \).

Proof. Let \( V(G_n) = \{v\} \cup \{v_1, v_2, \ldots, v_{2n}\} \) and \( E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n - 1\} \cup \{e''_i\} \) where \( e_i \) is the edge \( vv_{2i-1} \) (\( 1 \leq i \leq n \)), \( e'_i \) is the edge

Figure 4.10: Line graph of Gear Graph \( L(G_n) \).
Proof. \( V_i \) is the edge \( v_i v_{i+1} (1 \leq i \leq 2n-1) \), and \( e'_2 \) is the edge \( v_{2n-1}v_1 \). By the definition of line graph \( V(L(G_n)) = E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\} \).

Now, partition the vertex set of \( V(L(H_n)) \) as below.

\[ V_1 = \{e_1, e'_2, e'_{2n-1}\}; \quad V_i = \{e_i, e'_{2i}, e'_{2i-3} : 2 \leq i \leq n\}. \]

Clearly \( V_1, V_2, \ldots, V_n \) are independent sets of \( L(G_n) \). Also \( |V_1| = |V_i| = 3 \) \((2 \leq i \leq n)\), it holds the inequality \( |V_i| - |V_j| \leq 1 \) for every pair \((i, j)\). \( \chi(L(G_n)) \leq n \). Since \( e_i(1 \leq i \leq n) \) forms a clique of order \( n \), \( \chi(L(G_n)) \geq n \), \( \chi(L(G_n)) \geq \chi(L(G_n)) \geq n \). Therefore \( \chi(L(G_n)) = n \).

Theorem 4.5.3. If \( n \geq 5 \) the equitable chromatic number on middle graph of Gear graph \( M(G_n) \), \( \chi_e(M(G_n)) = n + 1 \).

Proof. Let \( V(G_n) = \{v\} \cup \{v_1, v_2, \ldots, v_{2n}\} \) and \( E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n-1\} \cup \{e''_i\} \) where \( e_i \) is the edge \( vv_{2i-1} \) \((1 \leq i \leq n)\), \( e'_i \) is the edge \( v_i v_{i+1} \) \((1 \leq i \leq 2n-1)\), and \( e''_{2n} \) is the edge \( v_{2n-1}v_1 \). By the definition of middle graph \( V(M(G_n)) = V(G_n) \cup E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\} \). Now, partition the vertex set of \( V(M(G_n)) \) as below.

\[ V_1 = \{e_1, v_2, e'_{2n-1}, e''_{2n-1}\}; \quad V_2 = \{e_2, v_1, v_4, e'_{2n-2}\}; \quad V_3 = \{e_3, v_3, e'_1, v_6, e'_2\}; \quad V_i = \{e_i, e'_{2i-5}, v_{2i-3}, v_{2i} : 4 \leq i \leq n\} \cup \{e''_{2i-8} : 5 \leq i \leq n\}; \quad V_{n+1} = \{v, v_{2n-6}, v_{2n-1}, e''_{2n-3}\}. \]

Clearly \( V_1, V_2, \ldots, V_n, V_{n+1} \) are independent sets of \( M(G_n) \). Also \( |V_1| = |V_2| = \)

Figure 4.11: Middle graph of Gear Graph \( M(G_n) \).
Proof. Let \( V(G_n) = \{ v \} \cup \{ v_1, v_2, \ldots, v_{2n} \} \) and \( E(G_n) = \{ e_i : 1 \leq i \leq n \} \cup \{ e'_i : 1 \leq i \leq 2n - 1 \} \cup \{ e'_n \} \) where \( e_i \) is the edge \( vv_{2i-1} \) (1 \( \leq i \leq n) \), \( e'_i \) is the edge \( v_iv_{i+1} \) (1 \( \leq i \leq 2n - 1) \), and \( e'_{2n} \) is the edge \( v_{2n-1}v_1 \). By the definition of total graph

\[
\begin{align*}
V(T(G_n)) &= V(G_n) \cup E(G_n) = \{ v \} \cup \{ v_i : 1 \leq i \leq 2n \} \cup \{ e_i : 1 \leq i \leq n \} \cup \{ e'_i : 1 \leq i \leq 2n \}.
\end{align*}
\]

Now, partition the vertex set of \( V(T(G_n)) \) as below.

\[
\begin{align*}
V_1 &= \{ e_1, e'_2, v_4, v_{2n-1} \}; \\
V_2 &= \{ e_2, v_1, e'_4, v_6 \}; \\
V_i &= \{ e_i, e'_{2i-5}, v_{2i-3}, e'_{2i}, v_{2i+2} : 3 \leq i \leq n-1 \}; \\
V_n &= \{ e_n, e'_{2n-5}, v_{2n-3}, e'_{2n} \}; \\
V_{n+1} &= \{ v, v_2, e'_{2n-1}, e'_{2n-3} \}.
\end{align*}
\]

Clearly \( V_1, V_2, V_i, V_n, V_{n+1} \) (3 \( \leq i \leq n - 1) \) are independent sets of \( T(G_n) \). Also \( |V_1| = |V_2| = |V_n| = |V_{n+1}| = 4 \) and \( |V_i| = 5 \) (3 \( \leq i \leq n - 1) \), it holds the inequality \( ||V_i| - |V_j|| \leq 1 \) for every pair \( (i, j) \), \( \chi_{e}(T(G_n)) \leq n + 1 \). Since \( ve_i(1 \leq i \leq n) \) forms a clique of order \( n + 1 \), \( \chi(T(G_n)) \geq n + 1 \), \( \chi_{e}(T(G_n)) \geq \chi(T(G_n)) \geq n + 1 \), \( \chi_{e}(T(G_n)) \geq n + 1 \). Therefore \( \chi_{e}(T(G_n)) = n + 1 \). \( \square \)